Mth 251: Calculus I

Midterm exam: Solutions

PRINT your name: A STUDENT

The exam has 6 problems and each problem is worth 10 points.

In order to receive full credit you must show all work.

Good luck!

Problem 1. Evaluate the limit

$$\lim_{x\to 2} \frac{x^2-3x+2}{2x^2-8}$$

Solution Notice that we have an indeterminate limit case of $\frac{0}{0}$.

With a few algebraic manipulations, we may simplify this limit as follows

$$\frac{x^2-3x+2}{2x^2-8} = \frac{(x-2)(x-1)}{2(x^2-4)} = \frac{(x-2)(x-1)}{2(x-2)(x+2)} = \frac{x-1}{2(x+2)}$$

Thus,

$$\lim_{x\to 2} \frac{x^2-3x+2}{2x^2-8} = \lim_{x\to 2} \frac{x-1}{2(x+2)} = \frac{2-1}{2(2+2)} = \frac{1}{8}$$

Problem 2. Evaluate the limit

$$\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}$$

Solution Notice that we have an indeterminate limit case of $\frac{0}{0}$, since $\cos 0 = 1$ and $\sin 0 = 0$

We may use the identity

$$\sin^2 x + \cos^2 x = 1$$

to obtain

$$\sin^2 x = 1 - \cos^2 x$$

and write the limit as

$$\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x} = \lim_{x\to 0}\frac{1-\cos x}{1-\cos^2 x} = \lim_{x\to 0}\frac{1-\cos x}{(1-\cos x)(1+\cos x)} = \lim_{x\to 0}\frac{1}{1+\cos x} = \frac{1}{2}$$

Problem 3. Evaluate the limit

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x+2}\right)$$

Solution Notice that we have an indeterminate limit case $\infty-\infty$.

We can resolve it with the help of the conjugate expression, as follows

$$\begin{aligned}\left(\sqrt{x+1}-\sqrt{x+2}\right) &= \frac{\left(\sqrt{x+1}-\sqrt{x+2}\right)\left(\sqrt{x+1}+\sqrt{x+2}\right)}{\left(\sqrt{x+1}+\sqrt{x+2}\right)} \\ &= \frac{(x+1)-(x+2)}{\left(\sqrt{x+1}+\sqrt{x+2}\right)}=\frac{-1}{\left(\sqrt{x+1}+\sqrt{x+2}\right)}\end{aligned}$$

Thus,

$$\lim_{x\to\infty}\left(\sqrt{x+1}-\sqrt{x+2}\right)=\lim_{x\to\infty}\frac{-1}{\sqrt{x+1}+\sqrt{x+2}}=0$$

since the denominator increases without bound to ∞ as $x\to\infty$.

Problem 4. Use the Intermediate Value Theorem to find an interval of length 1 containing a root of the equation

$$2^x - x^3 = 0$$

Solution Since

$$f(x) = 2^x - x^3$$

is a continuous function, to apply the IVT all we need is to find an interval (a, b) such that $b - a = 1$ (that is of length 1) and

$$f(a) \cdot f(b) < 0$$

Notice that

$$f(1) = 2 - 1 = 1 > 0, \qquad f(2) = 2^2 - 2^3 = 4 - 8 = -4 < 0$$

Thus (1, 2) is an interval of length 1 containing a root (solution) of the equation

Problem 5. Given the function

$$f(x) = x^2 - \sqrt{x} + 2$$

- find the value of the derivative $f'(1)$
- write the equation of the tangent line at $a = 1$.

Solution Notice that $\sqrt{x} = x^{\frac{1}{2}}$. Use the power rule to obtain the derivative function

$$f'(x) = 2x - \frac{1}{2\sqrt{x}}$$

Thus, at $x = 1$,

$$f'(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

The equation of the tangent line at $a = 1$ is

$$y = f(1) + f'(1) \cdot (x - 1)$$

Notice that $f(1) = 1 - 1 + 2 = 2$.

Therefore, the equation of the tangent line at $a = 1$ is

$$y = 2 + \frac{3}{2} \cdot (x - 1)$$

which may be also written as

$$y = \frac{3}{2}x + \frac{1}{2}$$

Problem 6. Given the function

$$f(x) = \begin{cases} x+2, & \text{for } x<1 \\ ax^2+b, & \text{for } x \geq 1 \end{cases}$$

find the constants a and b such that the function is continuous *and* differentiable at $x = 1$.

Solution Notice that

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-}(x+2) = 3, \quad f(1) = \lim_{x\to 1+} f(x) = \lim_{x\to 1+}(ax^2+b) = a+b$$

Therefore, the function is *continuous* at $x = 1$ if we require that

$$a+b=3$$

Notice that the derivative at $x = 1$ is defined as

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

We have

$$f'(1-) = \lim_{x\to 1-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1-} \frac{x+2-(a+b)}{x-1} \overset{use\ a+b=3}{=} \lim_{x\to 1-} \frac{x-1}{x-1} = 1$$

and

$$\begin{aligned} f'(1+) &= \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1} \\ &= \lim_{x\to 1+} \frac{ax^2+b-(a+b)}{x-1} = \lim_{x\to 1+} \frac{a(x^2-1)}{x-1} = \lim_{x\to 1+} \frac{a(x-1)(x+1)}{x-1} = 2a \end{aligned}$$

For the function to be *differentiable* at $x = 1$ we require that $f'(1-) = f'(1+)$,

$$2a = 1 \quad thus \quad a = \frac{1}{2}$$

Then, since $a+b=3$ we get the value of b as

$$b = 3 - \frac{1}{2} = \frac{5}{2}$$