Multivariate versions of binomial & hypergeometric ① 451 2/18

Multinomial Distribution

Each trial has k possible outcomes instead of 2.

Run n independent trials

X_1 = # outcomes of type 1	Probability	p_1
X_2 = " " 2	"	p_2
$\vdots$		
X_k = " " k	"	p_k
n		1

$$p(x_1, x_2, \ldots, x_k) = \binom{n}{x_1\ x_2 \cdots x_k} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

②

Example: A team plays 10 games.

Assume independent outcomes.

Assume that on each game,

there is a 50% chance of winning

40% " " losing

10% " " tie

Find the probability that the team wins 6, loses 2, and ties 2.

Multinomial $(n=10, p_1=.5, p_2=.4, p_3=.1)$

$$p(6,2,2) = \binom{10}{6,2,2} .5^6 .4^2 .1^2$$

(3)

Multivariate Hypergeometric Distribution

Population of N items	Take a sample of size n
a_1 of type 1	X_1 = # of type 1
a_2 " " 2	X_2 = " 2
$\vdots$	$\vdots$
a_k " " k	X_k = --- k
$\overline{N}$	$\overline{n}$

$$p(x_1, x_2, \ldots, x_k) = \frac{\binom{a_1}{x_1}\binom{a_2}{x_2}\cdots\binom{a_k}{x_k}}{\binom{N}{n}}$$

(4)

Example: 200 potential jurors

100 Caucasian
60 African/American
40 Asian/American

Select 12 jurors. Find the probability that all 12 are Caucasian

Multi Hypergeom($N = 200, n = 12, a_1 = 100, a_2 = 60, a_3 = 40$)

$$p(12,0,0) = \frac{\binom{100}{12}\binom{60}{0}\binom{40}{0}}{\binom{200}{12}}$$

Geometric Distribution

[Geometric Series: $a + ar + ar^2 + \dots = \dfrac{a}{1-r}$]

Run a sequence of Bernoulli trials.

(independent trials, 2 possible outcomes on each, same success probability throughout)

$X =$ trial on which the 1st success occurs.

$X = 1, 2, \dots \qquad p(x) = q^{x-1} p$

Check to see if this is a valid prob. distr.:

$$1 \stackrel{?}{=} \sum_{x=1}^{\infty} q^{x-1} p = p + pq + pq^2 + \dots$$

$$= \frac{p}{1-q} = \frac{p}{p} = 1$$

Find $\mu = E[X] = \sum_{\text{all } x} x\, p(x) = \sum_{x=1}^{\infty} x\, q^{x-1} p$

Let $y = x-1$ $\qquad = \sum_{y=0}^{\infty} (y+1) q^y p$

$$= \overset{①}{\sum_{y=0}^{\infty} y q^y p} + \overset{②}{\sum_{y=0}^{\infty} q^y p}$$

⑦

$$② = p + pq + pq^2 + \dots = \frac{p}{1-q} = 1$$

$$① = \sum_{y=1}^{\infty} y q^y p = q \sum_{y=1}^{\infty} y q^{y-1} p = q\mu$$

So $\mu = q\mu + 1$

$$\mu(1-q) = 1 \qquad \mu = \frac{1}{1-q} = \frac{1}{p}$$

The variance is found similarly.

$$\sigma^2 = \frac{q}{p^2}$$

⑧

<u>Negative Binomial Distribution</u> or <u>Pascal Distribution</u>

Run a sequence of Bernoulli trials

X = trial on which the k^{th} success occurs.

$X = k, k+1, \dots$

$p(k) = P(k \text{ successes in a row}) = p^k$

$p(k+1) = P($ $(k+1)^{st}$ trial was a success AND the 1^{st} k trials had $k-1$ successes and 1 failure $)$

$$= p \cdot \binom{k}{k-1} p^{k-1} q = \binom{k}{k-1} p^k q$$

$p(k+2) = P(\,(k+2)^{nd}$ trial was a success AND the 1^{st} $k+1$ trials had $k-1$ successes and 2 failures$)$

$$= p \cdot \binom{k+1}{k-1} p^{k-1} q^2 = \binom{k+1}{k-1} p^k q^2$$

In general, $\boxed{p(x) = \binom{x-1}{k-1} p^k q^{x-k}, \quad x = k, k+1, \ldots}$

Find μ and σ^2.

$$X = \underbrace{X_1 + X_2 + \ldots + X_k}_{\text{each is geometric, indep.}}$$

$$E[X] = E[X_1] + \ldots + E[X_k] = k \cdot \frac{1}{p}$$

$$V[X] = V[X_1] + \ldots + V[X_k] = k \cdot \frac{q}{p^2}$$

Coin toss. Keep going until you get 5 heads.

Find the expected number of trials.

$\text{NegBino}(k=5,\ p=\frac{1}{2}) \qquad E[X] = \frac{k}{p} = \frac{5}{1/2} = 10$

Find the probability that the 5^{th} head occurs on toss number 7. $\quad p(7) = \binom{7-1}{5-1}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^2 = .117$