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3-4

Gamma example:

We have a Poisson process with an average of 5 occurrences per minute. $(\lambda = 5)$

Find the probability that the waiting time for the 3^{rd} occurrence is more than 30 seconds.

T = waiting time for 3^{rd} occurrence.

$$T \sim \text{Gamma}\left(\alpha = 3,\ \beta = \frac{1}{\lambda} = \frac{1}{5}\right)$$

$$P(T > .5) = \int_{.5}^{\infty} \frac{1}{\left(\frac{1}{5}\right)^3 2!} x^2 e^{-5x}\,dx$$

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$$\boxed{\underbrace{P(T > t)}_{\text{Gamma}(\alpha,\ \beta = \frac{1}{\lambda})} = \underbrace{P(X \le \alpha - 1)}_{\text{Poisson}(\lambda t)}}$$

$$P(T > .5) = P(X \le 2) \qquad \mu = 5(.5) = 2.5$$

$$= p(0) + p(1) + p(2)$$

$$= \left(\frac{2.5^0}{0!} + \frac{2.5^1}{1!} + \frac{2.5^2}{2!}\right) e^{-2.5}$$

③

Same example, but find the probability that the waiting for the 4th occurrence is anywhere between 1 minute and 2 minuts.

$$T \sim \text{Gamma}(\alpha = 4,\ \beta = \frac{1}{\lambda} = \frac{1}{5})$$

$$P(1 < T < 2) = \int_1^2 \frac{1}{(\frac{1}{5})^4\, 3!} x^3 e^{-5x}\, dx$$

$$= P(T > 1) - P(T > 2)$$

④

$$P(T > 1) = P(X \le 3) \quad \text{where } X \sim \text{Poisson}$$

$$\mu = \lambda t = 5 \cdot 1 = 5$$

$$= p(0) + p(1) + p(2) + p(3)$$

$$= e^{-5}\left(\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} \right)$$

$$P(T > 2) = P(X \le 3) \quad \text{where } X \sim \text{Poisson}$$

$$\mu = \lambda \cdot t = 5 \cdot 2 = 10$$

$$= e^{-10}\left(\frac{10^0}{0!} + \frac{10^1}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} \right)$$

(5)

For the gamma distribution, $\mu = \alpha\beta$
$\sigma^2 = \alpha\beta^2$

Another special case of the gamma distribution:

Set $\alpha = \frac{v}{2}$, where v is a positive integer

and $\beta = 2$.

$$f(x) = \frac{1}{2^{v/2}\Gamma(\frac{v}{2})} x^{\frac{v}{2}-1} e^{-\frac{1}{2}x}$$

This is the Chi-squared density with parameter v.

(6)

Chp7 Sec.3

Moment-generating function

Definition: $M_X(t) = E[e^{tX}]$

Properties: $M_X(0) = E[e^0] = E[1] = 1$

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}E[e^{tX}]$$

$$= E\left[\frac{d}{dt}e^{tX}\right] = E[e^{tX} \cdot X]$$

$$\frac{d}{dt}M_X(t)\Big|_{t=0} = E[X]$$

(7)

$$\frac{d^2}{dt^2} M_x(t) = E\left[\frac{d^2}{dt^2} e^{tX} \right]$$
$$= E\left[e^{tX} \cdot X^2 \right]$$
$$\left. \frac{d^2}{dt^2} M_x(t) \right|_{t=0} = E[X^2]$$

Find the mgf of the exponential distribution.

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad x > 0$$

$$M_X(t) = E[e^{tX}] = \int_{\text{all } x} e^{tx} f(x)\, dx$$

(8)

$$M_X(t) = \int_0^\infty e^{tx} \frac{1}{\beta} e^{-x/\beta}\, dx$$
$$= \frac{1}{\beta} \int_0^\infty e^{-x\left(\frac{1}{\beta} - t\right)}\, dx$$
$$= \frac{1}{\beta} \left. \frac{e^{-x\left(\frac{1}{\beta} - t\right)}}{-\left(\frac{1}{\beta} - t\right)} \right|_{x=0}^{\infty}$$
$$= \frac{1}{\beta} \left[0 - \frac{1}{-\left(\frac{1}{\beta} - t\right)} \right] = \frac{1}{1 - \beta t}$$

provided $\frac{1}{\beta} - t > 0$

$t < \frac{1}{\beta}$

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Check $M_X(0) = \frac{1}{1-\beta \cdot 0} = 1$ ✓

$$\frac{d}{dt} M_X(t) = \frac{d}{dt}(1-\beta t)^{-1} = -(1-\beta t)^{-2}(-\beta)$$
$$= \beta(1-\beta t)^{-2}$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \beta = E[X]$$

$$\frac{d^2}{dt^2} M_X(t) = \beta(-2)(1-\beta t)^{-3}(-\beta) = 2\beta^2(1-\beta t)^{-3}$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = 2\beta^2 = E[X^2]$$

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$$\sigma^2 = E[X^2] - (E[X])^2 = 2\beta^2 - \beta^2 = \beta^2$$

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mgf for binomial:

$$M_X(t) = E[e^{tX}] = \sum_{\text{all } x} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

(11)

$$= (pe^t+q)^n \sum_{x=0}^{n} \binom{n}{x} \left(\frac{pe^t}{pe^t+q}\right)^x \left(\frac{q}{pe^t+q}\right)^{n-x}$$

$$= (pe^t+q)^n \underbrace{\sum_{x=0}^{n} \binom{n}{x} (p')^x (q')^{n-x}}_{1}$$

$$M_x(t) = (pe^t+q)^n$$

$$M_x(0) = (p+q)^n = 1 \checkmark$$

$$M_x'(t) = n(pe^t+q)^{n-1} \cdot pe^t$$

$$M_x'(0) = np = E[X]$$

$$M_x''(t) = np\left[(pe^t+q)^{n-1}e^t + e^t(n-1)(pe^t+q)^{n-2}pe^t\right]$$

(12)

$$M_x''(0) = np\left[1 + (n-1)p\right]$$

$$= np + n(n-1)p^2 = E[X^2]$$

$$\sigma^2 = np + n(n-1)p^2 - (np)^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np(1-p) = npq$$

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Mgf for gamma:

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \int_0^\infty \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x\left(\frac{1}{\beta}-t\right)}\, dx$$

$$= \frac{\left(\left(\frac{1}{\beta}-t\right)^{-1}\right)^\alpha}{\beta^\alpha} \underbrace{\int_0^\infty \frac{1}{\left[\left(\frac{1}{\beta}-t\right)^{-1}\right]^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\left(\frac{1}{\beta}-t\right)^{-1}}}\, dx}_{1}$$

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$$M_X(t) = \frac{\left(\frac{1}{\beta}-t\right)^{-\alpha}}{\beta^\alpha} = \frac{1}{\left[\beta\left(\frac{1}{\beta}-t\right)\right]^\alpha}$$

$$= (1-\beta t)^{-\alpha}$$

Check: $M_X(0) = 1$ ✓

$$M_X'(t) = -\alpha(1-\beta t)^{-\alpha-1}(-\beta)$$
$$= \alpha\beta(1-\beta t)^{-\alpha-1}$$

$$M_X'(0) = \alpha\beta = E[X]$$

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$$M_X''(t) = \alpha\beta(-\alpha-1)(1-\beta t)^{-\alpha-2}(-\beta)$$
$$= \alpha\beta^2(\alpha+1)(1-\beta t)^{-\alpha-2}$$

$$M_X''(0) = \alpha\beta^2(\alpha+1) = \alpha^2\beta^2 + \alpha\beta^2$$
$$= E[X^2]$$

$$\sigma^2 = \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2 = \alpha\beta^2$$