# The Number Sense

## How the Mind Creates Mathematics

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Chapter 4

# The Language of Numbers

I observe that when we mention any great number, such as a thousand, the mind has generally no adequate idea of it, but only a power of producing such an idea by its adequate idea of the decimals, under which the number is comprehended.
David Hume, *A Treatise of Human Nature*

Suppose that our only mental representation of number were an approximate accumulator similar to the rat's. We would have rather precise notions of the numbers 1, 2, and 3. But beyond this point, the number line would vanish into a thickening fog. We could not think of number 9 without confusing it with its neighbors 8 and 10. Even if we understood that the circumference of a circle divided by its diameter is a constant, the number $\pi$ would only be known to us as "about 3." This fuzziness would befuddle any attempt at a monetary system, much of scientific knowledge, indeed human society as we know it.

How did *Homo sapiens* alone ever move beyond approximation? The uniquely human ability to devise symbolic numeration systems was probably the most crucial factor. Certain structures of the human brain that are still far from understood enable us to use any arbitrary symbol, be it a spoken word, a gesture, or a shape on paper, as a vehicle for a mental representation. Linguistic symbols parse the world into discrete categories. Hence they allow us to refer to precise numbers and to separate them categorically from their closest neighbors. Without symbols, we might not discriminate 8 from 9. But with the help of our elaborate numerical notations, we can express thoughts as precise as "The speed of light is

299,792,458 meters per second." It is this transition from an approximate to a symbolic representation of numbers that I intend to describe in this chapter—a transition that occurs both in cultural history and in the mind of any child who acquires the language of numbers.

## A Short History of Number

When our species first began to speak, it may have been able to name only the numbers 1, 2, and perhaps 3. Oneness, twoness, and threeness are perceptual qualities that our brain computes effortlessly, without counting. Hence giving them a name was probably no more difficult than naming any other sensory attribute, such as red, big, or warm.

The linguist James Hurford has gathered considerable evidence for the antiquity and special status of the first three number words. In languages with case and gender inflections, "one," "two," and "three" are often the only numerals that can be inflected. For instance, in old German "two" can be *zwei*, *zwo*, or *zween* depending on the grammatical gender of the object that is being counted. The first three ordinals also have a particular form. In English, for instance, most ordinals end with "th" (fourth, fifth, etc.), but the words "first," "second," and "third" do not.

The numbers 1, 2, and 3 are also the only ones that can be expressed by grammatical inflections instead of words. In many languages, words do not just bear the mark of the singular or plural. Distinct word endings are also used to distinguish two items (*dual*) versus more than two items (*plural*), and a few languages even have special inflections for expressing three items (*trial*). In ancient Greek, for instance, "o hippos" meant the horse, "to hippo" the two horses, and "toi hippoi" an unspecified number of horses. But no language ever developed special grammatical devices for numbers beyond 3.

Finally the etymology of the first three numerals also bears testimony to their antiquity. The words for "2" and "second" often convey the meaning of "another," as in the verb *to second* or the adjective *secondary*. The Indo-European root of the word "three" suggests that it might have once been the largest numeral, synonymous with "a lot" and "beyond all others"—as in the French *très* (very) or the Italian *troppo* (too much), the English *through*, or the Latin prefix *trans-*. Hence perhaps the only numbers known to Indo-Europeans were "1," "1 and another" (2), and "a lot" (3 and beyond).

Today we find it hard to imagine that our ancestors might have been confined to numbers below three. Yet this is not implausible. Up to this very day, the Warlpiris, an aboriginal tribe from Australia, have names only for the quantities 1, 2, some, and a lot. In the domain of colors, some African languages distinguish only between black, white, and red. Needless to say, these limits are purely lexical. When Warlpiris come into contact with Occidentals, they easily learn English numerals. Thus, their ability to conceptualize numbers is not limited by the restricted lexicon of their language, nor (obviously) by their genes. Although there is a dearth of experiments on this topic, it seems likely that they possess quantitative concepts of numbers beyond three, albeit nonverbal and perhaps approximate ones.



How did human languages ever move beyond the limit of 3? The transition toward more advanced numeration systems seems to have involved the counting of body parts. All children spontaneously discover that their fingers can be put into one-to-one correspondence with any set of items. One merely has to raise one finger for the first item, two for the second, and so on. By this mechanism, the gesture of raising three fingers comes to serve as a symbol for the quantity three. An obvious advantage is that the required symbols are always "handy"—in this numeration system the digits are literally the speaker's digits!

Historically then, digits and other parts of the body have supported a body-based language of numbers, which is still in use in some isolated communities. Many aboriginal groups, who lack spoken words for numbers beyond three, possess a rich vocabulary of numerical gestures fulfilling the same role. The natives of the Torres straights, for instance, denote numbers by pointing to different parts of their body in a fixed order (Figure 4.1): from the pinkie to the thumb on the right hand (numbers 1 to 5), then up the right arm and down the left arm (6 to 12), through to the fingers of the left hand (13 to 17), the left toes (18 to 22), the left and right legs (23 to 28), and finally the right toes (29 to 33). A few decades ago, in a school in New Guinea, teachers were puzzled to see their aboriginal pupils wriggling during mathematics lessons, as if calculations made them itch. As a matter of fact, by rapidly pointing to parts of their body, the children were translating into their native body language the numbers and calculations being taught to them in English.

In more advanced numeration systems, pointing is not needed anymore: Naming a body part suffices to evoke the corresponding numeral. Thus in many societies in New Guinea, the word six is literally "wrist," while nine is "left breast." Likewise, in countless languages throughout the world, from Central Africa to Paraguay, the etymology of the word "five" evokes the word "hand."

A third step bridges the gap that separates these body-based languages from our modern "disembodied" number words. Body pointing suffers from a serious limitation: Our fingers form a finite set, indeed a rather small one. Even if we count toes and a few other salient parts of our bodies, the method is hopeless for

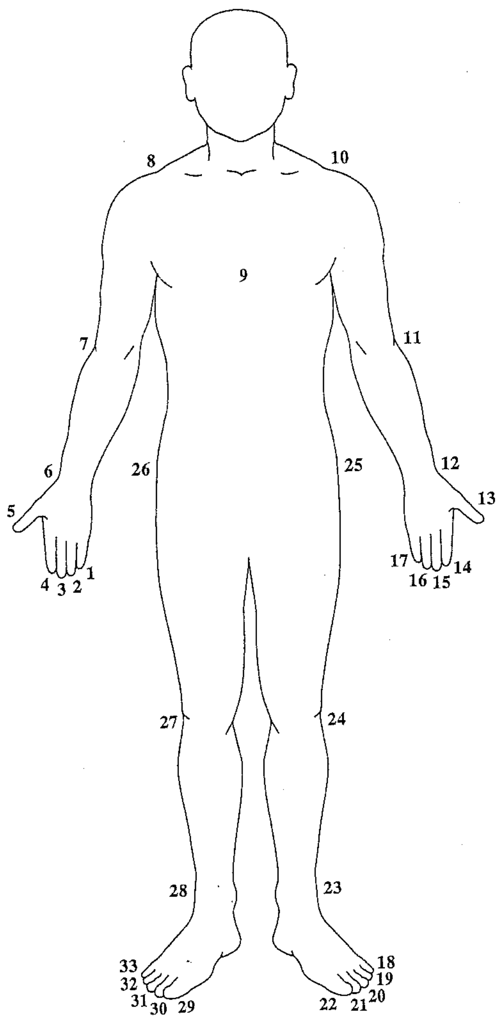


*Figure 4.1. The natives of the Torres Straight denote numbers by pointing towards a precise part of their body. (After Ifrah 1994.)*

numbers beyond thirty or so. It is highly impractical to learn an arbitrary name for each number. The solution is to create a syntax that allows larger numerals to be expressed by combining several smaller ones.

Number syntax probably emerged spontaneously from an extension of body-based numeration. In societies such as the native tribes of Paraguay, the number 6, instead of being given an arbitrary name such as "wrist," is expressed as "one on the other hand." Since the word "hand" itself means 5, by the very nature of their body language these people are led to express 6 as "5 and 1." Similarly, the



number 7 is "5 and 2," and so on all the way to 10, which is simply expressed as "two hands" (two 5s). Behind this elementary example lurk the basic organizing principles of modern number notations: the choice of a base number (here number 5), and the expression of larger numbers by means of a combination of sums and products. Once discovered, these principles can be extended to arbitrarily large numbers. Eleven, for instance, might be expressed as "two hands and a finger" (two 5s and 1), while 22 will be "four hands and two fingers."

Most languages have adopted a base number, such as 10 or 20, whose name is often a contraction of smaller units. In the Ali language, for instance, the word "mboona," which means 10, is a contraction of "moro boona"—literally "two hands." Once the new form is frozen, it can itself enter into more complex constructions. Thus the word for 21 could be expressed as "two 10s and 1". A similar process of contraction accounts for the irregular construction of some numerals such as 11, 12, 13, or 50 in present-day English. These words were once transparent compounds—"1 (and) 10," "2 (and) 10," "3 (and) 10," "5 10s"—before they were distorted and contracted.

As for base 20, it probably reflects an ancient tradition of counting on fingers and toes. This explains why the same word often denotes number 20 and "a man," as in some Mayan dialects or in Greenland Eskimo. A number such as 93 may then be expressed by a short sentence such as "after the fourth man, 3 on the first foot"—a twisted syntax indeed, but hardly more so than the modern French expression "quatre-vingt-treize" (4×20+13). It is through such means that humans eventually learned to express any number with perfect accuracy.

## Keeping a Permanent Trace of Numerals

Beyond giving numbers a name, to keep a durable record of them was also vital. For economical and scientific reasons, humans quickly developed writing systems that could maintain a permanent record of important events, dates, quantities, or exchanges—anything, in brief, that could be denoted by a number. Thus, the invention of written number notations probably unfolded in parallel with the development of oral numeration systems.

To understand the origins of number writing systems, we have to travel far back in time. Several bones from the Aurignacian period (35,000 to 20,000 B.C.) reflect the oldest method of writing numbers: the representation of a set by an identical number of notches. These bones are engraved with series of parallel notches, sometimes grouped in small packets. This might have been early humans' way of keeping a hunting record by carving one notch for each animal

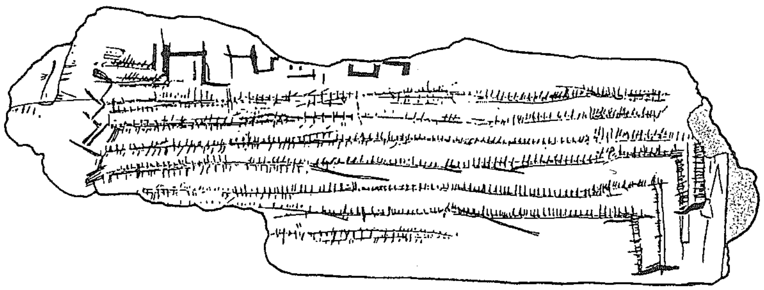

*Figure 4.2. This small bone plaque was unearthed in 1969 from the Grotte du Taï in southern France. Dated from the Upper Paleolithic (ca. 10,000 BC), it is engraved with regularly aligned marks. Because some of the notches are grouped into subsets of about 29, the plaque is thought to have recorded the number of days elapsing between two lunations. (Reprinted from Marshack 1991 by permission of the publisher; copyright © 1991 by Cambridge University Press.)*

captured. The patient decoding of the periodic structure of notches on a slightly more recent bone plaque even suggests that it might have been used as a lunary calendar that kept track of how many days had elapsed between two full moons (Figure 4.2).

The principle of one-to-one correspondence has been reinvented over and over again throughout the world as one of the simplest and most basic of numerical records. The Sumerians filled spheres of clay with as many marbles as the objects they counted; the Incas recorded numbers by tying knots on strings, which they kept as archives; and the Romans used vertical bars to form their first three digits. Even recently, some bakers still used notched sticks to keep track of their clients' debts. The word "calculation" itself comes from the Latin word *calculus*, which means "pebble," and draws us back to the time when numbers were manipulated by moving pebbles on an abacus.

Despite its deceptive simplicity, the one-to-one correspondence principle is a remarkable invention. It offers a durable, precise, and abstract representation of numbers. A series of notches can serve as an abstract numerical symbol and stand for any collection of items, be it livestock, people, debts, or full moons. It also enables humans to overcome the limitations of their perceptual apparatus. Humans, like pigeons, cannot distinguish forty-nine objects from fifty. Yet a stick engraved with forty-nine notches keeps a permanent track of this exact number. To verify whether a count is correct, one merely has to go through the objects one by one and move forward by one notch for each object. One-to-one correspondence therefore provides a precise representation of numbers too large to be accurately memorized on the mental number line.

Obviously, one-to-one correspondence also has its limitations. Series of notches are notoriously inconvenient to read or to write. As we have seen earlier, the



human visual system is unable to apprehend at a glance the numerosity of a set of more than three items. Hence an undifferentiated series of thirty-seven notches is as difficult to perceive as the set of thirty-seven sheep it stands for! Humans were therefore quickly drawn to breaking the monotony of number series by grouping the notches and by introducing novel symbols, in effect breaking a large number into something easier to read at a glance. This is exactly what we do when we strike out each group of five strokes, thus turning them into a visually salient group. Using this technique, the number 21 looks like 𝍸 𝍸 𝍸 𝍸 |, undeniably a more readable notation than |||||||||||||||||||||.

However, this system is convenient only on paper. When engraving a stick, carving in the wood's length is tedious. Cutting the wood at an angle is so much easier, and that is exactly the method that shepherds adopted thousands of years ago: They invariably selected symbols made up of oblique bars, such as V or X, to denote the numbers 5 and 10. As you may guess, this is the origin of Roman numerals. Their geometric shapes were determined by how easily they could be carved on a wooden stick. Other writing media have imposed different shapes. For instance, the Sumerians, who wrote on sheets of soft clay, adopted for their numerals the simplest shapes that could be formed with a pencil,—namely, round or cylindrical notches as well as the famous nail-shaped or "cuneiform" characters.

By adding together several of these symbols, other numbers may be formed. In Roman notation, 7 is written as 5+1+1 (VII). This additive principle, according to which the value of a number is equal to the sum of its component digits, underlies many number notations, including those of the Egyptians, Sumerians, and Aztecs. Additive notation saves time and space, because a number such as 38, which requires thirty-eight identical symbols in any concrete notation based on one-to-one correspondence, now mobilizes only seven Roman digits (38=10+10+10+5+1+1+1 or XXXVIII). Still, reading and writing remain a tedious chore. Compactness can be improved a bit by introducing special symbols such as numbers L (50) and D (500). Repetitions may be totally avoided if one is willing to use a distinct symbol for each of the numbers 1 to 9, 10 to 90, and 100 to 900. This solution was adopted by the Greeks and the Jews, who used letters of the alphabet instead of numbers. Using this trick, a number as complex as 345 can be written with only three letters (TME in Greek, or 300+40+5). The user, however pays a heavy cost: Considerable effort is needed to memorize the numerical value of the 27 symbols required to express all numbers from 1 to 999.

In retrospect, it seems obvious that addition alone cannot suffice to express very large numbers. Multiplication becomes indispensable. One of the first hybrid notations, mixing addition and multiplication, appeared in Mesopotamia over four millennia ago. Instead of expressing a number such as 300 by repeating the symbol for 100 three times, as in Roman numerals (CCC), the inhabitants of the city of Mari simply wrote down the symbol for "three" followed by the symbol for "hundred." Unfortunately, they still wrote units and decades using the addition principle, so their notation remained far from concise. The number 2,342, for instance, was literally written down as "1+1 thousand, 1+1+1 hundred, 10+10+10+10, 1+1".

The power of the multiplication principle was refined in later number notations. In particular, five centuries ago the Chinese invented a perfectly regular notation that has been preserved up to this day. It consists of only 13 arbitrary symbols for the digits 1 through 9 and the numbers 10, 100, 1,000, and 10,000. The number 2,342 is simply written down as "2 1000 3 100 4 10 2", a word-for-word transcription of the oral expression "two thousand three hundred forty-two" (forty being "four ten" in Chinese). Thus writing, at this stage, becomes a direct reflection of the oral numeration system.

## The Place-Value Principle

One final invention greatly expanded the efficacy of number notations: the place-value principle. A number notation is said to obey the place-value principle when the quantity that a digit represents varies depending on the place it occupies in the number. Thus the three digits that make up number 222, though identical, refer to different orders of magnitude: two hundreds, two tens, and two units. In a place-value notation, there is a privileged number called the base. We now use base 10, but this is not the only possibility. Successive places in the number represent successive powers of the base, from units ($10^0 = 1$), to tens ($10^1 = 10$), hundreds ($10^2 = 100$), and so on. The quantity that a given number expresses is obtained by multiplying each digit by the corresponding power of the base and then adding up all the products. Hence number 328 represents the quantity $3\times100 + 2\times10 + 8\times1$.

Place-value coding is a must if one wants to perform calculations using simple algorithms. Just try to compute XIV × VII using Roman numerals! Calculations are also inconvenient in the Greek alphabetical notation, because nothing betrays that number N (50) is ten times greater than number E (5). This is the main reason the Greeks and the Romans never performed computations without the help of an abacus. By contrast, our Arabic numerals, based on the place-value principle, make the magnitude relations between 5, 50, 500, and 5,000 completely transparent. Place-value notations are the only ones that reduce the complexity of multiplication to the mere memorization of a table of products from $2\times2$ up to $9\times9$. Their invention revolutionized the art of numerical computation.



While four civilizations seem to have discovered place-value notation, three of them never quite reached the simplicity of our current Arabic numerals. For this notation only becomes highly efficient in conjunction with three other inventions: a symbol for "zero," a unique base number, and the discarding of the addition principle for the digits 1 through 9. Consider, for instance, the oldest place-value system known, devised by Babylonian astronomers eighteen centuries before Christ. Their base number was 60. Hence a number such as 43,345, which is equal to $12\times60^2 + 2\times60 + 25$, was expressed by concatenating the symbols for 12, 2, and 25.

In principle, sixty distinct symbols would have been needed, one for each of the "digits" 0 to 59. Yet obviously it would have been impractical to learn sixty arbitrary symbols. Instead, the Babylonians wrote down these numbers using an additive base-10 notation. For instance, the "digit" 25 was expressed as 10+10 + 1+1+1+1+1. Eventually, the number 43,345 was thus rendered by an obscure sequence of cuneiform characters that literally meant 10+1+1 [implication $\times60^2$], 1+1 [implication ×60], 10+10 + 1+1+1+1+1. Such a mixture of additive and place-value coding, with two bases 10 and 60, turned the Babylonian notation into an awkward system understandable only to a cultivated elite. Still, it was a remarkably advanced numeration for its time. The Babylonian astronomers used it very skillfully for their celestial calculations, whose accuracy remained unsurpassed for more than a thousand years. Its success was due in part to its simple representation of fractions: Because 2, 3, 4, 5, and 6 are divisors of the base 60, the fractions ½, ⅓, ¼, ⅕, and ⅙ all had a simple sexagesimal expression.

Judged by today's standards, the Babylonian system had one final drawback: For fifteen centuries, it lacked a zero. What is a zero good for? It serves as a placeholder that denotes the absence of units of a given rank in a multidigit numeral. For instance, in Arabic notation the number "503" means five hundreds, no tens, and three units. Lacking a zero, Babylonian scientists simply left a blank at the place where a digit should have appeared. This meaningful void was a recurring source of ambiguities. The numbers 301 ($5\times60+1$), 18,001 ($5\times60^2+1$) and 1,080,001 ($5\times60^3+1$) were confusedly expressed by similar strings: 51, 5 1 (with one blank), and 5  1 (with two blanks). Hence the absence of a zero was the cause of many errors in calculation. Worse, an isolated digit such as "1" had multiple meanings. It could mean quantity 1, of course, but also "1 followed by a blank" or 1×60, or even "one followed by two blanks" or $1\times60^2 = 3,600$, and so on. Only the context could determine which interpretation was correct. Not until the third century before Christ did the Babylonians finally introduce a symbol to fill this gap and explicitly denote absent units. Even then, this symbol served only as a placeholder. It never acquired the meaning of a "null quantity" or of "the integer immediately below 1" which we attribute to it today.

While Babylonian astronomers' place-value notation was apparently lost in the collapse of their civilization, three other cultures later reinvented remarkably similar systems. Chinese scientists, in the second century before Christ, devised a place-value code devoid of the digit 0 and using the bases 5 and 10. Mayan astronomers, in the second half of the first millennium, computed with numbers written in a mixture of base 5 and 20 and with a full-fledged digit 0. And Indian mathematicians, finally, bequeathed humanity the place-value notation in base 10 that is now in use throughout the world.

It seems a bit unfair to call "Arabic numerals" an invention originally due to the ingenuity of the Indian civilization. Our number notation is called "Arabic" merely because the Western world discovered it for the first time through the mathematical writings of the great Persian mathematicians. Many of the modern techniques of numerical calculation derived from the work of Persian scientists. The word "algorithm" was named after a work by one of the them, Mohammed ibn Musa al-Khuwarizmi. His most famous book was a treatise for solving linear equations, *Al-jabr w'al muqâbala (On Reducing and Simplifying)*, one of the few books whose publication founded a new science, "algebra." Yet for all their inventiveness, the discoveries of the Persians could not have seen the light without the help of the Indian number notation.

A particular homage should be paid to a unique innovation in the Indian notation, one that was lacking in all other place-value systems: the selection of ten arbitrary digits whose shapes are unrelated to the numerical quantities they represent. At first glance, one might think that using arbitrary shapes should be a disadvantage. A series of strokes seem to provide a more transparent way of denoting numbers, one that is easier to learn. And perhaps this was the implicit logic of the Sumerian, Chinese, and Mayan scientists. However we have seen in the preceding chapter that it is incorrect. The human brain takes longer to count five objects than to recognize an arbitrary shape and associate it with a meaning. The peculiar disposition of our perceptual apparatus for quickly retrieving the meaning of an arbitrary shape, which I have dubbed the "comprehension reflex," is admirably exploited in the Indian-Arabic place-value notation This numeration tool, with its ten easily discernible digits, tightly fits the human visual and cognitive system.

## An Exuberant Diversity of Number Languages

Nowadays when people of almost any country write down a number, they adopt the same convention and employ the base-ten Arabic notation. Only the shape of digits remains slightly variable. Instead of our Arabic digits, some Middle Eastern countries such as Iran use another set of shapes referred to as "Indian digits." Even there, however, the standard Arabic notation is gaining ground. Its victory has little to do with imperialism or the establishment of commercial norms. If the evolution of written numeration converges, it is mainly because place-value coding is the best available notation. So many of its characteristics can be praised: its compactness, the few symbols it requires, the ease with which it can be learned, the speed with which it can be read or written, the simplicity of the calculation algorithms it supports. All justify its universal adoption. Indeed, it is hard to see what new invention could ever improve on it.



No such convergence is found for oral numeration. Although the vast majority of human languages possess a number syntax based on a combination of sums and products, in detail the diversity of numeration systems is striking. First of all, a variety of bases are used. In the Queensland district of Australia, some aborigines are still confined to base 2. Number 1 is "ganar," 2 is "burla," 3 "burla-ganar," and 4 "burla-burla." In old Sumer, by contrast, bases 10, 20, and 60 were concurrently used. Hence number 5,546 was expressed as "sàr (3,600) ges-u-es (60×10×3) ges-min (60×2) nismin (20×2) às (6)", or 3600 + 60×10×3+ 2×60 + 2×20 + 6 = 5546. Base 20 also had its adepts: It ruled the Aztec, Mayan, and Gaelic languages and is still in use in Eskimo and Yoruba. Traces of it can still be found in French, in which 80 is "quatre-vingt" (four twenties), and in Elizabethan English, which often counted in scores (twenty).

Although base 10 has now taken over most languages, number syntax remains highly variable. The prize for simplicity goes to Asian languages such as Chinese, whose grammar is a perfect reflection of decimal structure. In such languages there are only nine names for numbers 1 through 9 (yi, èr, san, si, wu, liù, qi, ba, and jiu), to which one should add four multipliers 10 (shi), 100 (bai), 1,000 (qian), and 10,000 (wàn). In order to name a number, one just reads its decomposition in base 10. Thus 13 is "shi san" (ten three), 27 "èr shi qi" (two ten seven), and 92,547 "shi wàn ér qian wu bai si shi qi" (nine myriads two thousands five hundreds four tens seven).

This elegant formalism contrasts sharply with the 29 words needed to express the same numbers in English or in French. In these languages, the numbers 11 through 19 and the decades from 20 to 90 are denoted by special words (eleven, twelve, twenty, thirty, etc.) whose appearance is not predictable from that of other numerals. No need to mention the even stranger peculiarities of French, with its awkward words "soixante-dix" (sixty-ten, or 70) and "quatre-vingt-dix" (four-twenty-ten, or 90). French also has confusing elision and conjunction rules involving the number 1: one says "vingt-*et*-un" (twenty-and-one) rather than "vingt-un," yet 22 is "vingt-deux" rather than "vingt-*et*-deux," and 81 is "quatre-vingt-un" and not "quatre-vingt-*et*-un." Likewise, 100 is "cent" rather than "un cent." Another eccentricity is the systematic reversal of decades and units in Germanic languages, where 432 becomes "vierhundertzweiunddreißig" (four hundred two and thirty).

What are the practical consequences of this exuberant diversity of numerical languages? Are all languages equivalent? Or are some number notations better adapted to the structure of our brains? Do certain countries, by virtue of their numeration system, start out with an advantage in mathematics? This is no trivial matter in the current period of fierce international competition, in which numeracy is a key factor to success. As adults, we are largely unaware of the complexity of our numeration system. Years of training have tamed us into accepting that 76 should be pronounced "seventy-six" rather than, say, "seven ten six" or "sixty-sixteen." Hence we can't objectively compare our language with others anymore. Rigorous psychological experiments are needed to measure the relative efficacy of various numeration systems. Surprisingly, these experiments repeatedly demonstrate the inferiority of English or French over Asian languages.

## The Cost of Speaking English

Read the following list aloud: 4, 8, 5, 3, 9, 7, 6. Now close your eyes and try to memorize the numbers for twenty seconds before reciting them again. If your native language is English, you have about a 50% chance of failure. If you are Chinese, however, success is almost guaranteed. As a matter of fact, memory span in China soars to about nine digits, while it averages only seven in English. Why this discrepancy? Are speakers of Chinese more intelligent? Probably not, but their number words happen to be shorter. When we try to remember a list of digits, we generally store it using a verbal memory loop (this is why it is difficult to memorize numbers whose names sound similar, such as "five" and "nine" or "seven" and "eleven"). This memory can hold data only for about two seconds, forcing us to rehearse the words in order to refresh them. Our memory span is thus determined by how many number words we can repeat in less that two seconds. Those of us who recite faster have a better memory.

Chinese number words are remarkably brief. Most of them can be uttered in less than one-quarter of a second (for instance, 4 is "si" and 7 "qi"). Their English equivalents—"four," "seven"—are longer: pronouncing them takes about one-third of a second. The memory gap between English and Chinese apparently is entirely due to this difference in length. In languages as diverse as Welsh, Arabic, Chinese, English, and Hebrew, there is a reproducible correlation between the time required to pronounce numbers in a given language and the memory span



of its speakers. In this domain, the prize for efficacy goes to the Cantonese dialect of Chinese, whose brevity grants residents of Hong Kong a rocketing memory span of about 10 digits.

In summary, the "magical number 7," which is so often heralded as a fixed parameter of human memory, is not a universal constant. It is merely the standard value for digit span in one special population of *Homo sapiens* on which more than 90% of psychological studies happen to be focused, the American college undergraduate! Digit span is a culture- and training-dependent value, and cannot be taken to index a fixed biological memory size parameter. Its variations from culture to culture suggest that Asian numerical notations, such as Chinese, are more easily memorized than our Western systems of numerals because they are more compact.

If you do not speak any Chinese, there is still hope. Several tricks are available to increase your memory for digits. First of all, always memorize numbers using the shortest possible sequence of words. A long number such as 83,412 is often best recalled by reciting it digit by digit, as with a phone number. Second, try grouping the digits into small blocks of two or three. Your working memory will jump to about twelve digits if you group them in four blocks of three. Phone numbers in the United States, with their division into a three-digit area code and then three groups of three, two, and two digits, as in "503 485 98 31," already make use of these stratagems. In France, by contrast, we have the bad habit of expressing phone numbers with two-digit numerals. For instance, we read 85 98 31 as "eighty-five ninety-eight thirty-one"—probably the most memory-inefficient method that one could think of!

A third trick is to bring the number back to familiar ground. Look for increasing or decreasing series of digits, familiar dates, zip codes, or any other information that you already know. If you can recode the number using only a few familiar items, you should easily remember them. After about 250 hours of training under the guidance of psychologists William Chase and K. Anders Ericsson, an American student was able to extend his memory span up to an extraordinary eighty digits using this recoding method. He was an excellent long-distance runner and had compiled a large mental database of record running times. He therefore stored the eighty digits to be remembered, broken down into groups of three or four, as a series of record times in long-term memory!

Using these guidelines, you should have little difficulty memorizing phone numbers. But unless you are Chinese, you are still in for a hard time. Number names also play a critical role in counting and calculating, and here again bad marks can be attributed to languages with the longest number names. For instance, it takes a Welsh pupil one second and a half more than an English pupil, on average, to compute 134+88. For equal age and education, this difference seems solely due to the time taken to pronounce the problem and the intermediate results: Welsh numerals happen to be considerably longer than the English. English is certainly not the optimum, though, because several experiments have shown that Japanese and Chinese children calculate much faster than their American peers.

It can be difficult, of course, to tease apart the effects of language from those of education, number of hours at school, parental pressure, and so on (in fact, good evidence exists that the organization of Japanese mathematics lessons is in many ways superior to that of the standard U.S. school system). However, many such variables can be left aside by studying language acquisition in children who have not yet been to school. All children are confronted with the challenging task of discovering, by themselves, the lexicon and grammar of their maternal language. How do they ever acquire the rules of French or German by mere exposure to phrases such as "soixante-quinze" or "fünfundsiebzig"? And how can a French child discover the meanings of "cent deux" and "deux cent"? Even if the child is a born linguist and if, as postulated by Noam Chomsky and Steven Pinker, the brain comes equipped with a language organ that makes learning the most abstruse linguistic rules a matter of instinct, the induction of number formation rules is by no means instantaneous and varies from language to language.

In Chinese, for instance, once you have learned the number words up to ten, the others are easily generated by a simple rule (11 = ten one, 12 = ten two . . . , 20 = two ten, 21 = two ten one, etc.). In contrast, American children have to learn by rote, not just the numerals from 1 to 10, but also those from 11 to 19, and also the tens numbers from 20 to 90. They must also discover for themselves the multiple rules of number syntax that specify, for instance, that "twenty forty" or "thirty eleven" are invalid sequences of number words.

In a fascinating experiment, Kevin Miller and his colleagues asked matched groups of American and Chinese children to recite the counting sequence. Startlingly, the linguistic difference caused American children to lag as much as one year behind their Chinese peers. When they were four, Chinese children already counted up to 40 on average. At the same age, American children painfully counted up to 15. It took them one year to catch up and reach 40 or 50. They were not just globally slower than the Chinese; up to the number 12, both groups stood on an equal footing. But when they reached the special numbers "13" and "14," American children suddenly stumbled, while the Chinese, helped by the unfailing regularity of the language, moved right along with much less trouble (Figure 4.3).

The Miller experiment shows beyond a doubt that the opacity of a numeration system takes an important toll on language acquisition. Another proof comes from the analysis of counting errors. Haven't we all heard American children recite "twenty-eight, twenty-nine, twenty-ten, twenty-eleven," and so on?



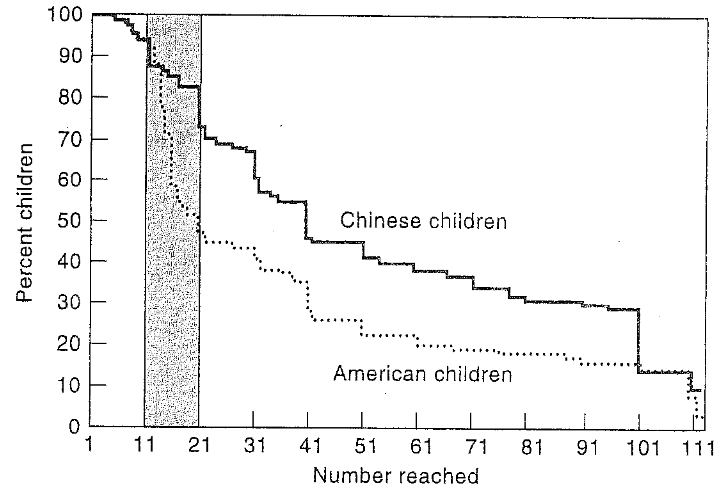


*Figure 4.3. Kevin Miller and his colleagues asked American and Chinese children to recite numbers as far as they could. At a matched age, Chinese children could count much farther than their American counterparts. (Adapted from Miller et al. 1995 by permission of the publisher; copyright © 1995 by Cambridge University Press.)*

Such grammatical errors, telltale signs of a poor induction of the rules of number syntax, are unheard of in Asian countries.

The influence of numeration systems carries through into subsequent school years. The organization of spoken Chinese numerals directly parallels the structure of written Arabic numerals. Hence, Chinese children experience much less difficulty than their American counterparts in learning the principles of place-value notation in base ten. When asked to form number 25 using some unit cubes and some bars of 10, Chinese schoolboys readily select two bars of 10 and five units, suggesting that they understand base ten. At a matched age, American children behave differently. Most of them laboriously count twenty-five units, thus failing to take advantage of the shortcut provided by the groups of 10. Worse yet, if one provides them with a bar comprising twenty units, they use it more frequently than two bars of ten. Thus they seem to attend to the surface form of the word "twenty-five," while the Chinese already master their deeper base-10 structure. Base 10 is a transparent concept in Asian languages, but is a real headache for Western children.

These experimental findings impose a strong conclusion: Western numeration systems are inferior to Asian languages in many respects—they are harder to keep in short-term memory, slow down calculation, and make the acquisition of counting and of base ten more difficult. Cultural selection should long have eliminated constructions as absurd as the French "quatre-vingt-dix-sept." Unfortunately, the normalization efforts of our schools and academies have put a stop to the natural evolution of languages. If children could vote, they would probably favor a widespread reform of numerical notations and the adoption of the Chinese model. Would such a revision be less utopian than the ill-fated spelling reforms? We have at least one historical example of a successful major linguistic reform. At the beginning of the twentieth century, the Welsh willingly relinquished their old numeration system, which was more complex even than present-day French, and selected instead a simplified notation quite similar to Chinese. Unfortunately, Welsh changed only to fall prey to another error: The new Welsh number words, while grammatically regular and thus easy to learn, are so long that memory suffers! Psychological experiments would probably dictate the adoption of a well-tested numeration system such as Mandarin Chinese, but national interests make this a rather distant and unlikely prospect.

## Learning to Label Quantities

Acquiring a number lexicon and syntax is not everything. It is not particularly useful to know that "two hundred and thirty" is a valid English phrase while "two thirty and hundred" is not. Above all, children must learn what these numerals mean. The power of numeration systems stems from their ability to establish precise links between linguistic symbols and the quantities they express. A child may well recite numerals up to 100, but is only parroting unless he or she also knows what magnitudes they stand for. How then do children ever learn the meaning of /wan/, /siks/ or /eit/?

A first basic problem confronting a child is to recognize that these words refer to number rather than to color, size, shape, or any other dimension of the environment. Consider the phrases "the three sheep" and "the big sheep." A child who hears them for the first time and who does not know the meaning of the words "three" and "big" has no way of telling that "big" refers to the physical size of each sheep, while "three" refers to the cardinal of the set of sheep.

Experiments show that by two and a half years of age, American children already differentiate number words from other adjectives. When given a choice between a picture of a single red sheep and another showing three blue sheep, children readily point to the first when they are told, "Show me the red sheep," and to the second when told, "Show me the three sheep." By that age, children already know that "three" applies to a collection of items rather than to a single item. At the same age, children also order number words and other adjectives correctly. They say "three little sheep," but never "little three sheep." Early on, then, children know that number words belong to a special category distinct from other words.



How did they find this out? Probably by exploiting all the available cues, be they grammatical or semantic. Grammar alone may be of precious help. Suppose that a mother tells her baby, "Look, Charlie, three little doggies." Baby Charlie may then infer that the word "three" is a special kind of adjective because other adjectives such as "nice," are always said with an article—"the nice little doggies." The fact that the word "three" does not require an article may suggest that "three" applies to the entire collection of little doggies, and that therefore it may be a number or a quantifier like "some" or "many."

Of course such reasoning is of little help for determining the precise quantity to which the word "three" refers. Indeed, it appears that for a whole year, children realize that the word "three" is a number without knowing the precise value it refers to. When they are ordered, "Give me three toys," most of them simply grasp a pile without caring about the exact number. If one lets them choose between a group of two and a group of three toys, they also respond at random —although they never select a card showing a single object. They know how to recite number words, and they sense that these words have to do with quantity, but they ignore their exact meaning.

Semantic cues are probably critical in order to overcome this stage and to determine the precise quantity that is meant by the word "three." With a little luck, Baby Charlie will see the three little doggies his mom is talking about. His perceptual system, whose sophistication we have discussed in Chapter 2, may then analyze the scene and identify the presence of several animals, of a small size, noisy, moving, and numbering about three. (By this I do not mean, of course, that Charlie already knows that the word "three" applies to this numerosity; I only mean that Charlie's internal non-verbal accumulator has reached the state of fullness which is typical of sets of three items.)

In essence, all Charlie has to do, then, is to correlate these preverbal representations with the words he hears. After a few weeks or months, he should realize that the word "three" is not always uttered in the presence of small things, of animals, of movement, or of noise; but that it is very often mentioned when his mental accumulator is in a particular state that accompanies the presence of three items. Thus, correlations between number words and his prior nonverbal numerical representations can help him determine that "three" means 3.

This correlation process can be accelerated by the "principle of contrast," which stipulates that words that sound different have different meanings. If Charlie already knows the meaning of the words "doggie" and "small," the principle of contrast guarantees him that the unknown word "three" cannot refer to the size or the identity of the animals. Narrowing down the set of hypotheses enables him to find out even faster that this word refers to numerosity three.

## Round Numbers, Sharp Numbers

Once children have acquired the exact meaning of number words, they still have to grasp some of the conventions governing their use in language. One of them is the distinction between round numbers and sharp numbers. Let me introduce it with a joke:

> At the museum of natural history, a visitor asks the curator, "How old is this dinosaur over here?" "Seventy million and thirty-seven years" is the answer. As the visitor marvels at the accuracy of the dating, the curator explains: "I've been working here for 37 years, you know, and when I arrived I was told that it was 70 million years old"!

Lewis Carroll, well-known for his ingenious word games based on logic and mathematics, often spiced his stories with "numerical non-sense." Here is an example from his little-known book *Sylvie and Bruno Concluded*:

> "Don't interrupt," Bruno said as we came in. "I'm counting the Pigs in the field!"
> "How many are there?" I enquired.
> "About a thousand and four," said Bruno.
> "You mean 'about a thousand,'" Sylvie corrected him. "There's no good saying 'and four': you ca'n't be sure about the four!"
> "And you're as wrong as ever!" Bruno exclaimed triumphantly. "It's just the four I can be sure about; 'cause they're here, grubbling under the window! It is the thousand I isn't pruffickly sure about!"

Why do these exchanges sound eccentric? Because they violate an implicit and universal principle governing the use of numerals. The principle stipulates that certain numerals, called "round numbers," can refer to an approximate quantity while all other numerals necessarily have a sharp and precise meaning. When one states that a dinosaur is 70 million years old, this value is implicitly understood to within 10 million years. The rule is that a number's accuracy is given by its last non-zero digit starting from the right. If I maintain that the population of Mexico City is 39,000,000, I mean that this number is correct to within a million, whereas if I give you a value of 39,452,000 inhabitants, I implicitly admit that it is correct to give or take a thousand.

This convention sometimes leads to paradoxical situations. If a precise quantity happens to fall exactly on a round number, just asserting it is not sufficient. One must supplement it with an adverb or locution that makes its accuracy explicit—for example, "Today, Mexico has *exactly* 39 million inhabitants." For the same reason, the sentence "nineteen is about 20" is acceptable, while "twenty is about 19" isn't. The phrase "about 19" is a contradiction in terms, for why use a sharp number such as 19 if one wants to state an estimation?



All the languages of the world seem to have selected a set of round numbers. Why this universality? Probably because all humans share the same mental apparatus and are therefore all confronted with the difficulty of conceptualizing large quantities. The larger a number, the less accurate our mental representation of it. Language, if it wants to be a faithful vehicle for thought, must incorporate devices that express this increasing uncertainty. Round numbers are such a device. Conventionally, they refer to approximate quantities. The sentence "There are twenty students in this room" remains true even if there are eighteen or twenty-two students because the word "twenty" can refer to an extended region of the number line. This is also why speakers of French find it so natural that "fifteen days" means "two weeks," although the exact number should be fourteen.

Approximation is so important to our mental life that many other linguistic mechanisms are available to express it. All languages possess a rich vocabulary of words for expressing various degrees of numerical uncertainty—about, around, circa, almost, roughly, approximately, more or less, nearly, barely, and so on. Most languages have also adopted an interesting construction in which two juxtaposed numbers, often linked by the conjunction "or," express a confidence interval: two or three books, five or ten people, a boy age twelve or fifteen years, 300 or 350 dollars. This construction allows us to communicate not just an approximate quantity, but also the degree of accuracy that should be granted to it. Thus the same central tendency can be expressed with increasing uncertainty by saying 10 or 11, 10 or 12, 10 or 15, or 10 or 20.

A linguistic analysis by Thijs Pollmann and Carel Jansen shows that two-number constructions follow certain implicit rules. Not all intervals are equally acceptable. At least one of the numbers must be round: One can say "twenty or twenty-five dollars" but not "twenty-one or twenty-six dollars." The other number must be of a similar order of magnitude: "Ten or one thousand dollars" sounds very strange indeed. Another Lewis Carroll quote illustrates this point:

> "How far have you come, dear?" the young lady persisted.
> Sylvie looked puzzled. "A mile or two, I think," she said doubtfully.
> "A mile or three," said Bruno.
> "You should not say 'a mile or three,'" Sylvie corrected him.

The young lady nodded approval. "Sylvie's quite right. It isn't usual to say 'a mile or three.'"

"It would be usual—if we said it often enough," said Bruno.

Bruno is wrong—"a mile or three" would never sound right, because it violates the basic rules of the two-number construction. These rules are understandable if one considers which representations we intend to communicate. These representations are fuzzy intervals on a mental number line. When we say "twenty, twenty-five dollars," we actually mean "a certain fuzzy state of my mental accumulator, somewhere around 20 and with a variance of about 5." Neither the interval from 21 to 26, nor that from 10 to 1,000, or from 1 to 3 are plausible states of the accumulator because the former is too accurate while the latter two are too imprecise.

## Why Are Some Numerals More Frequent Than Others?

Would you like to try a bet? Open a book at random and note the first digit that you encounter. If this digit is either 4, 5, 6, 7, 8, or 9, you win ten dollars. If it is 1, 2, or 3, I win this amount. Most people are ready to take this bet, because they believe that the odds are 6:3 for them to win. And yet the bet is a loser. Believe it or not, the digits 1, 2, and 3 are about twice as likely to appear in print than all other digits combined!

This is a strongly counterintuitive finding because the nine digits seem equivalent and interchangeable. But we forget that numbers that appear in print are not drawn from a random number generator. Each of them represents an attempt to transmit a piece of numerical information from one human brain to another. Hence, how frequently each numeral is used is determined in part by how easily our brain can represent the corresponding quantity. The decreasing precision with which numbers are mentally represented influences not just the perception, but also the production of numerals.

Jacques Mehler and I have systematically looked for number words in tables of word frequency. Such tables tally up how often a certain word, say "five," appears in written or spoken texts. Frequency tables are available in a great variety of languages, from French to Japanese, English, Dutch, Catalan, Spanish, and even Kannada, a Dravidian language spoken in Sri Lanka and southern India. In all of these languages, despite enormous cultural, linguistic, and geographic diversity, we have observed the same results: The frequency of numerals decreases systematically with number size.



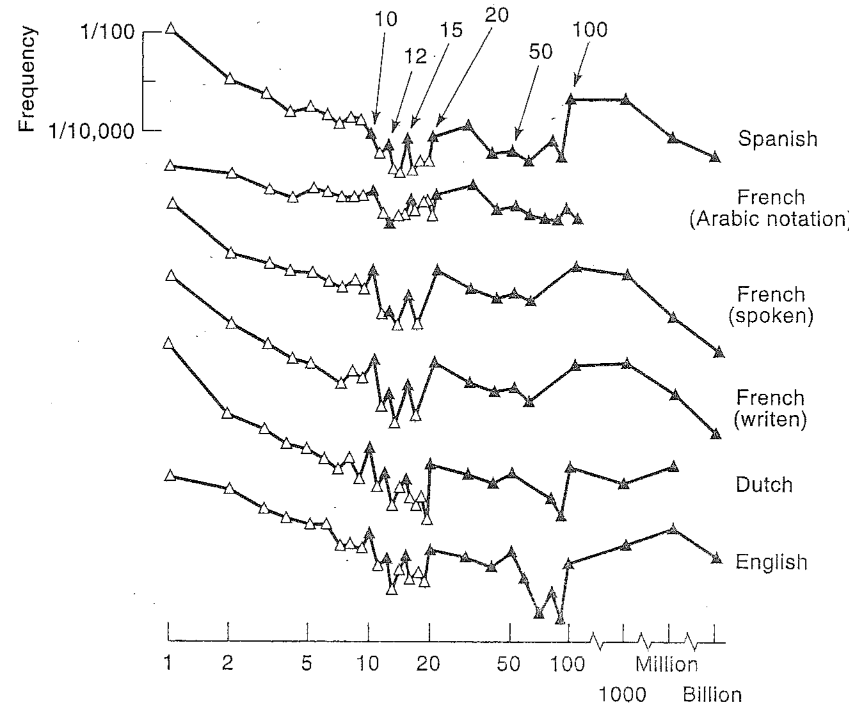


*Figure 4.4. In all languages, the frequency with which number words are printed or uttered decreases with magnitude, aside from local increases for the round numbers 10, 12, 15, 20, 50, and 100. For instance, we read or hear the word* two *about ten times more often than the word* nine. *(After Dehaene and Mehler 1992.)*

In French, for instance, the word "un" appears once every seventy words or so, the word "deux" once every six hundred words, the word "trois" once every one thousand, seven hundred words, and so on. Frequency decreases from 1 to 9, but also from 11 to 19, and for tens numbers from 10 to 90. A similar decrease is observed for written or spoken numerals, for Arabic numerals, and even for ordinals from "first" to "ninth." It is accompanied by a few deviations that are also universal: the very low frequency of the word "zero," and the elevated peaks for 10, 12, 15, 20, 50, and 100 (Figure 4.4). Remarkably, such cross-linguistic regularities persist in the face of pronounced differences in the way numbers are expressed, such as the absence of teen words in Japanese, the inversion of tens and units in Dutch, or the cryptic base 20 of the French words 70, 80, and 90.

I contend that once again, these linguistic regularities reflect the way our brain represents numerical quantities. Yet before jumping to this conclusion, several alternative explanations have to be examined. Ambiguity may be a possible source of this finding. In many languages the word for "one" is indistinguishable from the indefinite article "a." This probably contributes to the elevated frequency of the word "un" in French—but obviously not in English, where "one" can only be a number word. Ambiguity is also not a problem beyond "two," and yet frequency decreases sharply beyond this point.

Another contributing factor is our propensity for counting, which implies that many objects in our environment are numbered starting at 1. In any city, more houses bear number 1 than number 100, merely because all streets have a number 1, but some don't reach number 100. Again this effect certainly contributes to the elevated frequency of small numerals, but quick calculation shows that by itself it cannot account for the exponential drop of number frequency, even in the interval from 1 to 9.

Purely mathematical explanations of the effect should also be given some consideration. Few people know the following *very* counterintuitive mathematical law: If you draw several random numbers from essentially any smooth distribution, the numbers will start more often with 1 than with 9. This singular phenomenon is called Benford's law. Frank Benford, an American physicist, made a curious observation: At his university's library, the first pages of the tables of logarithms were more worn out than the last. Now surely people did not read tables of logarithms like a bad novel, stopping halfway through. Why did his colleagues have to consult the beginning of the table more often than the end? Could it be that small numbers were used more often than large ones? To his own bewilderment, Benford discovered that numbers of all origins—the surface of American lakes, his colleagues' street addresses, the square root of integers, and so on—were about six times more likely to start with digit 1 than with digit 9. About 31% of numbers started with 1, 19% with 2, 12% with 3, and the percentages decreases with each successive number. The probability that a number started with digit $n$ was very accurately predicted by the formula $P(n) = \log_{10}(n+1) - \log_{10}(n)$.

The exact origin of this law is still poorly understood, but one thing is certain: This is a purely formal law, due solely to the grammatical structure of our numerical notations. It has nothing to do with psychology: A computer reproduces it when it prints random numbers in Arabic notation or even spells them out. The only constraint seems to be that the numbers be drawn from a sufficiently smooth distribution spread over several orders of magnitude—for instance, from 1 to 10,000.

Benford's law certainly contributes to amplifying the frequency of small numbers in natural language. Yet its explanatory power is limited. The law applies only to the frequency of the leftmost digit in a multidigit numeral, and so it does not have any influence on how frequently we refer to the quantities 1 through 9.



But the measurements that Jacques Mehler and I have performed show, quite straightforwardly, that the human brain finds it more important to talk about quantity 1 than about quantity 9. Contrary to Benford's law, this fact has nothing to do with the production of large multidigit numerals.

If it is not the grammar of numerical notations that drives us to produce small numbers, could it be Mother Nature herself? Aren't small collections of objects exceptionally frequent in our environment? To take just one example, discussing the number of one's children nowadays usually only requires number words below 3 or 4! Yet as a general explanation of the decreasing frequency of numerals, this account is misguided. Philosophers Gottlob Frege and W.V.O. Quine have long demonstrated that, objectively speaking, small numerosities are no more frequent than larger ones in our environment. In any situation, a potential infinity of things might be enumerated. Why do we prefer to speak of *one* deck of cards rather than fifty-two cards? The notion that the world is mostly made up of small sets is an illusion imposed on us by our perceptual and cognitive systems. Nature isn't made that way, no matter what our brain may think.

To prove this point without resorting to philosophical arguments, consider the distribution of words with a numerical prefix, such as "*bi*cycle" or "*tri*angle." Just as the word "two" is more frequent than "three," there are more words that begin with the prefix *bi* (or *di* or *duo*) that with *tri*. Crucially, this remains true even in domains where there is arguably little or no environmental bias for small numbers. Consider time. My English dictionary lists fourteen temporal words with the prefix *bi* or *di* (from "biannual" to "diestrual"), five words with the prefix *tri* (from "triennial" to "triweekly"), five words with a prefix expressing fourness, and only two expressing fiveness (the uncommon words "quinquennial" and "quinquennium"). Hence, increasingly fewer words express increasingly large numbers. Could this be due to an environmental bias? In the natural world, events do not recur particularly often with a two-month period. No, the culprit is our brain, which pays more attention to events when they concern small or round numbers.

If a lexical bias for small numbers can emerge in the absence of any environmental bias, conversely there are situations in which an objective bias fails to be incorporated in the lexicon. Many more vehicles have four wheels than two, yet we have a number-prefixed word for the latter (bicycle) but not for the former (quadricycle?). Numerical regularities in the world seem to be lexicalized only if they concern a small enough numerosity. For instance, we have number-prefixed words for plants with three leaves (trifoliate, trifolium; trèfle in French), but not for the many other plants or flowers with a fixed but large number of leaves or petals. Words like "octopus" that explicitly refer to a precise large numerosity are rare. As a final example, *Scolopendra morsitans*, an arthropod with twenty-one body segments and forty-two legs, is commonly called a centipede (one hundred feet) in English and a "mille-pattes" (thousand-legs) in French! Clearly, we pay attention to the numerical regularities of nature only inasmuch as they fit in with our cognitive apparatus, which is biased toward small or round numerosities.

Human language is deeply influenced by a nonverbal representation of numbers that we share with animals and infants. I believe that this alone explains the universal decrease of word frequency with number size. We express small numbers much more often than large ones because our mental number line represents numbers with decreasing accuracy. The larger a quantity is, the fuzzier our mental representation of it, and the less often we feel the need to express that precise quantity.

Round numbers are exceptions because they can refer to an entire range of magnitudes. This is why the frequency of the words "ten," "twelve," "fifteen," "twenty," and "hundred" is elevated compared to their neighbors. All in all, both the global decrease and the local peaks in number frequency can be explained by a labeling of the internal number line (Figure 4.5). As children acquire language, they learn to put a name on each range of magnitudes. They discover that the word "two" applies to a percept that they know from birth; that "nine" pertains only to the precise quantity 9, which is difficult to represent exactly; and that people often use the word "ten" to mean any quantity somewhere between 5 and 15. In turn, they therefore utter the words "two" and "ten" more often than "nine," hence perpetuating the lawful distribution of number frequencies.

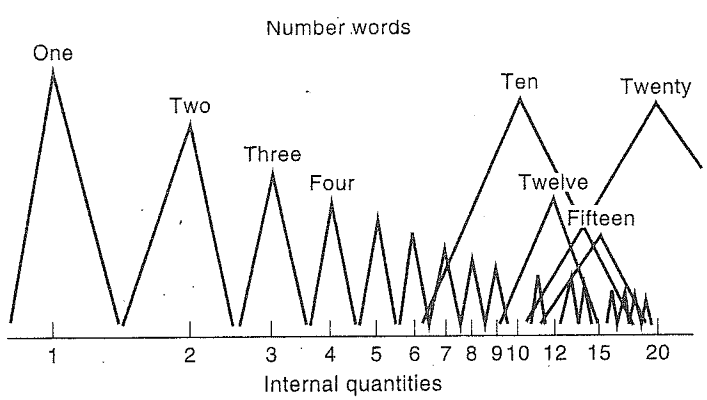


*Figure 4.5. The decreasing frequency of numerals is due to the organization of our mental representation of quantities. The larger a number, the less accurate our mental representation of it; hence, the less often we need to use the corresponding word. As for round numbers like 10, 12, 15 or 20, they are uttered more frequently than others because they can refer to a greater range of quantities. (After Dehaene and Mehler 1992.)*



One last detail: Our study showed that, in all Western languages, the frequency of number 13 was lower than that of 12 or 14. This seems the result of the "Devil's dozen" superstition, which assigns a maleficent power to number 13 and is known to such a degree that many American skyscrapers have no 13th floor. In India, where this superstition is unknown, the frequency of the number 13 does not show any notable drop. The frequency of numerals appears to faithfully reflect their importance in our mental lives, even in their most trivial details.

## Cerebral Constraints on Cultural Evolution

What does the analysis of numerical languages reveal about the relationship between mathematics and the brain? It shows that numeration systems have evolved both *through* the brain and *for* the brain. Through the brain, because the history of number notations is clearly limited by the inventiveness of the human brain and its ability to fathom new principles of numeration. For the brain, because numerical inventions have been transmitted from generation to generation only when they closely matched the limits of human perception and memory, and therefore increased humankind's computational potential.

The history of numerals is obviously not driven merely by random factors. It exhibits discernible regularities that transcend the fortunes of history. Across borders and oceans, men and women of all colors, cultures, and religions have regularly reinvented the same notation devices. The place-value principle was rediscovered, with an interval of about three thousand years, in the Middle East, on the American continent, in China, and in India. In all languages, frequency decreases with number size. In all languages too, round numbers are contrasted with sharp numbers. The explanation of these striking cross-cultural parallels does not reside in dubious exchanges between remote civilizations. Rather, they discovered similar solutions because they were confronted with the same problems and have been endowed with the same brain for solving them.

Let me sketch a summary of humankind's slow march toward greater numerical efficacy—a summary that must remain highly schematic, given that history is rarely linear and that some cultures may have skipped several steps.

### 1. *Evolution of oral numeration*

STARTING POINT: The mental representation of numerical quantities that we share with animals

PROBLEM: How to communicate these quantities through spoken language?
SOLUTION: Let the words "one," "two," and "three" refer directly to the subitized numerosities 1, 2, and 3.

PROBLEM: How to refer to numbers beyond 3?
SOLUTION: Impose a one-to-one correspondence with body parts (12 = pointing to the left breast).

PROBLEM: How to count when the hands are busy?
SOLUTION: Turn the names of body parts into number names (12 = "left breast").

PROBLEM: There is only a limited set of body parts, compared with an infinity of numbers.
SOLUTION: Invent number syntax (12 = "two hands and two fingers").

PROBLEM: How to refer to approximate quantities?
SOLUTION: Select a set of "round numbers" and invent the two-word construction (e.g., ten or twelve people).

*2. Evolution of written numeration*

PROBLEM: How to keep a permanent trace of numerosities?
SOLUTION: One-to-one correspondence. Engrave notches on bone, wood, and so on (7 = IIIIIII).

PROBLEM: This representation is hard to read.
SOLUTION: Regroup the notches (7 = ~~IIII~~ II). Replace some of these groups with a single symbol (7 = VII).

PROBLEM: Large numbers still require many symbols (e.g., 37 = XXXVII).
IMPASSE 1: Add even more symbols (e.g., L instead of XXXXX).
IMPASSE 2: Use distinct symbols to denote units, tens, and hundreds (345 = TME).
SOLUTION: Denote numbers using a combination of multiplication and addition (345 = 3 hundreds, 4 tens, and 5).
PROBLEM: This notation still suffers from the repetition of the words "hundreds" and "tens."
SOLUTION: Drop these words, resulting in a shorter notation ancestral to modern place-value notation (437 = 4 3 7).



PROBLEM: This notation is ambiguous when units of a certain rank are lacking (407, denoted as 4 7, is easily confused with 47).
SOLUTION: Invent a placeholder, the symbol zero.

The cultural evolution of numeration systems testifies to the inventiveness of humanity. Across centuries, ingenious notation devices have been invented and constantly refined, the better to fit the human mind and improve the usability of numbers. The history of number notations is hard to reconcile with the Platonist conception of numbers as ideal concepts that transcend humankind and give us access to mathematical truths independent of the human mind. Contrary to what the Platonist mathematician Alain Connes has written, mathematical objects are not "untainted by cultural associations"—or at the very least this is not true of numbers, one of the most central of all mathematical objects. What has driven the evolution of numeration systems is obviously not an "abstract concept" of number, nor an ethereal conception of mathematics. If this were the case, as generations of mathematicians have noted, binary notation would have been a much more rational choice than our good old base 10. At least a prime number such as 7 or 11, or perhaps a number with many divisors such as 12, should have been selected as the base of numeration. But more prosaic criteria governed our ancestors' choices. The preponderance of base 10 is due to the contingent fact that we have ten fingers; the bounds of our subitization procedure account for the structure of Roman numerals; and the sharp limits of our short-term memory explain the constant drive toward a compact notation for large numbers. Let us leave the last word to the philosopher Karl Popper: "The natural numbers are the work of men, the product of human language and of human thought."