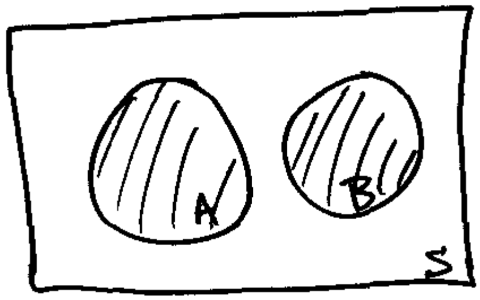
①

Suppose that events A and B are disjoint (or mutually exclusive). 451

1-12



$A \cup B$

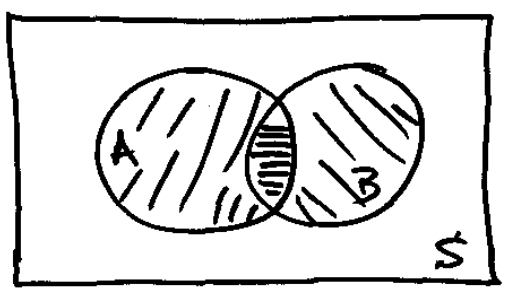
$$\frac{n(A\cup B)}{n(S)} = \frac{n(A)+n(B)}{n(S)}$$

$$\boxed{P(A\cup B) = P(A) + P(B)}$$

Union rule for disjoint events

②

In the general case:



$$A\cup B = (A\cap B') \cup (A\cap B) \cup (A'\cap B)$$

$P(A\cup B) = P(A\cap B') + P(A\cap B) + P(A'\cap B)$ by the union rule for disjoint events

③

$$A = (A \cap B') \cup (A \cap B)$$

$$P(A) = P(A \cap B') + P(A \cap B)$$

So $P(A \cup B) = P(A) + P(A' \cap B)$

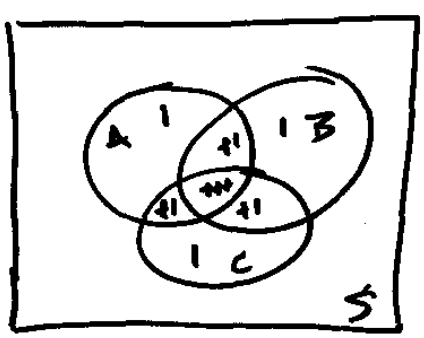
Also $P(B) = P(A \cap B) + P(A' \cap B)$

$$\Rightarrow P(A' \cap B) = P(B) - P(A \cap B)$$

$$\therefore \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Union rule

④



$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+ P(A \cap B \cap C)$$

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Some additional counting rules

n objects of k different types

n_1 are of type 1
n_2 " " " 2
$\vdots$
n_k " " " k

$\overline{}$
n

How many different ways can these items be arranged?

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Example: AAA B B C

How many different ways can they be arranged?

$$\frac{6!}{3!\,2!\,1!} = \binom{6}{3,2,1} = 60$$

In general, the answer is $\dfrac{n!}{n_1!\,n_2!\cdots n_k!}$

$$= \binom{n}{n_1, n_2, \ldots, n_k}$$

(7)

How many different ways can n items be arranged in a circle?

	A				D	
E		B	$\approx$	C		E
D		C		B		A

$$\frac{5!}{5} = 4!$$

In general, the answer is $(n-1)!$

(8)

Conditional probability

		Income			
		High	Med	Low	
Gender	M	30	40	30	100
	F	60	50	40	150
		90	90	70	250

"cross tab"

Experiment: Select 1 person at random from the 250.

$$P(M) = \frac{n(M)}{n(S)} = \frac{100}{250} = .4$$

$$P(H) = \frac{n(H)}{n(S)} = \frac{90}{250} = .36$$

$$P(M \cap H) = \frac{n(M \cap H)}{n(S)} = \frac{30}{250} = .12$$

Note: $P(M)$ and $P(H)$ are <u>marginal</u> probabilities

$P(M \cap H)$ is a <u>joint</u> probability

Given that the person is male, find the probability that he has a high income.

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$$P(H \mid M) = \frac{30}{100} = \frac{n(H \cap M)}{n(M)}$$

"given"

The <u>conditional probability</u> of A, given B

is $P(A \mid B) = \dfrac{n(A \cap B)}{n(B)}$

Notice $\dfrac{n(A \cap B) \div n(S)}{n(B) \div n(S)} = \dfrac{P(A \cap B)}{P(B)}$

(11)

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \Bigg| \quad P(B|A) = \frac{P(B\cap A)}{P(A)}$$

$$\boxed{\begin{aligned} P(A\cap B) &= P(B)P(A|B) \\ P(A\cap B) &= P(A)\,P(B|A) \end{aligned}}$$

Multiplicative rule

Experiment: Draw 2 cards from a deck of 52. Find the probability that they are both aces.

(12)

Solution 1: Keep track of order

$$n(S) = P^{52}_{2}$$

Let A be the event that both are aces

$$n(A) = P^{4}_{2}$$

$$P(A) = \frac{P^4_2}{P^{52}_2} = \frac{\left(\frac{4!}{2!}\right)}{\left(\frac{52!}{50!}\right)} = \frac{4.3}{52.51}$$

$$= .0045$$

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Solution 2: Don't keep track of order

$$n(S) = \binom{52}{2}$$

$$n(A) = \binom{4}{2}$$

$$P(A) = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{\left(\frac{4!}{\cancel{2!}\,2!}\right)}{\left(\frac{52!}{\cancel{2!}\,50!}\right)} = .0045$$

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Solution 3: Let A be the event that the 1^{st} card is an ace, and let B be the event that the 2^{nd} card is an ace.

Find $P(A \cap B)$

$$= P(A)\, P(B \mid A)$$

$$= \frac{4}{52} \cdot \frac{3}{51} = .0045$$

In the gender/income example,

$$P(H \mid M) = .3 \quad \text{and} \quad P(H) = .36$$

Note $P(H \mid M) \neq P(H)$.

Definition: If $P(A \mid B) = P(A)$, then

A and B are <u>independent</u>.

$P(A \cap B) = P(A)P(B)$ alternate definition