Laplace's equation: qualitative properties

Consider the Laplace's equation in a disk with radius a

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{1}$$

where $u = u(r,\theta)$ is the temperature, and (r,θ) are the polar coordinates $0 \le r \le a, -\pi \le \theta \le \pi$. Assume that the temperature is prescribed over the boundary and it is time independent

$$u(a,\theta) = f(\theta) \tag{2}$$

Using separation of variables, the general solution in the open disk $0 \le r < a, -\pi < \theta \le \pi$ is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta \tag{3}$$

where the coefficients are given by the formulae

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \tag{4}$$

$$A_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta, \;\; n \ge 1 \tag{5}$$

$$B_n = \frac{a^{-n}}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta, \;\; n \ge 1 \tag{6}$$

From (3) we notice that the temperature at the origin of the circle $r = 0$ is given by

$$u(0,\theta) = A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta \tag{7}$$

This property is called the *mean value property* of the Laplace's equation. Relation (7) may be used to deduce an important property of the Laplace's equation $\nabla^2 u = 0$ in an arbitrary bounded region Ω: *the maximum-minimum principle.*

The maximum-minimum principle: Let Ω be a bounded region with boundary S, and u the solution of the Laplace's equation $\nabla^2 u = 0$ in Ω. If M and m are respectively, the maximum and minimum values of $u(x)$ for x on S, then

$$m \le u(x) \le M \qquad \text{for all } x \text{ in } \Omega \cup S \tag{8}$$

More precisely,

$$\begin{aligned} &\textbf{either} \;\; m < u(x) < M \quad \text{for all } x \text{ in } \Omega \\ &\textbf{or else} \;\; m = u(x) = M \quad \text{for all } x \text{ in } \Omega \cup S \end{aligned}$$

The maximum-minimum principle may be used to show that the Laplace's equation

$$\nabla^2 u = 0 \quad \text{in } \Omega \tag{9}$$

with specified values on the boundary

$$u = f(x) \quad \text{on } S \tag{10}$$

is well posed. To show that there is an unique solution, notice that if u and $\hat{u}$ satisfy (9-10) then the difference $u - \hat{u}$ must satisfy the homogeneous problem, and therefore must be identically zero.

To show continuous dependence on data $f(x)$, consider the problem

$$\nabla^2 v = 0 \quad \text{in } \Omega \tag{11}$$

with specified values on the boundary

$$v = f(x) + \epsilon(x) \quad \text{on } S \tag{12}$$

where $\epsilon(x)$ is a small quantity for every x on S. The difference $w = v - u$ is then the solution of the problem

$$\nabla^2 w = 0 \quad \text{in } \Omega \tag{13}$$

with specified values on the boundary

$$w = \epsilon(x) \quad \text{on } S \tag{14}$$

The maximum-minimum principle tells us that we must have

$$\min_{x \in S} \epsilon(x) \leq w \leq \max_{x \in S} \epsilon(x) \tag{15}$$

Therefore, if ϵ approaches 0 for every x in S, then $w = v - u$ approaches zero for every x in Ω.

Solvability condition (Neumann boundary conditions)

Consider the Laplace's equation

$$\nabla^2 u = 0 \tag{16}$$

in a domain Ω with specified heat flow on the boundary S

$$-K_0 \nabla u \cdot n = \Phi \tag{17}$$

where K_0 is a positive constant. Integrating (16) in Ω we have

$$0 = \int_\Omega \nabla^2 u = \int_\Omega \nabla \cdot (\nabla u) dV \tag{18}$$

Using the *divergence theorem* we have

$$\int_\Omega \nabla \cdot (\nabla u) dV = \int_S \nabla u \cdot n ds \tag{19}$$

Therefore, we must have

$$\int_S \nabla u \cdot n ds = 0 \tag{20}$$

which implies

$$\int_S \Phi dS = 0 \tag{21}$$

Equation (21) (or (20)) is called *the solvability condition* or *compatibility condition* for Laplace's equation with specified flow on the boundary.

Q: Show that the solution to (16-17) is unique up to an additive constant.

Example: For what value of the constant α there is a solution to the problem

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad 0 < x < 1, \quad 0 < y < 1 \\ u_x(0, y) &= \sin(\pi y), \quad 0 \le y \le 1 \\ u_x(1, y) &= \sin(\pi y), \quad 0 \le y \le 1 \\ u_y(x, 0) &= \alpha x(1 - x), \quad 0 \le x \le 1 \\ u_y(x, 1) &= \sin(\pi x), \quad 0 \le x \le 1 \end{aligned}$$

Chapter 3: Fourier Series

The Fourier series of a function $f(x)$ over the interval $-L \le x \le L$ is defined to be the infinite series

$$\text{Fourier series} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{22}$$

where the coefficients are given by the formulae

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \tag{23}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \tag{24}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \tag{25}$$

There are a couple of questions to ask

- does the Fourier series exists for any function f ?
- if the Fourier series (22) converges, does it converge to $f(x)$?

The answer to the first question is :"No". For example, notice that the coefficient a_0 exists only if

$$|\int_{-L}^{L} f(x)dx| < \infty \tag{26}$$

An example of a function that does not satisfy (26) is $f(x) = 1/x^2$.

To answer the second question, first we need to recall some mathematical concepts.

A function f defined on the interval I is continuous at the point $x_0 \in I$ if the limit $\lim_{x\to x_0} f(x)$ exists, is finite, and is equal to $f(x_0)$.
We say that f has a jump discontinuity at x_0 if the left and respectively, right limits $f(x_0^-)$, $f(x_0^+)$ both exist and $f(x_0^-) \ne f(x_0^+)$.

A function f is *piecewise smooth* in an interval I of finite length if we can find a finite partition $I_1, I_2, \dots I_k$ of I, $I_1 \cup I_2 \cup \dots \cup I_k = I$ such that $f(x)$ and its derivative df/dx are continuous in each subinterval $I_i, i = 1, 2, \dots, k$. A function f is *piecewise smooth* in $(-\infty, \infty)$ if is piecewise smooth in any interval of finite length.

Notice that a piecewise smooth function in I is not necessary continuous at every point in I, but it is only allowed to have a finite number of jump discontinuities.

An example of the function that is not piecewise smooth on any interval $I = (-a, a)$ is $f(x) = x^{1/3}$ since the derivative $df/dx = 1/3x^{-2/3}$ is ∞ at $x = 0$.

A function f is *periodic* with the period $T > 0$ if $f(x + T) = f(x)$ for all x.

The Fourier series (22) is periodic with period $2L$.

If f is a function defined in $-L \leq x \leq L$, then the *periodic extension* of f is a function $\hat{f}$ such that $\hat{f}(x) = f(x), -L \leq x < L$, and $\hat{f}(x + 2L) = \hat{f}(x)$, for all x. Notice that the periodic extension of f is continuous if and only if f continuous and periodic: $f(-L) = f(L)$.

The following property holds:

Convergence theorem for Fourier series

If $f(x)$ is piecewise smooth on the interval $-L \leq x \leq L$, then the Fourier series (22) of $f(x)$ converges

1. to the periodic extension of $f(x)$, where the periodic extension is continuous
2. to the average of the left- and right-hand limits
$$\frac{1}{2}[f(x+) + f(x-)]$$
at the points x where the periodic extension has a jump discontinuity.

Mathematically, for $-L < x < L$ we have

$$\frac{f(x+) + f(x-)}{2} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{27}$$

If f is continuous at $x, -L < x < L$, then $f(x+) = f(x-)$ and (27) becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{28}$$

At the end points $x = L$ or $x = -L$ the infinite series converges to the average of the two values of the periodic extension.

Fourier sine series

Consider a function $f(x)$ over the interval $0 \leq x \leq L$. The series (29) with the coefficients (30) is called the *Fourier sine series of* $f(x)$ on the interval $0 \leq x \leq L$.

$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} dx \tag{29}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{30}$$

Example of Fourier sine series

The Fourier sine series of $f(x) = x, 0 \leq x \leq L$ is

$$x = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \quad 0 < x < L \tag{31}$$

where the Fourier coefficients are

$$B_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} = \frac{2L}{n\pi}(-1)^{n+1}$$

Fourier cosine series

Consider a function $f(x)$ over the interval $0 \leq x \leq L$. The series (32) with the coefficients (33) is called the *Fourier cosine series of* $f(x)$ on the interval $0 \leq x \leq L$.

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L} dx \tag{32}$$

where

$$a_0 = \frac{1}{L} \int_0^L f(x)dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \ n = 1, 2, \ldots \tag{33}$$

Continuous Fourier Series

The following properties hold:

1. For piecewise smooth functions $f(x)$, the *Fourier series* of $f(x)$ is continuous for $-L \leq x \leq L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$.
2. For piecewise smooth functions $f(x)$, the *Fourier sine series* of $f(x)$ is continuous for $0 \leq x \leq L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ and $f(L) = 0$.
3. For piecewise smooth functions $f(x)$, the *Fourier cosine series* of $f(x)$ is continuous for $0 \leq x \leq L$ if and only if $f(x)$ is continuous.

Term by term differentiation of Fourier series

The interchange of operations of differentiation and infinite summation is not always justified such that term by term differentiation is not always valid. It is not always true that

$$\frac{d}{dx} \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n \frac{du_n}{dx}$$

Counterexample: Consider the Fourier sine series of $f(x) = x, 0 \leq x \leq L$

$$x = 2 \sum_{n=1}^{\infty} \frac{L}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{L}, \quad 0 \leq x < L \tag{34}$$

Term by term differentiation leads to

$$1 = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos\frac{n\pi x}{L} \tag{35}$$

which is not true! (notice that the right hand side series is not even convergent since the n^{th} term does not approach zero).

For Fourier series the following properties hold:

1. *A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth.*
2. If $f'(x)$ is piecewise smooth then the *Fourier sine series* of a continuous function $f(x)$ can be differentiated term by term only if $f(0) = 0$ and $f(L) = 0$.
3. If $f'(x)$ is piecewise smooth then the *Fourier cosine series* of a continuous function $f(x)$ can be differentiated term by term.

In this case it follows that if

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}, \quad 0 \le x \le L \tag{36}$$

then

$$f'(x) \sim -\sum_{n=0}^{\infty}\left(\frac{n\pi}{L}\right) A_n \sin\frac{n\pi x}{L}, \quad 0 \le x \le L \tag{37}$$

Q: Why we used $\sim$ in (37) which means "$f'(x)$ has the Fourier sine series ..." and not "="?
Term by term differentiation with respect to the parameters

Consider a continuous function $u(x,t)$. We may view $u(x,t)$ as a function of x depending on the parameter t. The Fourier series (in x) of $u(x,t)$ is written

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty}\left[a_n(t)\cos\frac{n\pi x}{L} + b_n(t)\sin\frac{n\pi x}{L}\right] \tag{38}$$

with coefficients depending on t. If $u_t(x,t)$ is piecewise smooth, then the series (38) can be differentiated term by term with respect to the parameter t to obtain

$$u_t(x,t) = a_0'(t) + \sum_{n=1}^{\infty}\left[a_n'(t)\cos\frac{n\pi x}{L} + b_n'(t)\sin\frac{n\pi x}{L}\right] \tag{39}$$

To derive the wave equation we consider vertical vibrations of a string with fixed ends. Let $u(x,t)$ represent the vertical displacement of the string from the equilibrium position and assume that the horizontal displacement can be neglected.

If $\rho(x)$ denotes the density of the string, then the total mass of a small portion of the string from x to $x+\Delta x$ is approximately

$$m \approx \rho(x)\Delta x \tag{40}$$

The displacement of the string is determined by the action of various forces such as

1. body forces, e.g., gravity $m\vec{g}$
2. tangential (tension) forces $T(x,t)$
3. friction

Let $\theta(x,t)$ denote the angle of the string with the horizontal at location x and time t. The vertical component of the tension force is expressed as

$$\text{Tension vertical component} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) \tag{41}$$

The equation of the vertical motion is determined by Newton's law of motion $\vec{F} = ma$. We have (neglecting friction)

$$\rho(x)\Delta x u_{tt}(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) - (\rho(x)\Delta x)g \tag{42}$$

Letting $\Delta x \to 0$ in the equation above, we get

$$\rho(x)u_{tt}(x,t) = [T(x,t)\sin\theta(x,t)]_x - \rho(x)g \tag{43}$$

The relationship between the angle $\theta(x,t)$ and the displacement $u(x,t)$ is

$$u_x(x,t) = \tan\theta(x,t) \tag{44}$$

For small vibrations the angle θ is small and we may approximate

$$\sin\theta \approx \tan\theta = u_x \tag{45}$$

in (43) to obtain

$$\rho u_{tt} = [Tu_x]_x - \rho g \tag{46}$$

If we assume that the density and tension are constant and neglect gravitation, we obtain the standard form of the *one dimensional wave equation*

$$u_{tt} = c^2 u_{xx} \tag{47}$$

where $c^2 = T/\rho$. Equation (47) is a second order linear hyperbolic PDE.

Remark: If friction is considered, then we obtain the damped 1-D wave equation

$$u_{tt} = c^2 u_{xx} - ku_t \tag{48}$$

As initial conditions to the PDE (47) we must specify the initial configuration (position) of the string

$$u(x,0) = f(x) \tag{49}$$

and the initial velocity

$$u_t(x,0) = g(x) \tag{50}$$

In addition boundary conditions must be specified. We assume that the string has length L and that both ends of the string are fixed, such that there is no displacement at $x = 0, x = L$. This corresponds to the boundary conditions

$$u(0,t) = 0 \tag{51}$$

$$u(L,t) = 0 \tag{52}$$

Q: Show that the problem (47), (49-50), (51-52) has at most one solution.

We have to solve a linear and homogeneous problem and attempt to find a solution using the method of separation of variables. We seek for product solutions of the form

$$u(x,t) = \Phi(x)h(t) \tag{53}$$

After replacing (53) in (47) the variables may be separated to obtain

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2h}{dt^2} = \frac{1}{\Phi}\frac{d^2\Phi}{dx^2} = -\lambda \tag{54}$$

The differential equation for h is

$$\frac{d^2h}{dt^2} = -\lambda c^2 h \tag{55}$$

whereas for Φ we obtain from (54) and (51), (52) the boundary value problem

$$\frac{d^2\Phi}{dx^2} = -\lambda\Phi \tag{56}$$

$$\Phi(0) = 0 \tag{57}$$

$$\Phi(L) = 0 \tag{58}$$

We already know that for this problem the eigenvalues are all positive,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \;\; n = 1, 2, \ldots \tag{59}$$

and the corresponding eigenfunctions are

$$\Phi(x) = \sin\frac{n\pi x}{L} \tag{60}$$

Since $\lambda > 0$, the general solution of the equation (55) is

$$h(t) = c_1 \cos c\sqrt{\lambda}t + c_2 \sin c\sqrt{\lambda}t \tag{61}$$

Therefore, we obtain product solutions,

$$u_n(x,t) = \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right), \quad n = 1, 2, \ldots \tag{62}$$

which are also known as *standing waves*. To solve the initial value problem, we use the principle of superposition and consider the solution of the general form

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right) \tag{63}$$

From (49) we obtain

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \tag{64}$$

whereas from (50) we obtain

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L} \tag{65}$$

Using the orthogonality properties of $\sin \frac{n\pi x}{L}$ in $[0, L]$, we determine the coefficients as

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \tag{66}$$

$$B_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx \Rightarrow B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} dx \tag{67}$$

The solution of the problem (47-50) is therefore given by (63) where the coefficients are evaluated according to (66), (67).

Remark: The n^{th} term of the solution (63) is given by (62) and is called the n^{th} *mode of vibration* or the n^{th} *harmonic*. The period of the n^{th} mode is $2L/(nc)$ and its frequency (rad/sec) is $\omega_n = (n\pi c)/L$. For $n = 1$ we obtain *the fundamental* frequency $\omega_1 = \pi c/L$ and notice that $\omega_n = n\omega_1$: all frequencies are a multiple of the fundamental frequency.

Boundary conditions associated to the wave equation

For simplicity, in the discussion above we considered homogeneous boundary conditions (51-52). In practice various types of boundary conditions may be specified as follows:

1. Controlled end points (first kind, Dirichlet)

$$u(0, t) = g_1(t), \quad u(L, t) = g_2(t) \tag{68}$$

2. Force specified on the boundaries (second kind, Neumann)

$$u_x(0, t) = g_1(t), \quad u_x(L, t) = g_2(t) \tag{69}$$

3. Ellastic attachment on the boundaries (third kind, Robin)

$$u_x(0, t) - \alpha u(0, t) = g_1(t), \quad u_x(L, t) - \beta u(L, t) = g_2(t) \tag{70}$$

Q: Show that the wave equation (47) with initial conditions (49-50) and boundary conditions (69) has at most one solution. Then solve using separation of variables.

Q: Show that the wave equation (47) with initial conditions (49-50) and boundary conditions (70) has at most one solution. Then solve using separation of variables.