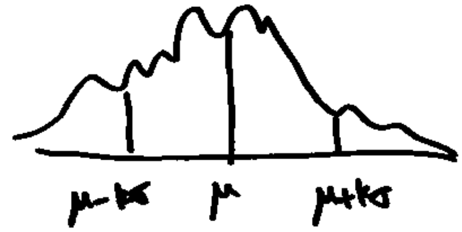
① 451 2-9

Sec 4.4 (not for midterm)

Chebyshev's Inequality



$$\sigma^2 = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \qquad \text{(continuous case)}$$

$$= \underbrace{\int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx}_{1} + \underbrace{\int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)\,dx}_{2} + \underbrace{\int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)\,dx}_{3}$$

$$\geq ① + ③$$

②

Look at ①: Everywhere in the range of integration, $X < \mu - k\sigma$

$$X - \mu < -k\sigma$$

$$(X-\mu)^2 > k^2\sigma^2$$

$$① = \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)\,dx \geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)\,dx$$

$$= k^2\sigma^2 \int_{-\infty}^{\mu-k\sigma} f(x)\,dx$$

$$= k^2\sigma^2 P(X < \mu - k\sigma)$$

$$① \geq k^2\sigma^2 P(X < \mu - k\sigma)$$

③

Look at ③: Everywhere in the range of integration, $X > \mu + k\sigma$

$$X - \mu > k\sigma$$

$$(X-\mu)^2 > k^2\sigma^2$$

$$③ = \int_{\mu+k\sigma}^{\infty} (X-\mu)^2 f(x)\,dx \geq \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)\,dx$$

$$= k^2\sigma^2 P(X > \mu + k\sigma)$$

$$③ \geq k^2\sigma^2 P(X > \mu + k\sigma)$$

④

Put this together:

$$\sigma^2 \geq ① + ③$$

$$\geq k^2\sigma^2 \left[P(X < \mu - k\sigma) + P(X > \mu + k\sigma) \right]$$

$$= k^2\sigma^2 \left[P(X - \mu < -k\sigma) + P(X - \mu > k\sigma) \right]$$

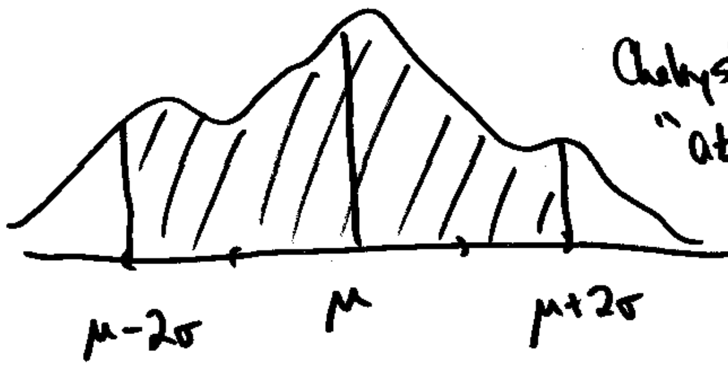
$$= k^2\sigma^2 \left[P(|X-\mu| > k\sigma) \right]$$

$$P(|X-\mu| > k\sigma) \leq \frac{1}{k^2}, \text{ OR}$$

$$\boxed{P(|X-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}}$$

Chebyshev's Inequality

For example, say that X has a continuous distribution, and we wish to know how much of the area lies within 2 standard deviations of the mean.



Chebyshev says "at least $\underbrace{1-\frac{1}{2^2}}_{75\%}$"