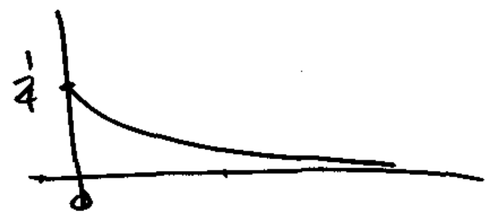
① 451 1-26

The cumulative distribution function (cdf)
(Continuous case)

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = P(X \le x)$$

Example: $f(x) = \frac{1}{4} e^{-\frac{1}{4}x} \quad x \ge 0$



$F(x) = 0$ if $x \le 0$

②

For $x > 0$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} \frac{1}{4} e^{-\frac{1}{4}t} dt$

let $u = -\frac{1}{4}t$

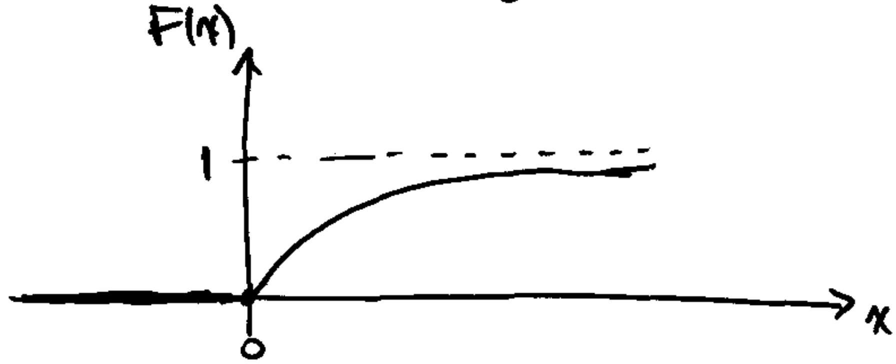
$\frac{du}{dt} = -\frac{1}{4}$

$du = -\frac{1}{4}dt$

$$F(x) = \int_{0}^{-\frac{1}{4}x} -e^{u} du = -e^{u}\Big]_{0}^{-\frac{1}{4}x}$$

$$= -e^{-\frac{1}{4}x} - (-1) = 1 - e^{-\frac{1}{4}x}$$

Altogether: $F(x) = \begin{cases} 0 & x \le 0 \\ 1 - e^{-\frac{1}{7}x} & x > 0 \end{cases}$



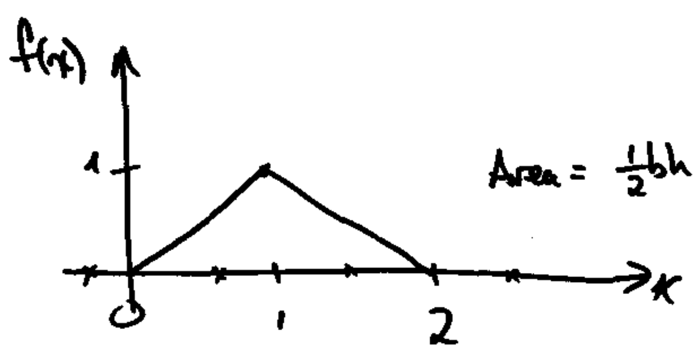
Properties of the cdf: ① $F(x) = \int_{-\infty}^{x} f(t)\,dt$

② $\frac{d}{dx}F(x) = f(x)$

③ $P(a < X < b) = \int_a^b f(x)\,dx$

$= F(b) - F(a)$

Example: $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$



Area $= \frac{1}{2}bh = \frac{1}{2}(2)(1) = 1$ ✓

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$ (5)

Case 1: $x<0$: $F(x)=0$

Case 4: $x>2$: $F(x)=1$

Case 2: $0<x<1$: $F(x) = \int_0^x t\,dt = \frac{t^2}{2}\Big|_0^x$

$$= \frac{x^2}{2} - 0 = \frac{x^2}{2}$$

Case 3: $1 \le x \le 2$: $F(x) = \int_0^1 t\,dt + \int_1^x (2-t)\,dt$

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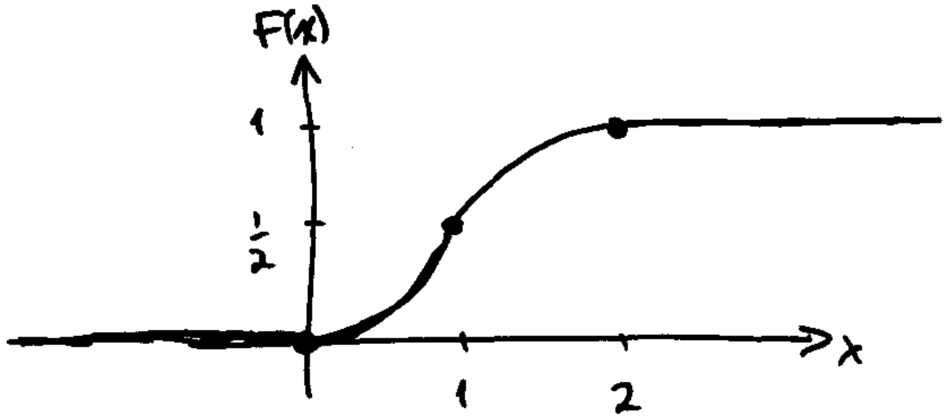
$$= \frac{t^2}{2}\Big|_0^1 + \left(2t - \frac{t^2}{2}\right)\Big|_1^x$$

$$= \frac{1}{2} - 0 + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \frac{3}{2} = 2x - \frac{x^2}{2} - 1$$

$$F(x) = \begin{cases} 0 & x \le 0 \\ x^2/2 & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 \le x \le 2 \\ 1 & x > 2 \end{cases}$$



Use this to find $P(\frac{1}{2} < x < 3/2)$

$$= F(3/2) - F(\tfrac{1}{2})$$

$$= 2(\tfrac{3}{2}) - \frac{(3/2)^2}{2} - 1 - \frac{(\frac{1}{2})^2}{2}$$

$$= 3 - \tfrac{9}{8} - 1 - \tfrac{1}{8} = 2 - 5/4 = 3/4$$

Expectations in the continuous case:

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

In general, $E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$

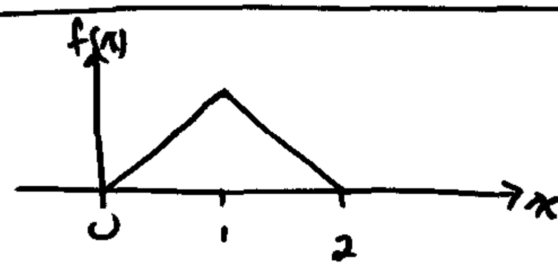
In particular,

$$\sigma^2 = Var(X) = V(X)$$
$$= E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

As in the discrete case,

$$\sigma^2 = E[X^2] - (E[X])^2$$

Back to our example:



$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx$$

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$$= \frac{x^3}{3}\Big|_0^1 + \left(x^2 - \frac{x^3}{3}\right)\Big|_1^2$$

$$= \frac{1}{3} - 0 + \left[\left(4 - 8/3\right) - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{1}{3} + 4 - 8/3 - 1 + \frac{1}{3}$$

$$= 3 - 6/3 = 1$$

$$\sigma^2 = E[X^2] - (E[X])^2 = E[X^2] - 1$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 1$$

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$$= \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 (2-x) \, dx - 1$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3) dx - 1$$

$$= \frac{x^4}{4} \Big|_0^1 + \left(\frac{2x^3}{3} - \frac{x^4}{4} \right) \Big|_1^2 - 1$$

$$= \frac{1}{4} + \left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - \frac{1}{4} \right) - 1$$

$$= \frac{1}{2} + \frac{14}{3} - 5 = \frac{3 + 28 - 30}{6} = \frac{1}{6}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{1}{6}}$$

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Return to discrete distributions:

Consider 2 random variables observed at the same time.

<u>Example</u>: Roll 2 dice

let X = minimum of the 2 dice

Y = maximum " " "

		Y (max)						
		1	2	3	4	5	6	
X (min)	1	1/36	2/36	2/36	2/36	2/36	2/36	11/36
	2	0	1/36	2/36	2/36	2/36	2/36	9/36
	3	0	0	1/36	2/36	2/36	2/36	7/36
	4	0	0	0	1/36	2/36	2/36	5/36
	5	0	0	0	0	1/36	2/36	3/36
	6	0	0	0	0	0	1/36	1/36
		1/36	3/36	5/36	7/36	9/36	11/36	1

The body of the table contains the joint probability distribution $p(x,y) = P(X=x \cap Y=y)$

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Notice that $P(X=x) = \sum_y p(x,y)$

and $P(Y=y) = \sum_x p(x,y)$

Also, $\sum_x \sum_y p(x,y) = 1$

From our table, find $p(5,6) = 2/36$

$= P(X=5 \cap Y=6)$

$P(X=5) = 3/36 \quad P(Y=6) = 11/36$