①

451
2-4

4.26 $f(x,y) = 4xy \quad \begin{matrix} 0<x<1 \\ 0<y<1 \end{matrix}$

$$E\left(\sqrt{x^2+y^2}\right) = \int_0^1 \int_0^1 \sqrt{x^2+y^2}\, 4xy \, dx \, dy$$

Let $u = x^2+y^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$$= \int_0^1 2y \int_{y^2}^{y^2+1} u^{1/2} \, du \quad dy$$

$$= \int_0^1 2y \left[\frac{2}{3} u^{3/2} \right]_{y^2}^{y^2+1} dy = \int_0^1 \frac{4y}{3}\left((y^2+1)^{3/2} - y^3\right) dy$$

②

$$= \frac{4}{3}\int_0^1 y(y^2+1)^{3/2} dy - \frac{4}{3}\int_0^1 y^4 dy$$

$$= \frac{4}{3}\int_1^2 u^{3/2} \frac{1}{2} du - \frac{4}{3} \frac{y^5}{5}\Big|_0^1$$

$$= \frac{2}{3}\frac{2}{5} u^{5/2}\Big|_1^2 - \frac{4}{15}$$

$$= \frac{4}{15}(2^{5/2} - 1) - \frac{4}{15} = .975$$

Let $u = y^2+1$

$\frac{du}{dy} = 2y \quad du = 2y\,dy$

③

Properties of $E[X]$ and $V[X]$

$$E[X] = \begin{cases} \sum_{\text{all } x} x\,p(x) & \text{discrete} \\ \int_{\text{all } x} x\,f(x)\,dx & \text{continuous} \end{cases}$$

$$E[c] = \int_{-\infty}^{\infty} c\,f(x)\,dx = c\int_{-\infty}^{\infty} f(x)\,dx = c$$

$$E[cX] = \int_{-\infty}^{\infty} c\,x\,f(x)\,dx = c\int_{-\infty}^{\infty} x\,f(x)\,dx = c\,E[X]$$

$$E[X+Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\,f(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(x,y)\,dx\,dy$$

④

$$= E[X] + E[Y]$$

That is, the expectation is a <u>linear</u> <u>operator</u>

<u>Example</u>: Find $E[2X - 3Y + 7]$

$$= 2E[X] - 3E[Y] + 7$$

<u>Example</u>. Suppose that $X_1, X_2, \ldots, X_n$ are random variables, all with $E[X_i] = \mu$.

Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ Find $E[\bar{X}]$

⑤

$$E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

$$V[X] = \begin{cases} \sum_{\text{all } x} (x-\mu)^2 p(x) & \text{discrete} \\ \int_{\text{all } x} (x-\mu)^2 f(x)\,dx & \text{continuous} \end{cases}$$

$$= E[X^2] - (E[X])^2$$

⑥

$$V[c] = E[c^2] - (E[c])^2$$
$$= c^2 - c^2 = 0$$

$$V[cX] = E[c^2X^2] - (E[cX])^2$$
$$= c^2E[X^2] - (c\,E[X])^2$$
$$= c^2\left[E[X^2] - (E[X])^2\right]$$
$$= c^2V[X]$$

⑦

$$V[X+Y] = E[(X+Y)^2] - [E(X+Y)]^2$$

$$= E[X^2+Y^2+2XY] - [E[X]+E[Y]]^2$$

$$= E[X^2]+E[Y^2]+2E[XY]$$

$$-\left((E[X])^2+(E[Y])^2+2E[X]E[Y]\right)$$

$$= \underbrace{E[X^2]-(E[X])^2} + \underbrace{E[Y^2]-(E[Y])^2}$$

$$+ 2(E[XY] - E[X]E[Y])$$

$$= V[X] + V[Y] + 2\,\mathrm{Cov}[X,Y]$$

⑧

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]E[Y]$$

$$\mathrm{Cov}[X,X] = E[X^2] - (E[X])^2 = V[X]$$

$$\mathrm{Cov}[X,c] = E[cX] - E[c]E[X]$$

$$= cE[X] - cE[X] = 0$$

$$\mathrm{Cov}[aX,bY] = E[abXY] - E[aX]E[bY]$$

$$= ab\,E[XY] - ab\,E[X]E[Y]$$

$$= ab\,\mathrm{Cov}[X,Y]$$

(9)

$$\mathrm{Cov}[X+Y, Z] = E[XZ + YZ] - E[X+Y]E[Z]$$

$$= \underline{E[XZ]} + \underline{\underline{E[YZ]}} - \underline{E[X]E[Z]} - \underline{\underline{E[Y]E[Z]}}$$

$$= \mathrm{Cov}[X,Z] + \mathrm{Cov}[Y,Z]$$

<u>Example</u>: Find $V[\underbrace{2X - 3Y} + \underbrace{7}]$

$$= V[2X-3Y] + \underbrace{V[\cancel{7}]}_{0} + 2\,\underbrace{\mathrm{Cov}(2X-3Y, \cancel{7})}_{0}$$

$$= V[2X] + V[-3Y] + 2\,\mathrm{Cov}(2X, -3Y)$$

$$= 4V[X] + 9V[Y] - 12\,\mathrm{Cov}[X,Y]$$

(10)

<u>Example</u>: Find $V[X+Y]$ if X and Y are independent.

$$V[X+Y] = V[X] + V[Y] + 2\,\mathrm{Cov}[X,Y]$$

If X and Y are independent, then

$$f(x,y) = g(x)h(y)$$

So $$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, g(x)h(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \left[y\, h(y) \int_{-\infty}^{\infty} x\, g(x)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} y\, h(y)\, E[X]\, dy$$

$$= E[X] \int_{-\infty}^{\infty} y\, h(y)\, dy$$

$$= E[X]\, E[Y]$$

$$\therefore\ X \text{ \& } Y \text{ indp} \Rightarrow \begin{cases} E[XY] = E[X]\,E[Y] \\ \mathrm{Cov}[X,Y] = 0 \\ V[X+Y] = V[X] + V[Y] \end{cases}$$

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Example: Assume $X_1, X_2, \ldots, X_n$ are random variables independent, all with the same $V[X_i] = \sigma^2$.

Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ Find $V[\bar{X}]$

$$V[\bar{X}] = V\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} V[X_i]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{1}{n}\sigma^2$$

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HW #5 due 2-11

p. 134 # 58, 62, 70, 72

Midterm exam Thursday Feb 11