Stat 451
10-13-16

(1)

For the hypergeometric distribution,

$$\mu = n\frac{N_1}{N}, \quad \sigma^2 = n\frac{N_1}{N}\left(1-\frac{N_1}{N}\right)\left(\frac{N-n}{N-1}\right)$$

finite population correction

Geometric distribution

Geometric series $= a + ar + ar^2 + \dots = T$

$$T - a = ar + ar^2 + \dots$$
$$= r(a + ar + ar^2 + \dots) = rT$$
$$T(1-r) = a \qquad T = \frac{a}{1-r}$$

(2)

Run a sequence of Bernoulli trials

(independent trials, 2 possible outcomes on each trial, same probability of success " " ")

X = trial on which the 1st success occurs

$$\boxed{X = 1, 2, 3, \dots \qquad p(x) = q^{x-1}p}$$

Check that the probs sum to 1:

$$\sum_{x=1}^{\infty} q^{x-1}p = p + pq + pq^2 + \dots$$
$$= \frac{p}{1-q} = \frac{p}{p} = 1$$

③

Find $\mu = E[X] = \sum_{\text{all } x} x\, p(x)$

$$= \sum_{x=1}^{\infty} x\, q^{x-1} p \qquad \text{let } y = x-1, \quad x = y+1$$

$$= \sum_{y=0}^{\infty} (y+1) q^{y} p = \underbrace{\sum_{y=0}^{\infty} y q^{y} p}_{①} + \underbrace{\sum_{y=0}^{\infty} q^{y} p}_{②}$$

①: $\sum_{y=1}^{\infty} y q^{y} p = q \sum_{y=1}^{\infty} y q^{y-1} p = q\mu$

②: $p + pq + pq^2 + \dots = 1$

$$\mu = q\mu + 1$$
$$\mu - q\mu = 1$$
$$\mu = \frac{1}{1-q} = \frac{1}{p}$$

④

To find σ^2, find $E[X^2] - \mu^2$

know this already

$E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p$ let $y = x-1$ and proceed as before

Answer: $\sigma^2 = \dfrac{q}{p^2}$

Negative binomial (Pascal) distribution

Same setup as the geometric distribution,

but X = trial on which the r^{th} success occurs

(5)

$$X = r, r+1, r+2, \ldots$$

$p(x) = P(r^{\text{th}}$ success happens on trial # $x)$

$= P($ r−1 successes in the 1^{st} x−1 trials $\cap$ success on trial $x)$

$$= \binom{x-1}{r-1} p^{r-1} q^{(x-1-(r-1))} \cdot p$$

$$\boxed{p(x) = \binom{x-1}{r-1} p^r q^{x-r}, \quad X = r, r+1, \ldots}$$

(6)

$$X = X_1 + X_2 + \ldots + X_r$$

X_1 ↑ trial on which 1^{st} success occurred

X_2 ↑ trial on which 2^{nd} occurs, starting with new count

Each X_i has a geometric distribution, + they are all independent.

So $\mu = E[X] = E[X_1] + \ldots + E[X_r]$

$$= \frac{r}{p}$$

$$\sigma^2 = Var[X] = V[X_1] + \ldots + V[X_r] = \frac{rq}{p^2}$$

(7)

Example: Each attack has a 20% success rate
It takes 3 successes to eliminate the threat. (Must assume independence)

✓Find the prob. distr. Find the expected value.

$$X \sim NB(r=3, p=.2)$$

$$p(x) = \binom{x-1}{2}(.2)^3(.8)^{x-3}, \quad X = 3, 4, \ldots$$

$$\mu = \frac{r}{p} = \frac{3}{.2} = 15$$

(8)

Find the probability that the 3rd success occurs on trial 10

$$p(10) = \binom{9}{2}(.2)^3(.8)^7 = .0604$$

Find the prob. that the 3rd success occurs on or before trial 20.

$$P(X \le 20) = p(3) + p(4) + \ldots + p(20)$$

(9)

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

Let $y=\left(1+\frac{x}{n}\right)^n$

$$\ln y = n\ln\left(1+\frac{x}{n}\right) = \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}}$$

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{\left(-\frac{1}{n^2}\right)}$$

by L'Hôpital

(10)

$$= \lim_{n\to\infty}\left(\frac{x}{1+\frac{x}{n}}\right) = x$$

$$\lim_{n\to\infty}\ln y = x$$

$$\parallel$$

$\ln\left(\lim_{n\to\infty} y\right)$ because $\ln y$ is continuous

$$\therefore \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

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Start with the binomial distribution

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

Take the limit as $n \to \infty$ and $p \to 0$

and hold np constant

Set $np = \lambda$ so $p = \frac{\lambda}{n}$

$$p(x) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$\lim_{n\to\infty} p(x) = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!}\,\frac{\lambda^x}{n^x}\,\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

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$$= \frac{\lambda^x}{x!}\lim_{n\to\infty} \underbrace{\frac{n(n-1)\cdots(n-x+1)}{n\cdot n\cdots n}}_{1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}\,\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\lambda^x}{x!}e^{-\lambda}$$

$$\boxed{p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \text{ is the Poisson distribution} \quad X = 0, 1, 2, \ldots}$$

(13)

Check: $$\sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \underbrace{\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right]}_{\text{Taylor series for } e^{\lambda}}$$

$$= 1$$

$\mu = E[X] = \lambda$ because $np = \lambda$ was held constant

$\sigma^2 = \lambda$ $\qquad npq = \lambda\left(1 - \frac{\lambda}{n}\right)$

(14)

Poisson process or Poisson experiment

- observe a continuous process over a fixed amount of time or space
- X counts the # of occurrences of a certain type
- Observation in non-overlapping intervals are independent
- The probability of an occurrence in a given interval is proportional to the length of the interval
- In a very small interval, the prob. of more than 1 occurrence is approx. zero

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Example: A fire station gets an average of 2.83 calls per day.
On a particular day, find the prob that it gets 4 or fewer calls.

$X \sim \text{Poisson}(\lambda = 2.83)$

$$p(x) = \frac{2.83^x e^{-2.83}}{x!}$$

$$P(X \le 4) = p(0) + p(1) + \dots + p(4)$$
$$= e^{-2.83}\left(1 + 2.83 + \frac{2.83^2}{2} + \frac{2.83^3}{6} + \frac{2.83^4}{24}\right)$$
$$= .8429$$

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HW #3 due Oct 20

p. 89 #9, 12

p. 97 #10

p. 122 #4

p. 145 #5, 7, 13, 16