Complete Expanding Interval Maps are Mixing

Emma O'Neil

Portland State University

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Outline

Example: The Gauss Map

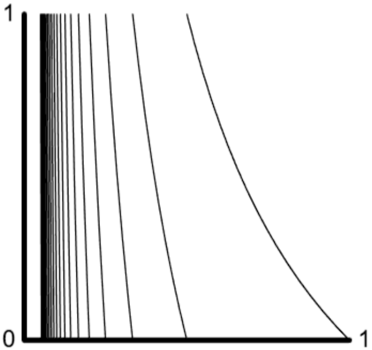

Figure 1: First 20 branches of the Gauss map, $x \to \frac{1}{x} \mod 1$

Invariant Measures

Invariant Measure

An invariant measure associated with a map T is a measure m such that

$$m(T^{-1}(A)) = m(A)$$

for m-measurable sets A. The invariant density μ associated with an invariant measure satisfies

$$\int_A \mu(x)dx = m(A)$$

The maps we will be working with have a unique invariant measure, so we will often just say "the" invariant measure. The invariant measure of the Gauss map, for example, is given by the density $\rho(x) = \frac{1}{\ln 2}\frac{1}{1+x}$.

Definition: Complete Expanding Interval Map (CEIM)

A CEIM is a C^2 map $T : [0, 1] \to [0, 1]$ such that:

- $[0, 1]$ can be partitioned into countably many intervals I_j.
- Each $I_j \to [0, 1]$ has a C^2 inverse.
- T is eventually expanding, that is, there exist $C > 0$ and $\Delta > 1$ such that $|\partial T^n(x)| \geq C\Delta^n$.
- T has bounded nonlinearity.

Examples:

- The Gauss map: $x \to \frac{1}{x} \mod 1$
- Generalized Gauss maps: $x \to x^{-p} \mod 1$
- $x \to cx \mod 1$ $(c \geq 2, c \in \mathbb{N})$

The intervals I_j are called fundamental intervals.

Definition: Nonlinearity

The nonlinearity of a CEIM T is defined as

$$\sup_{I_j} \sup_{z,u \in I_j} \left| \frac{\partial^2 T(z)}{\partial T(z) \partial T(u)} \right|,$$

where the first supremum is over the set of fundamental intervals of T.

Note that bounded nonlinearity of T over the first level fundamental intervals implies bounded nonlinearity of T^n over the nth level fundamental intervals.

Existence of invariant density

Theorem (Existence of invariant density)

Let $T : [0, 1) \to [0, 1)$ be a CEIM. Then:
a) T must have a unique invariant probability measure m with density μ
b) There exists $C > 1$ such that for all $x \in [0, 1]$, $C^{-1} \leq \mu(x) \leq C$

The important thing to note here that will be needed later is that this invariant density is bounded above and below.

Example

The generalized Gauss maps $T_p : x \mapsto x^{-p}$ for $p > 1$ are complete interval maps and have bounded nonlinearity.

Proof idea:

- T_p is expanding, as we have $|\partial T(x)| > 1$ for all x
- Bounded nonlinearity follows from a short but tedious computation of the nonlinearity $\sup_{I_j} \sup_{z,u \in I_j} \left| \frac{\partial^2 T(z)}{\partial T(z) \partial T(u)} \right|$, where we make a "worst-case-scenario" estimate to bound.

The pushforward operator

Definition: The pushforward operator

The pushforward operator under T, denoted T_*, on test functions ϕ is defined as

$$(T_*\phi)(y) := \sum_{x \in T^{-1}(y)} \frac{\phi(x)}{|\partial T(x)|}$$

An important property of the pushforward is that for the invariant density μ of a map T, $T_*(\mu) = \mu$.

Mixing

Definition: Mixing

We say a map T is mixing with respect to a measure m if m is its invariant measure, and

$$\lim_{n\to\infty} |m(T^{-n}(B) \cap A) - m(A)m(B)| = 0$$

for any two measurable sets A and B.

Intuitively, this says that as we iterate the inverse image of T on a set B, the proportion of B lying in A is the same as the proportion of B in the entire domain, i.e. B becomes evenly mixed throughout the interval.

Alternative form of mixing

Let 1_A be a characteristic function of an interval of m-length m_A. Then $\lim_{n\to\infty} ||T_*^n[1_A\mu - m_A\mu]||_\infty = 0$ implies T is mixing with respect to m.

Though we omit it here, this can be proven via the duality property of the pushforward below and boundedness of μ.

$$\int (T_*\phi)(y) \cdot \psi(y)dy = \int \phi(y) \cdot \psi(T(y))dy$$

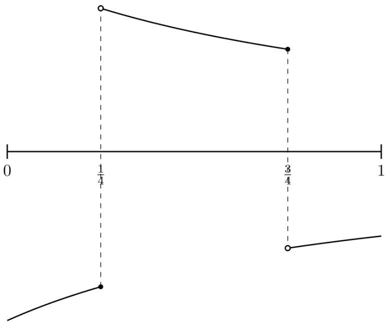

Figure 2: The function $1_A\mu - m_A\mu$, for $A = (0.25, 0.75)$ and $T = 1/x$ mod 1

$C(\alpha, J)$ functions

Definition ($C(\alpha, J)$ functions)

For $0 \leq \alpha \leq 1$ and $J \geq 0$, we define the class of functions $C(\alpha, J)$ to be functions $\phi : [0, 1] \to \mathbb{R}$, such that $\phi = j + h$ and
a) j is a step function with total variation bounded by J
b) The functions h are piecewise uniformly α-Hölder continuous (Lipshitz for $\alpha = 1$, equicontinuous for $\alpha = 0$).
c) $\int \phi dx = 0$

That is, there exists $K \geq 0$, such that $|h(x) - h(y)| < K|x - y|^{\alpha}$.
The following proposition, which utilizes this definition, is reminiscent of the Arzela-Ascoli theorem.

Proposition ("Arzelà-Ascoli-like")

Given a sequence f_n of $C(\alpha, J_n)$ functions such that $J_n \to 0$, there is a subsequence that is uniformly convergent to some continuous function g.

Theorem ($C(\alpha, J)$ goes to 0)

Let T be a CEIM. If a $\phi \in C(\alpha, J)$, then $\lim_{n\to\infty} ||T_*^n \phi||_\infty = 0$

Proof idea: Recall the invariant density μ of T is bounded by some constant, say K. Define $f_0 = \phi/\mu$, which is $C(0, KJ)$. We introduce a new operator based on the pushforward, $\tilde{T}_*\phi = \frac{1}{\mu} T_*\phi$. Writing the full sum, we can see this is a weighted average, in particular $\tilde{T}_*(1) = 1$:

$$(\tilde{T}_* 1)(y) = \frac{1}{\mu(y)} \sum_{x \in T^{-1}(y)} \frac{\mu(x)}{|\partial T(x)|} = 1$$

Defining $f_n = \tilde{T}_*^n(f_0)$, this tells us that $||f_{n+1}||_\infty \le ||f_n||_\infty$

Proof idea

- Now that $||f_{n+1}||_\infty \leq ||f_n||_\infty$, $\lim ||f_n||$ exists, so call this limit L and assume it is nonzero for the sake of contradiction.
- Write $f_n = h_n + j_n$, where h is continuous and j is a step function. The total variation J_n of j_n tends to 0, since in the pushforward operator, we divide by $|\partial T^n|$ at each preimage.
- h_n inherits the same modulus of continuity as h_0, so now f_n is $C(0, J_n)$, with $J_n \to 0$.
- By the earlier Arzela-Ascoli-like theorem, there is a subsequence of the f_n's uniformly convergent to some continuous function g.

Proof idea (continued)

- Since g is the limit of a subsequence of $\{f_n\}$, $\lim ||f_n||_\infty = L = ||g||_\infty$.
- Since g is continuous, and it has average 0, there is a point where it is 0 and arbitrarily small values around this point.
- The weighted average will eventually sum over some of these values, decreasing the sup norm.
- However, $||\tilde{T}_*^n f||_\infty$ becomes arbitrarily close to $||\tilde{T}_*^n g||_\infty$, so it also decreases below L
- This contradicts that L is the limit of $||f_n||_\infty$

Lemma (Lipshitz density)

The invariant density μ of T is Lipschitz.

Proof idea: We apply the previous theorem. Since $1-\mu \in C(0,0)$, we obtain $\lim_{n\to\infty} ||T_*^n[1-\mu]||_\infty = 0$, which gives us $T_*^n(1) \to \mu$ uniformly.
It can be shown that each function in this sequence is Lipschitz with the same Lipschitz constant by a direct computation with bounded nonlinearity, so the limit μ is Lipschitz with the same constant.

Mixing

Theorem (Mixing)

T is mixing with respect to its invariant density μ.

Proof Idea: Recall that 1_A is a characteristic function of an interval of m-length m_A. Since μ is Lipschitz, $1_A\mu - m_A\mu$ is $C(1, 2||\mu||)$, since there are two jumps contributed by 1_A. Then $\lim_{n\to\infty} ||T_*^n[1_A\mu - m_A\mu]||_\infty = 0$, which implies T is mixing with respect to μ.

Thank you!

References I

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