Find the mgf for the normal distribution (1)

451

3-11

$$M_X(t) = E[e^{tX}] = \int_{\text{all } x} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\left[tx - \frac{1}{2\sigma^2}x^2 + \frac{1}{\sigma^2}\mu x - \frac{1}{2\sigma^2}\mu^2\right]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2\sigma^2 tx - 2\mu x + \mu^2\right]} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2(\sigma^2 t + \mu)x + \mu^2\right]} dx$$ (2)

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left[x^2 - 2(\sigma^2 t + \mu)x + (\sigma^2 t + \mu)^2 - (\sigma^2 t + \mu)^2 + \mu^2\right]} dx$$

$$= e^{-\frac{1}{2\sigma^2}\left[-(\sigma^2 t + \mu)^2 + \mu^2\right]} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left[x - (\sigma^2 t + \mu)\right]^2} dx}_{1}$$

$$= e^{-\frac{1}{2\sigma^2}\left[-\sigma^4 t^2 - 2\mu\sigma^2 t - \mu^2 + \mu^2\right]}$$

③

$$= e^{-\frac{1}{2\sigma^2}[-\sigma^2(\sigma^2 t^2 + 2\mu t)]}$$

$$= e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$$

For the standard normal distribution,

$$M_X(t) = e^{\frac{1}{2}t^2} \quad (\text{Since } \mu = 0 \text{ and } \sigma = 1)$$

Properties of the m.g.f.

Start with a random variable X.

Let $Y = X + c$

④

$$M_Y(t) = E[e^{tY}]$$

$$= E[e^{t(X+c)}]$$

$$= E[e^{tX} \cdot e^{tc}]$$

$$= e^{tc} E[e^{tX}] = e^{tc} M_X(t)$$

Let $Y = aX$

$$M_Y(t) = E[e^{tY}] = E[e^{taX}]$$

$$= E[e^{(ta)X}] \quad \text{Let } u = ta$$

$$= E[e^{uX}] = M_X(u) = M_X(ta)$$

⑤

Suppose that X_1 and X_2 are independent random variables. Let $Y = X_1 + X_2$

$$M_Y(t) = E[e^{tY}] = E[e^{t(X_1+X_2)}]$$

$$= E[e^{tX_1} \cdot e^{tX_2}] = E[e^{tX_1}] \cdot E[e^{tX_2}]$$

(because of independence)

$$= M_{X_1}(t)\, M_{X_2}(t)$$

Example: Suppose $X_1 \sim Exp(\beta)$, $X_2 \sim Exp(\beta)$, $X_3 \sim Exp(\beta)$, independent.

⑥

Let $Y = X_1 + X_2 + X_3$. Find the distribution of Y.

$$M_Y(t) = M_{X_1}(t)\, M_{X_2}(t)\, M_{X_3}(t)$$

$$= \frac{1}{1-\beta t} \cdot \frac{1}{1-\beta t} \cdot \frac{1}{1-\beta t} = (1-\beta t)^{-3}$$

This is the mgf of a gamma distribution with $\alpha = 3$, β

⑦

Let $X_1, X_2, \ldots, X_n$ be independent random variables, each with the same distribution with mean μ and standard deviation σ.

Let $Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$.

Find the m.g.f. of Y and take its limit as $n \to \infty$.

⑧

Let $Z_i = \frac{X_i - \mu}{\sigma}$ $\quad E[Z_i] = 0$ and $V[Z_i] = 1$

$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$$

$$M_Y(t) = M_{\Sigma Z_i}\left(\frac{t}{\sqrt{n}}\right)$$

$$= M_{Z_1}\left(\frac{t}{\sqrt{n}}\right) M_{Z_2}\left(\frac{t}{\sqrt{n}}\right) \cdots M_{Z_n}\left(\frac{t}{\sqrt{n}}\right)$$

(9)

$$M_Y(t) = \left[M_{Z_1}\left(\tfrac{t}{\sqrt{n}}\right)\right]^n = \left[E\left[e^{\frac{t}{\sqrt{n}}Z_1}\right]\right]^n$$

Now take the limit as $n \to \infty$

$$\lim_{n\to\infty} M_Y(t) = \lim_{n\to\infty}\left[E\left[1 + \frac{tZ_1}{\sqrt{n}} + \frac{t^2 Z_1^2}{2n} + \dots\right]\right]^n$$

$$= \lim_{n\to\infty}\left[1 + 0 + \frac{t^2}{2n}\cdot 1 + \dots\right]^n$$

$$\lim_{n\to\infty} \ln M_Y(t) = \lim_{n\to\infty} n \ln\left[1 + \frac{t^2}{2n} + \dots\right]$$

(10)

$$= \lim_{n\to\infty} \frac{\ln\left[1 + \frac{t^2}{2n} + \dots\right]}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{1 + t^2/2n + \dots}\left[-\frac{t^2}{2n^2} + \dots\right]}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1 + \frac{t^2}{2n} + \dots}\left[\frac{t^2}{2} + \dots\right]$$

$$= \frac{t^2}{2}$$

$$\ln M_Y(t) \to \frac{t^2}{2}$$

so $M_Y(t) \to e^{t^2/2}$

$e^{t^2/2}$ is the mgf of the standard normal distribution. (11)

$$Y = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) = \frac{1}{\sigma\sqrt{n}} \left(\sum_{i=1}^{n} X_i - n\mu \right)$$

$$= \frac{n}{\sigma\sqrt{n}} \left(\frac{1}{n} \sum_{i=1}^{n} X_i - \mu \right) = \frac{\sqrt{n}}{\sigma} \left(\bar{X} - \mu \right)$$

$$= \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

(12)

Central Limit Theorem:

Assume $X_1, X_2, \ldots, X_n$ are independent and identically distributed with mean μ and standard deviation σ.

Let $Y = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$. Then as $n \to \infty$ the distribution of Y approaches the standard normal distribution.

HW #9 due Thurs, March 18

p. 226 #18

p. 252 #24, 26

Final exam is Thurs, March 18

10:15 - 12:05

1 page of notes + tables + calculator

Covers all material since the midterm (after 4.3)