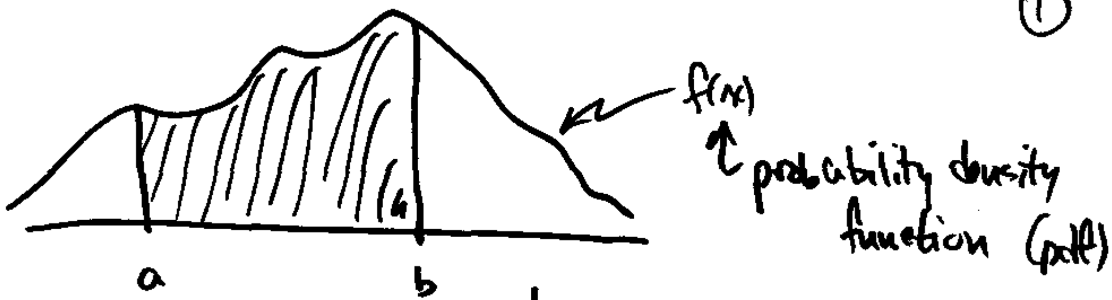
Stat 451
10-18-16

Continuous random variables

①



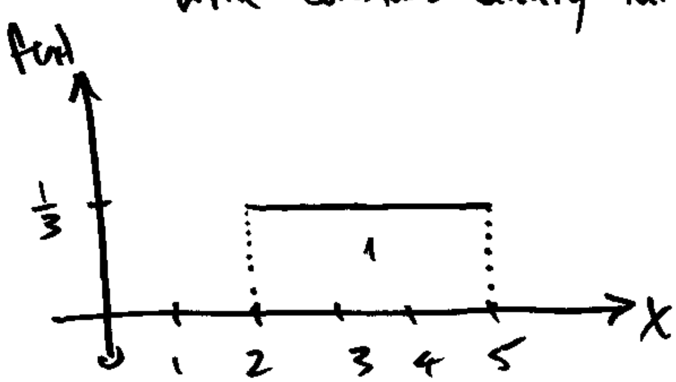
$$P(a < X < b) = \int_a^b f(x)\,dx$$

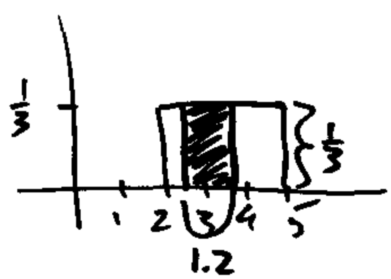
Properties of pdfs: 1. $f(x) \geq 0 \quad \forall x$

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

②

Example: Suppose that X takes on all possible values from 2 to 5, with constant density throughout.





Find the prob. that X lies between 2.5 and 3.7

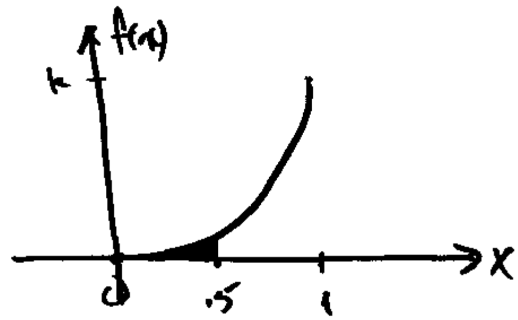
$$P(2.5 < X < 3.7) = \int_{2.5}^{3.7} \frac{1}{3}\,dx = \frac{1}{3}x\Big|_{2.5}^{3.7} = \frac{1}{3}(3.7 - 2.5)$$
$$= .4$$

③

Example: $f(x) = kx^2 \quad 0<x<1 \quad (f(x)=0 \text{ elsewhere})$

① Find k

② $P(X<.5)$



① $\displaystyle\int_0^1 kx^2\,dx = 1 = k\,\frac{x^3}{3}\Big|_0^1$

$\displaystyle = \frac{k}{3}(1-0) = \frac{k}{3} \qquad \therefore k=3$

② $\displaystyle P(X<.5) = \int_0^{.5} 3x^2\,dx = x^3\Big|_0^{.5} = (.5)^3 - 0 = .125$

④

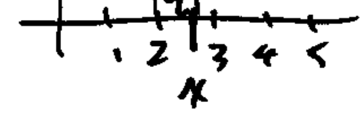
The cumulative distribution function (cdf)

was $P(X \le x) = F(x)$

For continous random variables, $\displaystyle F(x) = \int_{-\infty}^{x} f(t)\,dt$

Look at the graphs of our 2 examples.

Ex1. $f(x) = \frac{1}{3}, \quad 2<x<5$



$$F(x) = \begin{cases} 0 & x \le 2 \\ \int_2^x \frac{1}{3}\,dt & 2<x<5 \\ 1 & x \ge 5 \end{cases}$$

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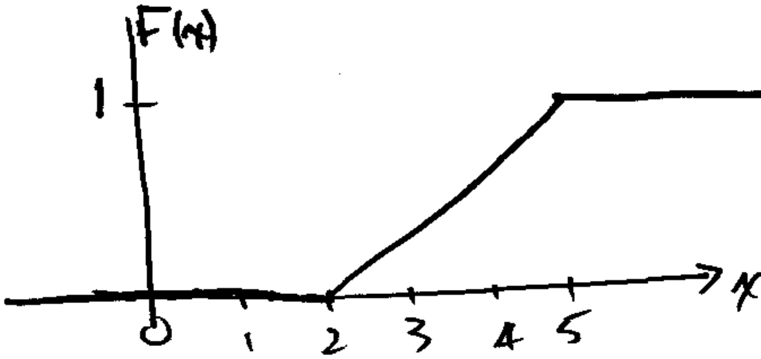
$$F(x) = \begin{cases} 0 & x \le 2 \\ \frac{1}{3}(x-2) & 2 < x < 5 \\ 1 & x \ge 5 \end{cases} \qquad \frac{1}{3}t\Big|_2^x = \frac{1}{3}(x-2)$$



Note: this nondecreasing & continuous

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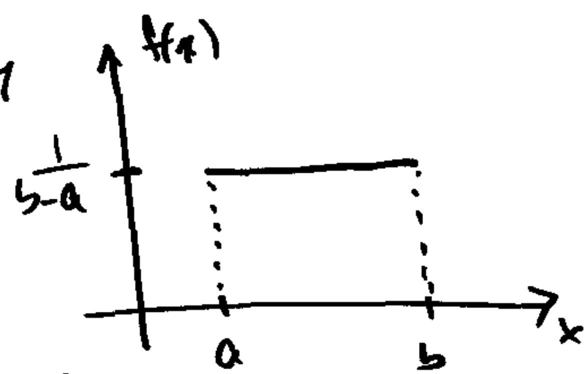
The uniform distribution

X takes on values in the interval (a, b)

with constant density

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$



Recall $\mu = E(X) = \begin{cases} \sum_{\text{all } x} x\, p(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x f(x)\, dx & X \text{ continuous} \end{cases}$

(7)

For the uniform distribution,

$$\mu = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{b-a}\frac{1}{2}(b^2-a^2)$$

$$= \frac{1}{\cancel{b-a}}\frac{1}{2}\cancel{(b-a)}(b+a) = \frac{a+b}{2}$$

Recall $\sigma^2 = Var(X) = E(X^2) - \mu^2$

and $E(X^2) = \begin{cases} \sum_{\text{all } x} x^2 p(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} x^2 f(x)\,dx & X \text{ continuous} \end{cases}$

(8)

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b = \frac{1}{b-a}\frac{1}{3}(b^3-a^3)$$

$$= \frac{1}{\cancel{b-a}}\cdot\frac{1}{3}\cancel{(b-a)}(b^2+ab+a^2) = \frac{1}{3}(b^2+ab+a^2)$$

$$\sigma^2 = \frac{1}{3}(b^2+ab+a^2) - \left[\frac{1}{2}(a+b)\right]^2$$

$$= \frac{1}{3}(b^2+ab+a^2) - \frac{1}{4}(a^2+2ab+b^2)$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

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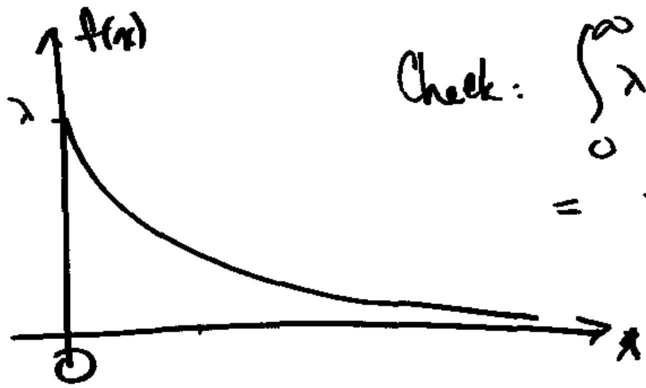
The exponential distribution

$$f(x) = \lambda e^{-\lambda x} \qquad x > 0$$



Check: $\int_0^\infty \lambda e^{-\lambda x}\, dx$

$$= \lambda \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty$$

$$= -(0 - 1)$$

$$= 1 \checkmark$$

10

Find $F(x)$, μ, σ^2

$$F(x) = \begin{cases} 0 & x < 0 \\ \underbrace{\int_0^x \lambda e^{-\lambda t}\, dt}_{} & x \geq 0 \end{cases}$$

$$\lambda \frac{e^{-\lambda t}}{-\lambda} \Big|_{t=0}^{x} = -(e^{-\lambda x} - 1)$$

$$= 1 - e^{-\lambda x}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

(11)

$$\mu = \int_0^\infty x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x\, e^{-\lambda x} dx$$

$$= \cancel{\lambda}\left[x \frac{e^{-\lambda x}}{-\cancel{\lambda}} \Big|_0^\infty - \int_0^\infty \frac{e^{-\lambda x}}{-\cancel{\lambda}} dx \right]$$

$u = x$

$dv = e^{-\lambda x} dx$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$= -\frac{x}{e^{\lambda x}}\Big|_0^\infty + \frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty$$

$du = dx$

$v = \frac{e^{-\lambda x}}{-\lambda}$

$$= 0 + \left(-\frac{1}{\lambda}\right)(0-1)$$

same limit as $\frac{1}{\lambda e^{\lambda x}}$

$$\boxed{\therefore \mu = \frac{1}{\lambda}}$$

(12)

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x^2\, e^{-\lambda x} dx$$

$u = x^2 \quad du = 2x\,dx$

$dv = e^{-\lambda x} dx$

$v = \frac{e^{-\lambda x}}{-\lambda}$

$$= \cancel{\lambda}\left[x^2 \frac{e^{-\lambda x}}{-\cancel{\lambda}} \Big|_0^\infty - \int_0^\infty 2x \frac{e^{-\lambda x}}{-\cancel{\lambda}} dx \right]$$

$$= -\frac{x^2}{e^{\lambda x}}\Big|_0^\infty + 2\frac{1}{\lambda}\underbrace{\int_0^\infty \lambda x\, e^{-\lambda x} dx}_{\mu}$$

$$= 0 + 2 \cdot \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\boxed{\therefore \sigma^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}}$$

(13)

Relationship between the Poisson (discrete) and exponential (continuous) distributions

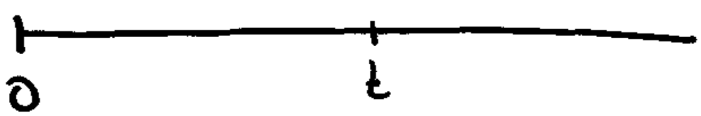
Assume $X \sim \text{Poisson}(\alpha)$ — expected # occurrences per unit time

X counts the # of occurrences of a certain type in a time interval.

Let T be time until the 1st occurrence.

(14)

Find $P(T > t) = \underbrace{P(X=0)}_{\text{in the } (0,t) \text{ interval}}$



$$P(X=0) = \frac{(\alpha t)^x e^{-\alpha t}}{x!}$$

$$= e^{-\alpha t}$$

[α was the expected # of occurrences per unit time, so the expected # in time t is αt]

$$P(T > t) = e^{-\alpha t}$$

$$\therefore F(t) = P(T \le t) = 1 - e^{-\alpha t}$$

$$\therefore f(t) = -e^{-\alpha t}(-\alpha)$$

$$= \alpha e^{-\alpha t}$$

, which is an exponential pdf.