Heat conduction in a thin circular ring

We consider a thin wire (with lateral sides insulated) of length $2L$ which is bent into the shape of a circle. The model problem is in this case

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -L < x < L \tag{1}$$

with the boundary conditions

$$u(-L,t) = u(L,t), \quad t > 0 \tag{2}$$

$$\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t), \quad t > 0 \tag{3}$$

and the initial condition

$$u(x,0) = f(x), \quad -L \leq x \leq L \tag{4}$$

We apply the method of separation of variables and seek for product solutions

$$u(x,t) = \Phi(x)G(t) \tag{5}$$

Using similar arguments as in the previous case, we obtain

$$G(t) = ce^{-\lambda kt} \tag{6}$$

and the boundary value problem for Φ

$$\frac{d^2\Phi}{dx^2} = -\lambda\Phi \tag{7}$$

$$\Phi(-L) = \Phi(L) \tag{8}$$

$$\frac{d\Phi}{dx}(-L) = \frac{d\Phi}{dx}(L) \tag{9}$$

The boundary conditions (8) and (9) are referred to as *periodic boundary conditions.*

Q: Show that there are no eigenvalues $\lambda < 0$ for the problem (7), (8), (9).

We must consider next two cases: $\lambda > 0$ and $\lambda = 0$.

If $\lambda > 0$, the general solution of equation (7) is again

$$\Phi = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \tag{10}$$

The boundary condition (8) implies that

$$c_2 \sin\sqrt{\lambda}L = 0 \tag{11}$$

and the boundary condition (9) implies that

$$c_1\sqrt{\lambda}\sin\sqrt{\lambda}L = 0 \tag{12}$$

Since we are looking for nontrivial solutions, we must have

$$\sin\sqrt{\lambda}L = 0 \tag{13}$$

such that we obtain the eigenvalues

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots \tag{14}$$

Notice that there are no restrictions on the constants c_1 and c_2. In this case, corresponding to each eigenvalue λ we have two eigenfunctions

$$\Phi(x) = \cos\frac{n\pi x}{L}, \ \Phi(x) = \sin\frac{n\pi x}{L}, \ \ n = 1, 2, 3, \ldots \tag{15}$$

and any linear combination

$$\Phi(x) = c_1 \cos\frac{n\pi x}{L} + c_2 \sin\frac{n\pi x}{L} \tag{16}$$

is also an eigenfunction for (7) subject to (8), (9).
Therefore, for $\lambda > 0$ we obtain two infinite families of product solutions of the PDE (1) subject to the B.C. (2) and (3)

$$u(x,t) = \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \quad \text{and} \quad u(x,t) = \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt}, \ \ n = 1, 2, 3, \ldots \tag{17}$$

If $\lambda = 0$, the general solution of (7) is

$$\Phi(x) = c_1 + c_2 x \tag{18}$$

and using the boundary conditions (8) and (9) we obtain

$$\Phi(x) = c_1 \tag{19}$$

Applying the principle of superposition, we obtain that the general solution of the problem (1), (2), (3) is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-(n\pi/L)^2 kt} \tag{20}$$

To satisfy the initial condition (4) we must have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \ \ -L \le x \le L \tag{21}$$

To determine the coefficients a_0, $a_i, b_i, i = 1, 2, \ldots$ first

Q: Prove that for any nonnegative integers n, m the following orthogonality relations are true:

$$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = \begin{cases} 0 & n \ne m \\ L & n = m \ne 0 \\ 2L & n = m = 0 \end{cases} \tag{22}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx = \begin{cases} 0 & n \ne m \\ L & n = m \ne 0 \end{cases} \tag{23}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L} dx = 0 \tag{24}$$

Q: Using (21) and the properties (22), (23), (24) show that the coefficients a_0, $a_i, b_i, i = 1, 2, \ldots$ are given by the formulae

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \tag{25}$$

$$a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{m\pi x}{L} dx, \ \ m = 1, 2, \ldots \tag{26}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx, \;\; m = 1, 2, \ldots \tag{27}$$

The solution of the problem (1-4) if therefore given by (20) where the coefficients are evaluated according to (25), (26), (27).

Laplace's equation: solutions and qualitative properties

Laplace's equation in a rectangular domain

We want to find the equilibrium temperature inside a rectangular region $0 \le x \le L, 0 \le y \le H$ when the temperature on the boundary is a prescribed function of the location only (time independent).

The problem to solve is the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{28}$$

with the boundary conditions

$$u(0, y) = g_1(y) \tag{29}$$

$$u(L, y) = g_2(y) \tag{30}$$

$$u(x, 0) = f_1(x) \tag{31}$$

$$u(x, H) = f_2(x) \tag{32}$$

Notice that for the problem above the boundary conditions are nonhomogeneous, so the method of separation of variables may not be applied directly in this case. The trick is to break problem (28-32) into four problems, each having only one nonhomogeneous boundary condition. We write

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y) \tag{33}$$

where each of $u_i, i = 1, 2, 3, 4$ satisfies the Laplace's equation (28), one nonhomogeneous boundary condition, and three homogeneous boundary conditions as shown in the table below

$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0$	$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$	$\frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} = 0$	$\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0$
$u_1(0, y) = 0$	$u_2(0, y) = 0$	$u_3(0, y) = 0$	$u_4(0, y) = g_1(y)$
$u_1(L, y) = 0$	$u_2(L, y) = g_2(y)$	$u_3(L, y) = 0$	$u_4(L, y) = 0$
$u_1(x, 0) = f_1(x)$	$u_2(x, 0) = 0$	$u_3(x, 0) = 0$	$u_4(x, 0) = 0$
$u_1(x, H) = 0$	$u_2(x, H) = 0$	$u_3(x, H) = f_2(x)$	$u_4(x, H) = 0$

Next we show that each of these problems may be solved using the separation of variables. Let's focus on solving the problem for u_4. You may do similar computations for u_1, u_2, and u_3.

We want to solve the problem

$$\frac{\partial^2 u_4}{\partial x^2} + \frac{\partial^2 u_4}{\partial y^2} = 0 \tag{34}$$

with the boundary conditions

$$u_4(0,y) = g_1(y) \tag{35}$$

$$u_4(L,y) = 0 \tag{36}$$

$$u_4(x,0) = 0 \tag{37}$$

$$u_4(x,H) = 0 \tag{38}$$

using the method of separation of variables. We seek for a solution of the form

$$u_4(x,y) = h(x)\Phi(y) \tag{39}$$

When we replace (39) in (34) we obtain

$$\Phi(y)\frac{d^2h}{dx^2} + h(x)\frac{d^2\Phi}{dy^2} = 0 \tag{40}$$

which may be written

$$\underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{\text{function of } x \text{ only}} = \underbrace{-\frac{1}{\Phi}\frac{d^2\Phi}{dy^2}}_{\text{function of } y \text{ only}} = \lambda \text{ (constant)} \tag{41}$$

Therefore, we obtained two differential equations

$$\frac{d^2h}{dx^2} = \lambda h \tag{42}$$

$$\frac{d^2\Phi}{dy^2} = -\lambda\Phi \tag{43}$$

and from the three homogeneous boundary conditions (36-38) we obtain

$$h(L) = 0 \tag{44}$$

$$\Phi(0) = 0 \tag{45}$$

$$\Phi(H) = 0 \tag{46}$$

We focus first on the boundary value problem (43), (45), (46) to determine the eigenvalues λ.

Three cases must be taken into consideration: $\lambda > 0, \lambda = 0, \lambda < 0$. We already now (see previous lectures) that for problem (43), (45), (46) the eigenvalues are all positive

$$\lambda = \left(\frac{n\pi}{H}\right)^2, \ \ n = 1, 2, 3, \ldots \tag{47}$$

and the associated eigenfunctions are

$$\Phi(y) = \sin\frac{n\pi y}{H} \tag{48}$$

Next we replace (47) in equation (42) to obtain

$$\frac{d^2h}{dx^2} = \left(\frac{n\pi}{H}\right)^2 h \tag{49}$$

The general solution of equation (49) is

$$h(x) = c_1 e^{\frac{n\pi x}{H}} + c_2 e^{-\frac{n\pi x}{H}} \tag{50}$$

which can be written in the equivalent form

$$h(x) = a_1 \cosh\frac{n\pi}{H}(x-L) + a_2 \sinh\frac{n\pi}{H}(x-L) \tag{51}$$

Q: Show that (50) and (51) are equivalent. (Hint: See page 74 in the book for details.)

Now, from (51) it is easy to see that the condition (44) implies $a_1 = 0$ (remember that $\cosh 0 = 1, \sinh 0 = 0$). Therefore,

$$h(x) = a_2 \sinh \frac{n\pi}{H}(x - L) \tag{52}$$

Replacing (52) and (48) in (5), we obtain the product solutions of the form

$$u_4(x, y) = A \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x - L) \tag{53}$$

Using the principle of superposition, we have that the general solution of the problem (34), (36), (37), (38) is

$$u_4(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(x - L) \tag{54}$$

To satisfy the nonhomogeneous boundary condition (35) we require that

$$g_1(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sinh \frac{n\pi}{H}(-L) \tag{55}$$

Q: Using (55) and the orthogonality of the functions $\sin \frac{n\pi y}{H}, 0 \leq y \leq H$, show that

$$A_n \sinh \frac{n\pi}{H}(-L) = \frac{2}{H} \int_0^H g_1(y) \sin \frac{n\pi y}{H} dy \tag{56}$$

and the coefficients A_n are given by the formula

$$A_n = \frac{2}{H \sinh \frac{n\pi(-L)}{H}} \int_0^H g_1(y) \sin \frac{n\pi y}{H} dy \tag{57}$$

Therefore, the solution $u_4(x, y)$ of the problem (34-38) is (54) where the coefficients are evaluated according to (57). Notice that to obtain the solution $u(x, y)$, we must repeat these computations for u_1, u_2, and u_3, then use (33).

Laplace's equation for a circular disk

We consider the Laplace's equation in a disk with radius a

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{58}$$

where $u = u(r, \theta)$ is the temperature, and (r, θ) are the polar coordinates $0 \leq r \leq a, -\pi \leq \theta \leq \pi$. We assume that the temperature is prescribed over the boundary and is time independent

$$u(a, \theta) = f(\theta) \tag{59}$$

In addition, we assume *boundedness at origin*

$$|u(0, \theta)| < \infty \tag{60}$$

and impose the *periodicity conditions*

$$u(r, -\pi) = u(r, \pi) \tag{61}$$

$$\frac{\partial u}{\partial \theta}(r,-\pi) = \frac{\partial u}{\partial \theta}(r,\pi) \tag{62}$$

We use the separation of variables technique and search for solutions of the product form

$$u(r,\theta) = \Phi(\theta)G(r) \tag{63}$$

Replacing (63) in (58) we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\Phi(\theta) + \frac{1}{r^2}G(r)\frac{d^2\Phi}{d\theta^2} = 0 \tag{64}$$

such that the variables may by separated to obtain

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\theta^2} = \lambda \tag{65}$$

From (65) and the periodicity conditions (61), (62) it results that Φ must satisfy the boundary value problem

$$\frac{d^2\Phi}{d\theta^2} = -\lambda\Phi \tag{66}$$

$$\Phi(-\pi) = \Phi(\pi) \tag{67}$$

$$\frac{d\Phi}{d\theta}(-\pi) = \frac{d\Phi}{d\theta}(\pi) \tag{68}$$

Q: Show that for the problem (66)-(68), the eigenvalues are $\lambda = n^2$ and to each eigenvalue correspond two eigenfunctions:

$$\Phi(\theta) = \sin n\theta \quad \text{and} \quad \Phi(\theta) = \cos n\theta \tag{69}$$

When $n = 0$, the eigenfunction is a constant.

From (65) the differential equation for G is

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \lambda = n^2 \tag{70}$$

which may be written as

$$r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2G = 0 \tag{71}$$

The condition (60) implies that

$$|G(0)| < \infty \tag{72}$$

Equation (71) is a second order, linear and homogeneous differential equation with variable coefficients. It also has the particular property that it remains invariant under the scale change $r \to cr$. For this reason this type of equations are often called *equidimensional* and may be solved by directly substituting the trial function $G = r^p$ into the differential equation.

We obtain

$$[p(p-1) + p - n^2]r^p = 0 \tag{73}$$

such that we obtain $p = \pm n$. If $n > 0$, the general solution of (71) is

$$G(r) = c_1 r^n + c_2 r^{-n} \tag{74}$$

whereas for $n = 0$ the general solution of (71) is

$$G(r) = \overline{c}_1 + \overline{c}_2 \ln r \tag{75}$$

Q: Show that condition (72) implies $c_2 = \overline{c}_2 = 0$.

Therefore, we have the solution

$$G(r) = c_1 r^n, n \geq 0 \tag{76}$$

Replacing (76) and (69) in (63) we obtain the product solutions of the homogeneous problem (58), (60), (61), (62) as

$$u(t,\theta) = r^n \cos n\theta, n \geq 0 \qquad \text{and} \qquad u(t,\theta) = r^n \sin n\theta, n \geq 1 \tag{77}$$

Using the principle of superposition, the general solution in the open disk $0 \leq r < a, -\pi < \theta \leq \pi$ is

$$u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta \tag{78}$$

To satisfy the nonhomogeneous boundary condition (59) we require that

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \cos n\theta + \sum_{n=1}^{\infty} B_n a^n \sin n\theta, \;\; -\pi < \theta \leq \pi \tag{79}$$

Q: Use the orthogonality properties of sin and cos (see also Lecture 4) to show that the coefficients are given by the formulae

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta \tag{80}$$

$$A_n = \frac{a^{-n}}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta d\theta, \;\; n \geq 1 \tag{81}$$

$$B_n = \frac{a^{-n}}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta d\theta, \;\; n \geq 1 \tag{82}$$

In conclusion, the steady-state temperature distribution inside the circle is given by (78) where the coefficients are evaluated according to (80-82).