Counting rules, continued

⑤ Arrangements of like objects.

Suppose there are n items of k different types.

n_1 of type 1, ..., n_k of type k.

The # of different possible arrangements is

$$\binom{n}{n_1\ n_2 \cdots n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Example: STATISTICS. How many different ways can these letters be arranged?

②

Type 1: A $\quad n_1 = 1 \qquad n = 10$

Type 2: C $\quad n_2 = 1$

Type 3: I $\quad n_3 = 2$

Type 4: S $\quad n_4 = 3$

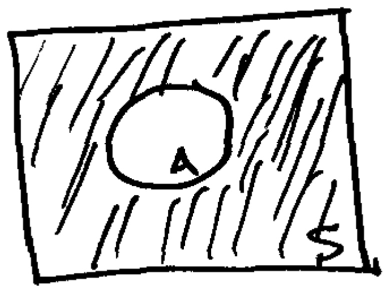
Type 5: T $\quad n_5 = 3$

$$\binom{10}{1\ 1\ 2\ 3\ 3} = \frac{10!}{1!\,1!\,2!\,3!\,3!} = \frac{10!}{72}$$

$$= 50{,}400$$

Defn: The complement of an event A is the set A^c consisting of all items in S that are not in A.

③



A^c

$$N(A) + N(A^c) = N$$

$$\frac{N(A)}{N} + \frac{N(A^c)}{N} = 1$$

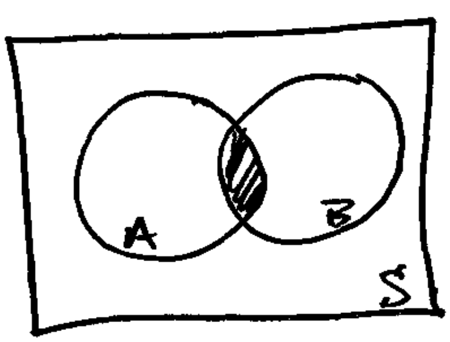
$$P(A) + P(A^c) = 1$$

$$\boxed{P(A^c) = 1 - P(A)}$$

<u>Complement</u> <u>rule</u>

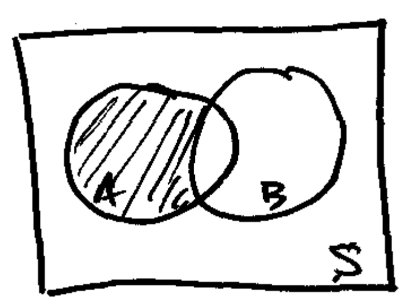
④

<u>Defn</u>: The <u>intersection</u> of event A and B is $A \cap B$, which is the set of items appearing in both A <u>and</u> B.



$A \cap B$

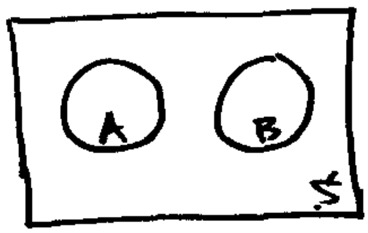
<u>Defn</u>: $A - B = A \cap B^c$



$A - B$

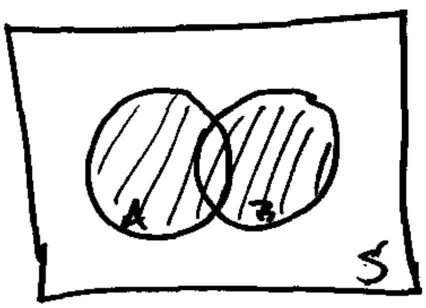
⑤

Defn: A and B are disjoint or mutually exclusive if $A \cap B = \phi$ ← empty set



Defn: The union of events A and B is the set of items that appear in either A or B (or possibly both)

⑥



$A \cup B$

Suppose A & B are disjoint.

Then $\frac{N(A \cup B)}{N} = \frac{N(A) + N(B)}{N}$

$\boxed{P(A \cup B) = P(A) + P(B)}$ Union rule for disjoint events

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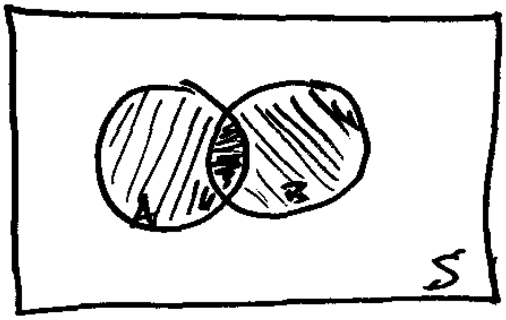
Consider the more general case



$$(A-B) \cup (A \cap B) \cup (B-A) = A \cup B$$

$$P(A \cup B) = P(A-B) + P(A \cap B) + P(B-A)$$

by our union rule for disjoint events

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Also $A = (A-B) \cup (A \cap B)$

$$P(A) = P(A-B) + P(A \cap B)$$

$$\Rightarrow P(A-B) = P(A) - P(A \cap B)$$

Similarly, $P(B-A) = P(B) - P(A \cap B)$

$$P(A \cup B) = \underbrace{P(A) - P(A \cap B)}_{P(A-B)} + P(A \cap B) + \underbrace{P(B) - P(A \cap B)}_{P(B-A)}$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$ Union rule

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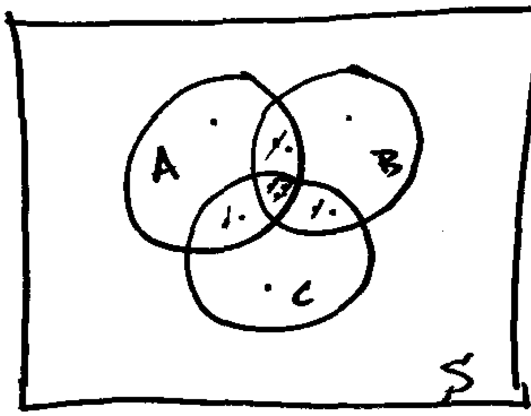
Generalize further to 3 events:



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

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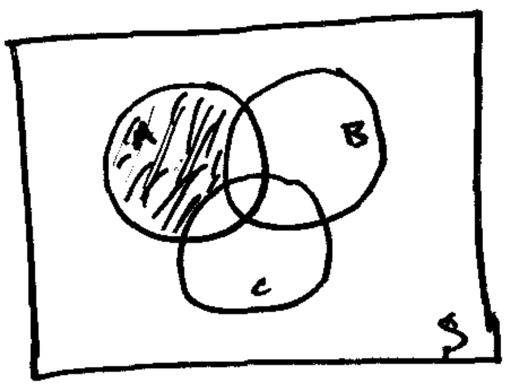
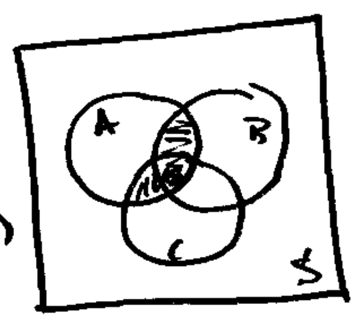
<u>Example</u>: $A \cap (B \cup C)^c = A - (B \cup C)$



$$P(A \cap (B \cup C)^c) = P(A) - P(A \cap (B \cup C))$$

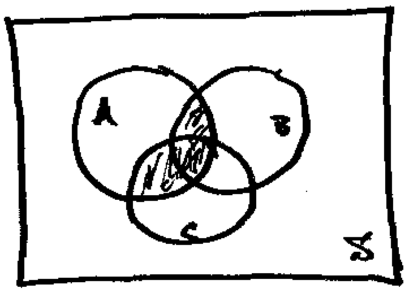
<u>Example</u>: $(A \cap B) \cup (A \cap C)$

$$P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$



(11)

<u>Example</u>: $A \cap (B \cup C)$



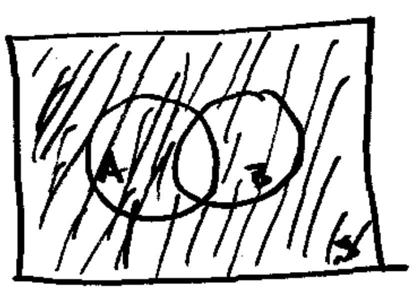
Note: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

<u>Distributive rule</u>

Also: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(12)

<u>Example</u>: $(A \cap B)^c = A^c \cup B^c$



De Morgan's Law

Also $(A \cup B)^c = A^c \cap B^c$

(13)

Conditional probability

		Income: High	Mid	Low	
Gender	M	30	40	30	100
	F	60	50	40	150
		90	90	70	250

Experiment: Select one of the 250 people

$$N = 250$$

Find $P(M) = \dfrac{N(M)}{N} = \dfrac{100}{250} = .4$

(14)

Find $P(M \cap H) = \dfrac{N(M \cap H)}{N} = \dfrac{30}{250} = .12$

Find $P(H) = \dfrac{N(H)}{N} = \dfrac{90}{250} = .36$

$P(M)$ & $P(H)$ are called <u>marginal probabilities</u>

$P(M \cap H)$ is called a <u>joint probability</u>

Given that the person who was selected is male, find the probability that he has a high income.

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$$P(H \mid M) = \frac{30}{100} = \frac{N(H \cap M)}{N(M)}$$

"given"

Defn: The conditional probability of A, given B,

is $$P(A \mid B) = \frac{N(A \cap B)}{N(B)}$$

Note: $$P(A \mid B) = \frac{N(A \cap B)/N}{N(B)/N} = \frac{P(A \cap B)}{P(B)}$$

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So $$\boxed{P(A \cap B) = P(B)\,P(A \mid B)}$$

Multiplication rule for intersections

OR: $$P(A \cap B) = P(A)\,P(B \mid A)$$

Defn: If $P(A \mid B) = P(A)$

then A & B are independent

Note: If A & B are independent, then

$$P(A \cap B) = P(B)\,P(A)$$