1

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2-2

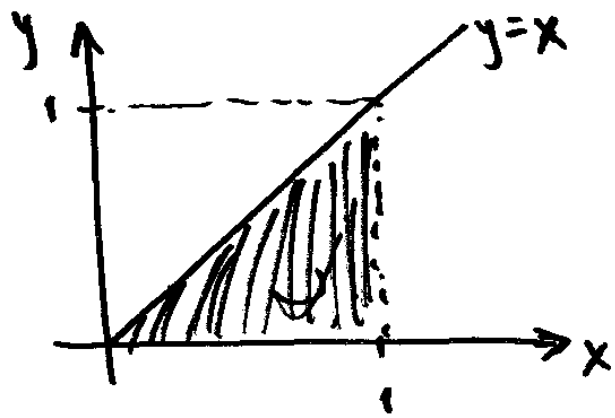
The joint density function for 2 Continuous random variables

<u>Example</u>: $f(x,y) = \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

Probabilities are found by double integration.

Check to see if the total probability is 1.

2



Check: $1 \stackrel{?}{=} \int_0^1 \int_y^1 8xy \, dx \, dy$ } use this option

OR

$\int_0^1 \int_0^x 8xy \, dy \, dx$

3

$$= \int_0^1 8y\,\frac{x^2}{2}\Big|_{x=y}^{1}\,dy$$

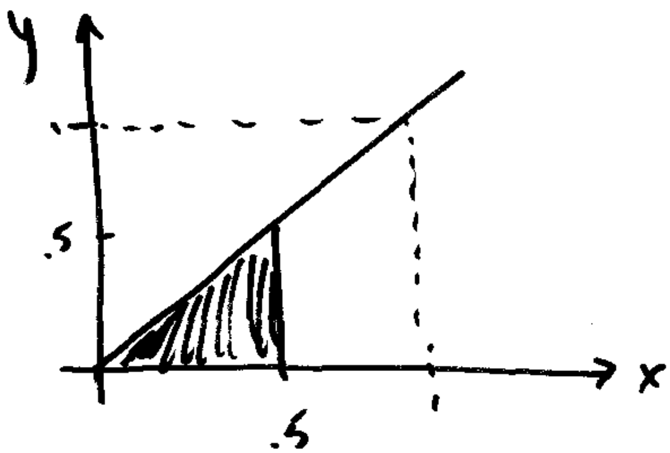
$$= \int_0^1 4y - 4y^3\,dy$$

$$= 2y^2 - y^4\Big|_0^1 = 2 - 1 - (0)$$

$$= 1 \checkmark$$

Find the probability that both X and Y are less than .5

4



$$P(X<.5 \cap Y<.5) = \int_0^{.5}\underline{\int_0^{x} 8xy\,dy}\,dx$$

$$= \int_0^{.5} 4xy^2\Big|_{y=0}^{x}\,dx = \int_0^{.5} 4x^3 - 0\,dx$$

$$= x^4 \Big|_0^{.5} = .5^4 - 0 = \frac{1}{16}$$ ⑤

If $f(x,y)$ is the joint density of X and Y, then the marginal density for X is

$$g(x) = \int_{\text{all } y} f(x,y)\,dy$$

and the marginal density for Y is

$$h(y) = \int_{\text{all } x} f(x,y)\,dx$$

⑥

Back to our example: $f(x,y) = 8xy$,

$0 \le y \le x \le 1$

Find $g(x) = \int_0^x 8xy\,dy = 8x\frac{y^2}{2}\Big|_0^x$

$= 4x^3, \quad 0 \le x \le 1$

and $h(y) = \int_y^1 8xy\,dx = 8y\frac{x^2}{2}\Big|_y^1$

$= 4y - 4y^3, \quad 0 \le y \le 1$

(7)

<u>Definition</u>: X and Y are <u>independent</u> if

$$f(x,y) = g(x)h(y) \quad \text{for all } x, y$$

In our example, are X and Y independent?

$$8xy \stackrel{?}{=} (4x^3)(4y-4y^3) \quad \text{No.}$$

<u>Definition</u>: The conditional density of X given Y is $f(x|y) = \frac{f(x,y)}{h(y)}$

(8)

In our example,

$$f(x|y) = \frac{8xy}{4y-4y^3} = \frac{2x}{1-y^2}$$

and $$f(y|x) = \frac{8xy}{4x^3} = \frac{2y}{x^2}$$

Both have $0 \le y \le x \le 1$, since that was the constraint on $f(x,y)$

(9)

<u>Definition</u>: Let $w(X,Y)$ be any function of the random variables X and Y.

Then $$E[w(X,Y)] = \int_{\text{all } x}\int_{\text{all } y} w(x,y) f(x,y)\, dx\, dy$$

Special cases: $$E[XY] = \int_{\text{all } x}\int_{\text{all } y} x\, y\, f(x,y)\, dx\, dy$$

(10)

$$E[X] = \int_{\text{all } x}\int_{\text{all } y} x\, f(x,y)\, dy\, dx$$

$$= \int_{\text{all } x}\left[x \int_{\text{all } y} f(x,y)\, dy \right] dx$$

$$= \int_{\text{all } x} x\, g(x)\, dx$$

The <u>covariance</u> of X and Y is

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

(11)

So $\sigma_{XY} = \iint\limits_{\substack{\text{all } x \\ \text{all } y}} (x-\mu_X)(y-\mu_Y) f(x,y)\,dx\,dy$

Alternate formula for covariance:

$$\sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= E[XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y]$$

$$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + E[\mu_X \mu_Y]$$

(12)

$$= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E[XY] - \mu_X \mu_Y$$

$$= E[XY] - E[X]E[Y]$$

<u>Defn:</u> The <u>correlation</u> between X and Y

is $\quad \mathrm{Corr}(X,Y) = \rho_{XY} = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)V(Y)}}$

$$= \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

(13)

Back to today's example: $f(x,y) = 8xy$

$0 \le y \le x \le 1$

Find ρ_{XY}.

$g(x) = 4x^3 \quad 0 \le x \le 1$

$h(y) = 4y - 4y^3 \quad 0 \le y \le 1$

$$E[XY] = \int_0^1 \int_0^x xy \, 8xy \, dy\, dx = \int_0^1 \int_0^x 8x^2y^2 \, dy\, dx$$

$$= \int_0^1 8x^2 \frac{y^3}{3} \Big|_{y=0}^{x} dx = \int_0^1 \frac{8}{3} x^5 \, dx$$

$$= \frac{8}{3} x^6/6 \Big|_0^1 = \frac{4}{9}$$

(14)

$$E[X] = \int_{\text{all } x} x\, g(x) dx = \int_0^1 x\, 4x^3 dx$$

$$= \int_0^1 4x^4 dx = \frac{4x^5}{5} \Big|_0^1 = \frac{4}{5}$$

$$E[Y] = \int_0^1 y(4y - 4y^3) dy = \int_0^1 4y^2 - 4y^4 \, dy$$

$$= \frac{4y^3}{3} - \frac{4y^5}{5} \Big|_0^1 = 4\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{8}{15}$$

$$\sigma_{XY} = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}$$

(15)

$$V[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_0^1 x^2\, 4x^3\, dx = \int_0^1 4x^5\, dx$$

$$= \frac{4x^6}{6}\Big|_0^1 = \frac{2}{3}$$

$$V[X] = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = 2/75$$

$$E[Y^2] = \int_0^1 y^2 (4y - 4y^3)\, dy = \int_0^1 4y^3 - 4y^5\, dy$$

$$= \frac{4y^4}{4} - \frac{4y^6}{6}\Big|_0^1 = \frac{1}{3}$$

(16)

$$V[Y] = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{4/225}{\sqrt{\frac{2}{75} \cdot \frac{11}{225}}} = .492$$