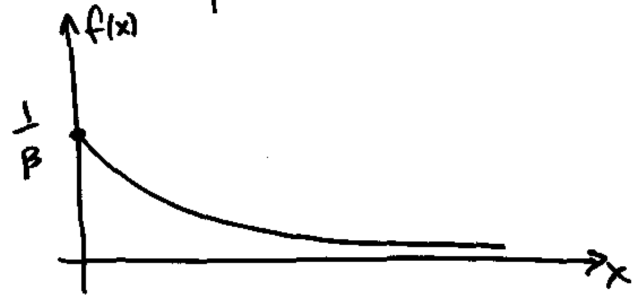
The exponential distribution

① 451 3-2

$$f(x) = \frac{1}{\beta} e^{-x/\beta} \qquad x > 0$$



To find probabilities, integrate:

$$P(a < X < b) = \int_a^b \frac{1}{\beta} e^{-x/\beta}\, dx$$

$$= F(x)\Big|_a^b = F(b) - F(a)$$ ②

But what is $F(x)$?

$$F(x) = P(X \le x) = \int_0^x \frac{1}{\beta} e^{-t/\beta}\, dt$$

$$= \int_0^{-x/\beta} -e^u\, du \qquad \text{let } u = -t/\beta$$

$$\frac{du}{dt} = -\frac{1}{\beta}$$

$$= -e^u\Big|_0^{-x/\beta} \qquad du = -\frac{1}{\beta} dt$$

$$= -e^{-x/\beta} - (-1) = 1 - e^{-x/\beta}$$

Example: Exponential distribution with $\beta = 2$
Find the probability that X is between 1 and 5.

③

Expon$(\beta=2)$ $\quad f(x) = \frac{1}{2} e^{-x/2}$

$$F(x) = 1 - e^{-x/2}$$

$$P(1 < X < 5) = \int_1^5 \frac{1}{2} e^{-x/2} dx$$

$$= F(5) - F(1)$$

$$= (1 - e^{-5/2}) - (1 - e^{-1/2})$$

$$= e^{-1/2} - e^{-5/2}$$

Find μ and σ^2.

④

$$\mu = E[X] = \int_{\text{all } x} x f(x)\, dx = \int_0^\infty x \frac{1}{\beta} e^{-x/\beta} dx$$

$u = x \qquad dv = \frac{1}{\beta} e^{-x/\beta} dx$

$du = dx \qquad v = -e^{-x/\beta}$

$$= uv \Big|_a^b - \int_a^b v\, du$$

$$= -xe^{-x/\beta} \Big|_0^\infty + \int_0^\infty e^{-x/\beta} dx$$

(5)

What is $\lim_{x\to\infty} x e^{-x/\beta}$?

$$\lim_{x\to\infty} \frac{x}{e^{x/\beta}} = \lim_{x\to\infty} \frac{1}{\frac{1}{\beta}e^{x/\beta}} = 0$$

$$\mu = 0 - (0) + \int_0^\infty e^{-x/\beta} dx$$

$$= -\beta e^{-x/\beta} \Big|_0^\infty = -\beta(0-1) = \beta$$

Similarly, $\sigma^2 = E[x^2] - \beta^2 = \int_0^\infty x^2 \frac{1}{\beta} e^{-x/\beta} dx - \beta^2$

$$= \beta^2$$

(6)

Recall the Poisson distribution:

$$p(x) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

= probability of x occurrences in a fixed amount of time t.

$$p(0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

$$= 1 - \left(1 - e^{-\lambda t}\right)$$

$= 1 - P(T \le t)$, where

T has an $\mathrm{Exp}(\beta = 1/\lambda)$ distribution

⑦

That is, $P(X=0) = P(T > t)$

$\uparrow$ Poisson $(\mu = \lambda t)$ $\qquad$ $\uparrow$ Exponential $\left(\beta = \frac{1}{\lambda}\right)$

Thus, the probability of getting 0 occurrences in time t is the same as the probability that the waiting time for the 1st occurrence is longer than t.

⑧

The gamma function is defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

$$u = x^{\alpha-1} \qquad dv = e^{-x}\, dx$$

$$du = (\alpha-1)x^{\alpha-2}\, dx \qquad v = -e^{-x}$$

$$\Gamma(\alpha) = \cancel{-x^{\alpha-1}e^{-x}\Big|_0^\infty}^{\,0} + \int_0^\infty (\alpha-1)x^{\alpha-2}e^{-x}\, dx$$

$$= (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\, dx = (\alpha-1)\Gamma(\alpha-1)$$

(9)

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 0-(-1)$$
$$= 1$$

Apply the recursion formula:

$$\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1!$$

$$\Gamma(3) = 2\cdot\Gamma(2) = 2 = 2!$$

$$\Gamma(4) = 3\cdot\Gamma(3) = 6 = 3!$$

$$\Gamma(5) = 4\cdot\Gamma(4) = 24 = 4!$$

In general $\Gamma(n) = (n-1)!$

The gamma probability distribution is (10)

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Notice that, when $\alpha = 1$, this is the exponential distribution.

Evaluate $\Gamma(\frac{1}{2})$ (11)

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx$$

Let $y = \sqrt{2x}$ $\qquad \frac{y^2}{2} = x$

$$\frac{dy}{dx} = \sqrt{2}\,\frac{1}{2}\,x^{-1/2}$$

$$dy = \frac{\sqrt{2}}{2}\,x^{-1/2}\,dx$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty \frac{2}{\sqrt{2}}\,e^{-y^2/2}\,dy = \sqrt{2}\int_0^\infty e^{-\frac{y^2}{2}}\,dy$$

(12)

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\left(\int_0^\infty e^{-y^2/2}\,dy\right)^2$$

$$= 2\int_0^\infty e^{-y^2/2}\,dy\int_0^\infty e^{-\frac{x^2}{2}}\,dx$$

$$= 2\int_0^\infty\int_0^\infty e^{-\frac{1}{2}(x^2+y^2)}\,\underline{dx\,dy}$$

Let $x = r\cos\theta$
$\quad\; y = r\sin\theta$

$$= 2\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-\frac{1}{2}r^2}\,\underbrace{r\,dr\,d\theta}_{\text{Jacobian}}$$

Let $u = -\frac{1}{2}r^2$

$du = -r\,dr$

$$[\Gamma(\tfrac{1}{2})]^2 = 2\int_0^{\pi/2}\int_0^{-\infty} e^u(-du)\,d\theta$$

$$= 2\int_0^{\pi/2} -e^u\Big|_0^{-\infty}\,d\theta$$

$$= 2\int_0^{\pi/2} 0-(-1)\,d\theta = 2\int_0^{\pi/2} d\theta$$

$$= 2\,\theta\Big|_0^{\pi/2} = 2(\tfrac{\pi}{2}-0) = \pi$$

(14)

So $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

For example $\Gamma(5/2) = \frac{3}{2}\Gamma(\frac{3}{2})$

$$= \frac{3}{2}\cdot\frac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \frac{3}{4}\sqrt{\pi}$$

Consider the case where α is an integer.

Let X have a Poisson distribution with parameter λt.

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Let T_1 be the waiting time until the 1^{st} occurrence. Then T_1 has an exponential distribution with parameter $\beta = \frac{1}{\lambda}$

Let T_2 be the waiting time between the 1^{st} occurrence and the 2^{nd}. $T_2 \sim Exp(\frac{1}{\lambda})$

$\vdots$

Let T_α be the waiting time between the $(\alpha-1)^{st}$ occurrence and the α^{th}.

Let $T = T_1 + T_2 + \ldots + T_\alpha$

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So T is the waiting time to get α occurrences.

T will have a gamma distribution with parameters α and $\beta = \frac{1}{\lambda}$.

$$\boxed{\underbrace{P(X \le \alpha - 1)}_{\text{Poisson}(\mu = \lambda t)} = \underbrace{P(T > t)}_{\text{Gamma}(\alpha,\ \beta = \frac{1}{\lambda})}}$$

HW #8 p. 193 #24

p. 205 #42, 46, 52

due March 9