Generalized Production Functions[1]

1. INTRODUCTION

In this paper generalized production functions (GPF's) are introduced with the generalization referring to what is assumed about not only the elasticity of substitution but also the behaviour of the returns to scale. That is, given a neo-classical production function with a given elasticity of substitution (constant or variable), we show below how this function can be transformed to yield a neo-classical GPF with the same elasticity of substitution and with the returns to scale variable and *satisfying a preassigned relationship to the output level.* Thus, for example, a GPF may have a constant elasticity of substitution and returns to scale which diminish in a preassigned fashion with the level of output. This contrasts with many current production functions used in applied studies which assume the same returns to scale at all levels of output. Since returns to scale may indeed be different at different scales of operation, GPF's may be useful in analyzing data relating to production. Also, obviously, average cost curves associated with GPF's can show decreasing costs at low levels of output and increasing costs at high levels of output.

The plan of the paper is as follows. In section 2 we show how GPF's can be generated and establish some of their salient properties. In section 3, we briefly consider implications of GPF's for the behaviour of labour's share. Estimates of the parameters of a GPF are provided in section 4. Finally, in section 5, we present some concluding remarks.

2. PROPERTIES AND EXAMPLES OF GENERALIZED PRODUCTION FUNCTIONS

For simplicity we shall carry forward the discussion in terms of production functions involving two inputs, labour, L, and capital, K.[2] Consider any neo-classical production function $f(L, K)$, of an arbitrary degree of homogeneity.[3] Then we consider[4]

$$V = g(f) \qquad \ldots(2.1)$$

where V denotes the real output rate and where the transformation function g satisfies $g(0) = 0$ and $\frac{dg}{df} > 0$ for $0 \leq f < \infty$.

First, we note that for $0 \leq f < \infty$

$$\frac{\partial V}{\partial L} = \frac{dg}{df}\frac{\partial f}{\partial L} > 0 \qquad \ldots(2.2a)$$

and

$$\frac{\partial V}{\partial K} = \frac{dg}{df}\frac{\partial f}{\partial K} > 0 \qquad \ldots(2.2b)$$

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2 Generalization to cases involving more inputs is direct.

3 It will become clear later in this section that for our purposes, we do require that $f(L, K)$ be homogeneous of some degree.

4 The function in (2.1) is a homothetic production function provided $f(L, K)$ is such a function, which latter we assume to be the case—cf. Shepard [7]. See also Clemhout [3].

since, by assumption $\frac{dg}{df} > 0, \frac{\partial f}{\partial L} > 0$ and $\frac{\partial f}{\partial K} > 0$. Thus the marginal physical products associated with the function in (2.1) are positive and it is easily verified that $V = g(f)$ satisfies other conditions required of a neo-classical production function.

Second, we note the following simple lemma : [1]

Lemma. *The elasticity of substitution associated with the GPF, $V = g(f)$, is the same as that associated with the function $f(L, K)$.*

Proof. The reciprocal of the elasticity of substitution, $\sigma(0 < \sigma < \infty)$, is defined as follows:

$$\frac{1}{\sigma} = \frac{x}{r}\frac{dr}{dx} \qquad \ldots(2.3)$$

where $X = K/L$ and $r = \frac{\partial g}{\partial L}\Big/\frac{\partial g}{\partial K}$, the marginal rate of substitution. Note

$$r = \left(\frac{dg}{df}\frac{\partial f}{\partial L}\right)\Big/\left(\frac{dg}{df}\frac{\partial f}{\partial K}\right) = \frac{\partial f}{\partial L}\Big/\frac{\partial f}{\partial K} \qquad \ldots(2.4)$$

which is the marginal rate of substitution for the function $f(L, K)$. Since r is the same for f and g, $1/\sigma$, the elasticity of the marginal rate of substitution with respect to the capital-labour ratio, will be the same for the two functions which proves the Lemma.

Third, we establish the following theorem:

Theorem. *Let $f(L, K)$ be a neo-classical production function homogeneous of degree α_f, a constant. Then a production function, $V = g(f)$, with preassigned returns to scale function $\alpha(V)$, can be obtained by solving the following differential equation:*

$$\frac{dV}{df} = \frac{V}{f}\frac{\alpha(V)}{\alpha_f}. \qquad \ldots(2.5)$$

Before we proceed to prove the Theorem, it may be helpful first to indicate a definition of the returns to scale function $\alpha(V)$. If $\alpha(V)$ is the returns to scale for the function in (2.1) when the level of output is V, we have, employing Euler's Theorem,

$$L\frac{\partial g}{\partial L} + K\frac{\partial g}{\partial K} = \alpha(V)V. \qquad \ldots(2.5a)$$

The relation in (2.5*a*) *defines* the returns to scale function $\alpha(V)$. A couple of other definitions of $\alpha(V)$ may also be noted. First, from (2.1) we have

$$dV = \frac{\partial g}{\partial L}dL + \frac{\partial g}{\partial K}dK, \text{ or}$$

$$L\frac{\partial g}{\partial L}\frac{dL}{L} + K\frac{\partial g}{\partial K}.\frac{dK}{K} = dV.$$

Assuming that $dL/L = dK/K = \lambda$, and using (2.5*a*), we obtain

$$\alpha(V) = dV/V\lambda \qquad \ldots(2.5b)$$

as an alternative definition of the returns to scale function. Yet another view would take (2.5) itself as a definition of $\alpha(V)$. Thus we write

$$\frac{dV}{V}\Big/\frac{df}{f\alpha_f} = \alpha(V). \qquad \ldots(2.5c)$$

[1] As is obvious, this lemma is closely related to properties of monotonic transformations of utility functions in the theory of consumption.

Given equal infinitesimal proportionate changes $dL/L = dK/K = \lambda$, $df/f\alpha_f = \lambda$ and thus $\alpha(V)$ is just equal to $dV/V\lambda$, the ratio of a proportionate change in output due to the equal proportionate change, λ, in both inputs. Since $\alpha(V)$ is a pure number, it is not affected by changes in units of measurement.

We now proceed to present the proof of the Theorem.

Proof. The proof is simple. Consider the definition of $\alpha(V)$ given in (2.5*a*). In view of (2.2*a*-*b*), and noting that $\frac{\partial V}{\partial L} = \frac{\partial g}{\partial L}$ and $\frac{\partial V}{\partial K} = \frac{\partial g}{\partial K}$, equation (2.5*a*) becomes

$$\frac{dg}{df}\left(L\frac{\partial f}{\partial L} + K\frac{\partial f}{\partial K}\right) = V\alpha(V).$$

But since $f(L, K)$ is by assumption homogeneous of degree α_f, we have by Euler's Theorem

$$L\frac{\partial f}{\partial L} + K\frac{\partial f}{\partial K} = \alpha_f f.$$

This last relation taken with the one immediately above establishes the assertion in (2.5).

Definition. A production function $V = g(f)$ which is a solution to (2.5) for a particular choice of the returns to scale function $\alpha(V)$ is termed a generalized production function (GPF).

We note that the function f in (2.5) might arise as the solution to the following differential equation involving a constant or variable elasticity of substitution σ:

$$\sigma = \frac{-f'(f - xf')}{xff''}$$

where f has been assumed to have constant returns to scale, $x = K/L$, and $f' = df/dx$.

As for examples illustrating the use of (2.5), suppose we first take

$$\alpha(V) = \alpha_f\left(1 - \frac{V}{k}\right) \text{ with } 0 \leq V < k. \qquad \ldots(2.6)$$

Here the return to scale is α_f when $V = 0$ and falls monotonically to zero as V approaches k, the limiting value for output. For the function $\alpha(V)$ in (2.6), the differential equation in (2.5) is:

$$\frac{dV}{df} = \frac{V}{f}\left(1 - \frac{V}{k}\right) \qquad \ldots(2.7)$$

with solution

$$V = g(f) = \frac{kf}{1+f}$$

$$= \frac{k}{1 + e^{-\log f}}. \qquad \ldots(2.8)$$

As is apparent from the second line of (2.8), V and $\log f$ are related by the logistic function. Below we shall consider two cases, namely (i) f is taken to be in the Cobb-Douglas (CD) form and (ii) f is taken in the CES form.

As another example, suppose we take

$$\alpha(V) = \alpha_f\left[\gamma + \frac{\beta}{g^{-1}(V)}\right] \quad \gamma > 0,\ \beta > 0$$

$$= \alpha_f\left(\gamma + \frac{\beta}{f}\right). \qquad \ldots(2.9)$$

Here at low levels of output, $\alpha(V)$ is extremely large while $\lim_{V\to\infty} \alpha(V) = \lim_{f\to\infty} \alpha(f) = \alpha_f$. With this form for $\alpha(V)$, the differential equation in (2.5) is:

$$\frac{dV}{df} = \frac{V}{f}\left(\gamma + \frac{\beta}{f}\right), \qquad \text{...(2.10)}$$

which has solution

$$V = g(f) = Cf^{\gamma}e^{-\beta/f} \qquad \text{...(2.11)}$$

where C is a constant of integration. Equation (2.11) thus defines a class of GPF's exhibiting returns to scale in accord with (2.9).

As a third example of the generation of a GPF, suppose that we wish the returns to scale to exhibit the following behaviour:

$$\alpha(V) = \alpha_f + h\left(\frac{a-V}{a+V}\right), \quad a_f > h \geqq 0, \qquad \text{...(2.12)}$$

where α_f, a and h are constant parameters. With (2.12), the return to scale is $\alpha_f + h$ when $V = 0$ and falls monotonically to reach a limiting value $\alpha_f - h$ as V grows in value. Inserting $\alpha(V)$, given by (2.12), in the differential equation shown in (2.5) and solving, we obtain

$$V\{(\alpha_f + h)a + (\alpha_f - h)V\}^{2h/(\alpha_f - h)} = k'f^{(\alpha_f + h)/\alpha_f}, \qquad \text{...(2.13)}$$

where k' is a constant of integration. Letting $\rho = h/\alpha_f$, we have $0 \leqq \rho < 1$ and can write (2.13) as:

$$V\{(1+\rho)a + (1-\rho)V\}^{2\rho/(1-\rho)} = kf^{1+\rho}, \qquad \text{...(2.14)}$$

where k is a constant. Upon choice of the form of f, say in the CD form, the parameters appearing in (2.14) can be estimated. Further, testing the hypothesis $\rho = 0$ is a test of hypothesis that the returns to scale parameter is independent of output. That is, $\rho = h/\alpha_f$ and thus, for $\alpha_f \neq 0$, $\rho = 0$ implies that $h = 0$ in which case (2.12) gives $\alpha(V) = \alpha_f$, a constant.

Lastly, in these examples and in general, since the elasticity of substitution associated with $V = g(f)$, is the same as that associated with f, by appropriate choice of f we can give $g(f)$, the solution to (2.5), whatever elasticity of substitution we desire, subject to the condition that f be a homogeneous neo-classical production function.

3. IMPLICATIONS FOR THE BEHAVIOUR OF LABOUR'S SHARE

We shall assume that labour is paid in accord with its marginal productivity, that is,[1]

$$\frac{\partial V}{\partial L} = \frac{w}{p}, \qquad \text{...(3.1)}$$

where w is the money wage rate, p is the price of output, and $V = g(f)$ is a GPF. Since $\frac{\partial V}{\partial L} = \frac{dg}{df}\frac{\partial f}{\partial L}$, (3.1) can be brought into the following form:

$$S_{L|g} = \frac{Lw}{pV} = \eta_{g|f}\eta_{f|L} \qquad \text{...(3.2)}$$

where $S_{L|g}$ is labour's share given production function g and

$$\eta_{g|f} \equiv \frac{f}{V}\frac{dg}{df} \qquad \text{...(3.3}a\text{)}$$

[1] Note from (2.2a) $\frac{\partial V}{\partial L} = \frac{dg}{df}\frac{\partial f}{\partial L}$ and thus (3.1) can be written as $\frac{\partial f}{\partial L} = Rw/p$ with $R \equiv \left(\frac{dg}{df}\right)^{-1}$. Thus an inappropriate choice of production function, here f rather than $V = g(f)$, might lead to the *incorrect* conclusion that entrepreneurs fail to optimize since $R \neq 1$. On this, cf. Hoch ([4], p. 568).

and

$$\eta_{f \mid L} = \frac{L}{f}\frac{\partial f}{\partial L}. \qquad \text{...(3.3}b\text{)}$$

Further, since from (2.5) $\eta_{g \mid f} = \frac{\alpha(V)}{\alpha_f}$ and since labour's share using production function f is $S_{L \mid f} = \eta_{f \mid L}$, we can express (3.2) as follows:

$$S_{L \mid g} = \frac{\alpha(V)}{\alpha_f} S_{L \mid f}. \qquad \text{...(3.4)}$$

The result in (3.4) conveniently relates the behaviour of labour's share for a GPF g to its behaviour for a production function f and to the returns to scale function $\alpha(V)$.

Some explicit examples of (3.4) follow. If we use the GPF in (2.8) with f in the *CD* form, that is $f = AL^{\alpha_1}K^{\alpha_2}$, labour's share is given by:

$$S_{L \mid g} = \left(1 - \frac{V}{k}\right)\alpha_1 \qquad \text{...(3.5)}$$

which asserts that labour's share will fall, ceteris paribus, as the output rate is increased.[1] On the other hand if we take f in (2.8) to be in the *CES* form, that is

$$f = \gamma[(1-\delta)L^{-\rho} + \delta K^{-\rho}]^{-C/\rho},$$

where $\sigma = 1/(1+\rho)$, labour's share is given by

$$S_{L \mid g} = \left(1 - \frac{V}{k}\right)\frac{C(1-\delta)}{1-\delta+\delta(K/L)^{(\sigma-1)/\sigma}} \qquad \text{...(3.6}a\text{)}$$

or alternatively,

$$S_{L \mid g} = \left(1 - \frac{V}{k}\right)\frac{C(1-\delta)^{\sigma}\gamma^{\sigma-1}}{(w/p)^{\sigma-1}}. \qquad \text{...(3.6}b\text{)}$$

From (3.6*a*) we see that labour's share depends on, among other things, the level of output, V, the capital-labour ratio, K/L, and σ, the elasticity of substitution. For $\sigma > 1$, labour's share will fall as V and K/L increase. On the other hand, with $0 < \sigma < 1$ and with increasing V and K/L, the first factor on the r.h.s. of (3.6*a*) decreases linearly, approaching zero as V approaches k while the second factor increases with increasing K/L and approaches C. Thus under these conditions, labour's share could rise, reach a maximum and then fall as V and K/L increase.

Clearly (3.5) and (3.6), as well as other GPF's embodying variable returns to scale, allow for a more varied behaviour of labour's share than is provided by, say, the usual *CD* or *CES* functions. On this point, we note that U.S. Census of Manufactures data for 1939 on the relation between labour's share and value of output for plants (by size class) in 87 industries have been presented in Steindl [8]. The data show a variety of patterns. In quite a few industries labour's share falls continuously as size of plant, measured by value of output, increases. In a few cases, labour's share is constant and independent of the size of plant, while for some others the pattern is quite irregular. Finally, for a number of industries labour's share rises to a maximum and then declines as plant size increases. It is beyond the scope of the present paper to provide explanations for this varied behaviour of labour's share in different industries. However, we would not rule out the possibility that variation in the returns to scale may be responsible in part for this observed behaviour of labour's share.

[1] In certain applications (3.5) can be used to obtain estimates of α_1 and k quite simply.

4. ESTIMATION OF GENERALIZED PRODUCTION FUNCTIONS

In this section we provide an example of a generalized production function which has been estimated with cross section data. The returns to scale function which we employ is given by [1]

$$\alpha(V) = \frac{\alpha'}{1+\theta'(V-b')}$$

$$= \frac{\alpha}{1+\theta V} \qquad \dots(4.1)$$

where $\alpha'>0$, $b'>0$, $b'\theta'<1$, $\alpha = \alpha' h$, $h = (1-b'\theta')^{-1}$ and $\theta = \theta' h$. We note that if $\theta>0$, (4.1) posits that the returns to scale fall from α, at $V = 0$, to zero as $V \to \infty$. If $\theta<0$, a less plausible case, $\alpha(V)$ increases from α, at $V = 0$, to α', at $V = b$, and approaches infinity as $V \to -1/\theta$. Last, if $\theta = 0$, $\alpha(V) = \alpha$, a constant. With empirical data, we shall be in a position to make inferences about which of these possibilities is in accord with the information in the data.

On inserting (4.1) in (2.5) and solving the differential equation, we are led to the following GPF:

$$Ve^{\theta V} = c^h f^h \qquad \dots(4.2)$$

where c is a constant of integration. Taking f in the Cobb-Douglas form, $f = \gamma' K^{\alpha'(1-\delta)} L^{\alpha'\delta}$, with returns to scale parameter α', we have from (4.2)

$$Ve^{\theta V} = \gamma K^{\alpha(1-\delta)} L^{\alpha\delta} \qquad \dots(4.3)$$

with $0<\delta<1$, $\gamma>0$ and $\alpha>0$.

If we introduce a multiplicative random error term in (4.3) and then take natural logarithms of both sides, we obtain

$$\log V_i + \theta V_i = \log \gamma + \alpha(1-\delta) \log K_i + \alpha\delta \log L_i + u_i \qquad \dots(4.4)$$

where the subscript i denotes variables pertaining to the ith unit ($i = 1, 2, \dots, N$) and u_i is a random error term. We assume that the u_i's are normally and independently distributed, each with mean zero and common variance σ^2. Under these assumptions and further assuming that $\log K_i$ and $\log L_i$ are distributed independently of the error terms or are fixed quantities,[2] the logarithm of the likelihood function, $\log l$, is given by

[1] See Revankar [6] for further analysis of this returns to scale function in the context of variable elasticity of substitution production functions.

[2] The estimation procedure described below involves the assumption that either the input variables are (i) fixed or if random are (ii) distributed independently of the disturbance term in the production function. In Zellner, Kmenta and Drèze [10], a Cobb-Douglas production model involving the maximization of the mathematical expectation of profits led to a model with (ii) satisfied. Further, with respect to the issue of " transmitted shocks ", treated extensively by Mundlak and Hoch [5], we can express our production function as follows

$$\log V_i + \theta V_i = \log \gamma + \alpha(1-\delta) \log K_i^* + \alpha\delta \log L_i^* + u_i^* \qquad \dots(a)$$

where L_i^* and K_i^* are " effective " labour and capital inputs, respectively, both random and *not* observed. To relate effective inputs to observed inputs and allow for the transmission of shocks, we posit

$$L_i^* = L_i e^{u_i^* + v_{1i}} \text{ and } K^* = K_i e^{u_i^* + v_{2i}} \qquad \dots(b)$$

where L_i and K_i are the actual amounts of labour and capital employed, amounts which are *fixed* by the entrepreneur and the quantities measured by our data. v_{1i} and v_{2i} are random disturbances in the relations connecting " effective " and actual inputs. On substituting from (*b*) into (*a*), we obtain

$$\log V_i + \theta V_i = \log \gamma + \alpha(1-\delta) \log K_i + \alpha\delta \log L_i + u_i \qquad \dots(c)$$

where $u_i = \alpha u_i^* + \alpha(1-\delta)v_{2i} + \alpha\delta v_{1i}$. Since we regard the entrepreneur as fixing K_i and L_i, we can regard these variables to be fixed in our estimation procedure—an approach which allows for transmission of shocks and which is close to Berkson's [1] case of the " errors in the variables " model.

$$\log l = \text{const.} - \frac{N}{2}\log\sigma^2 + \log J - \frac{1}{2\sigma^2}\sum_{i=1}^{N}\{z_i(\theta) - c_0 - c_1\log K_i - c_2\log L_i\}^2 \quad \text{...(4.5)}$$

where $z_i(\theta) = \log V_i + \theta V_i$, $c_0 = \log\gamma$, $c_1 = \alpha(1-\delta)$, $c_2 = \alpha\delta$, and J is the Jacobian of the transformation from the u_i's to the V_i's, that is

$$J = \prod_{i=1}^{N}\frac{\partial u_i}{\partial V_i} = \prod_{i=1}^{N}\left[\frac{1+\theta V_i}{V_i}\right]. \quad \text{...(4.6)}$$

Substituting from (4.6) in (4.5), we have

$$\log l = \text{const.} - \frac{N}{2}\log\sigma^2 + \sum_{i=1}^{N}\log(1+\theta V_i) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}\{z_i(\theta) - c_0 - c_1\log K_i - c_2\log L_i\}^2. \quad \text{...(4.7)}$$

Differentiating (4.7) partially with respect to σ^2 and setting the derivative equal to zero leads to

$$\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}\{z_i(\theta) - c_0 - c_1\log K_i - c_2\log L_i\}^2 \quad \text{...(4.8)}$$

as the conditional maximizing value for σ^2. When $\hat{\sigma}^2$ in (4.8) is substituted for σ^2 in (4.7), we obtain

$$\log l^* = \text{const.} - \frac{N}{2}\log\left[\sum_{i=1}^{N}\{z_i(\theta) - c_0 - c_1\log K_i - c_2\log L_i\}^2\right] + \sum_{i=1}^{N}\log[1+\theta V_i]. \quad \text{...(4.9)}$$

For given θ, say $\theta = \theta_0$, the conditional maximizing values of c_0, c_1 and c_2 can easily be obtained by regressing $z_i(\theta_0)$ on $\log K_i$ and $\log L_i$. That is for given $\theta = \theta_0$, $\log l^*$ will be maximized for the values of c_0, c_1 and c_2 which minimize

$$\sum_{i=1}^{N}\{z_i(\theta_0) - c_0 - c_1\log K_i - c_2\log L_i\}^2.$$

Then, using these estimates and $\theta = \theta_0$, we evaluate $\log l^* - \text{const.}$ as shown in (4.9). By repeating this procedure for various values of θ, we can find the values of the c's and θ associated with the global maximum of the likelihood function, a procedure which is very similar to that described by Box and Cox [2].

Having obtained the maximum likelihood estimates of θ and the c's by the procedure described above, these can be inserted in (4.8) to yield a maximum likelihood estimate for σ^2. Last, the square roots of diagonal elements of the inverse of the well-known information matrix, evaluated at the maximum likelihood estimates, provide large sample standard errors.

On applying the above procedure to 1957 state data for the U.S. Transportation Equipment Industry—see Appendix for data—we found that the maximum likelihood estimate of θ to be $\hat{\theta} = 0{\cdot}134$ with a large sample standard error of 0·0638.[1] For this value of θ, the regression of $z_i(\hat{\theta})$ on $\log K_i$ and $\log L_i$ yielded the following estimates of the c's:

$\hat{c}_0 = 3{\cdot}0129$	$\hat{c}_1 = 0{\cdot}3330$	$\hat{c}_2 = 1{\cdot}1551$.
(0·1343)	(0·1022)	(0·1228)
[0·3854]	[0·1023]	[0·1564]

The figures in parentheses below the point estimates are *conditional* standard errors, that is standard errors which are appropriate if it is assumed *a priori* that $\theta = \hat{\theta}$. The figures in square brackets are *unconditional* large sample standard errors, obtained from the

[1] The ratio of $\hat{\theta}$ to its standard error is 2·09, a value which suggests that it is possible to reject the hypothesis that $\theta = 0$ at a reasonable level of significance.

inverse of the estimated information matrix, which take account of the fact that the sample information has been used to estimate θ as well as the c's.

Using the fact that $\hat{c}_1+\hat{c}_2$ is an estimate of α, we obtain the following estimate of the returns to scale function: [1]

[1] Note that both α and $\alpha(V)$ are invariant with respect to changes in units of measurement for V, L, and K.

$$\text{Est. } \alpha(V) = \frac{1{\cdot}49}{1+0{\cdot}134V}.$$

Shown below are values of Est. $\alpha(V)$ by state for the U.S. Transportation Equipment Industry in 1957:

Values of Est. $\alpha(V)$ by State for U.S. Transportation Equipment Industry, 1957

State	V	Est. $\alpha(V)$	State	V	Est. $\alpha(V)$
Florida	0·193	1·45	Pa.	2·651	1·10
Maine	0·364	1·42	N.J.	2·701	1·09
Iowa	0·477	1·40	Md.	3·219	1·04
La.	0·638	1·37	Wash.	3·558	1·01
Mass.	1·404	1·25	Ind.	3·816	0·98
W. Va.	1·513	1·24	Ky.	4·031	0·97
Texas	1·712	1·21	Ga.	4·289	0·94
Ala.	1·855	1·19	Ohio	4·440	0·93
N.Y.	2·040	1·17	Conn.	4·485	0·93
Va.	2·052	1·17	Mo.	5·217	0·88
Cal.	2·333	1·13	Kansas	6·507	0·80
Wis.	2·463	1·12	Mich.	7·182	0·76
Ill.	2·629	1·10	...	...	...

V is value added per establishment measured in millions of dollars.

From this table, we see that our estimated GPF shows considerable variation in returns to scale. For example Est. $\alpha(V)$ is 1·45 for Florida and 0·76 for Michigan. These results indicate rather well that it is important to allow for variation in returns to scale in analyzing production relations for the transportation equipment industry. However, in interpreting these results it is important to bear in mind that the data pertain to an industry which is not very homogeneous since transportation equipment is a rather broad category. Further work using disaggregated data would be of interest.

5. CONCLUDING REMARKS

In this paper we have presented an operational method for generating production functions exhibiting returns to scale which vary with the scale of operations in a preassigned manner and having a preassigned elasticity of substitution which can be constant or variable. We refer to such production functions as generalized production functions. Some implications of several such functions for the behaviour of labour's share have been explored. In addition, we have provided an illustrative example to show how the parameters of a generalized production function can be estimated by the maximum likelihood method. This approach can be utilized quite readily in the analysis of the particular functional form utilized herein and of other generalized production functions.

University of Chicago A. ZELLNER.
State University of New York at Buffalo N. S. REVANKAR

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APPENDIX A

TABLE A

1957 U.S. Annual Survey of Manufactures Data for the Transportation Equipment Industry Employed in Calculations *

State	Aggregate value added, V_s	Aggregate capital service flow,† K_s	Aggregate man-hours worked,‡ L_s	No. of establishments, n
	(millions of dollars)		(millions of man-hours)	
Alabama	126·148	3·804	31·551	68
California	3201·486	185·446	452·844	1372
Connecticut	690·670	39·712	124·074	154
Florida	56·296	6·547	19·181	292
Georgia	304·531	11·530	45·534	71
Illinois	723·028	58·987	88·391	275
Indiana	992·169	112·884	148·530	260
Iowa	35·796	2·698	8·017	75
Kansas	494·515	10·360	86·189	76
Kentucky	124·948	5·213	12·000	31
Louisiana	73·328	3·763	15·900	115
Maine	29·467	1·967	6·470	81
Maryland	415·262	17·546	69·342	129
Massachusetts	241·530	15·347	39·416	172
Michigan	4079·554	435·105	490·384	568
Missouri	652·085	32·840	84·831	125
New Jersey	667·113	33·292	83·033	247
New York	940·430	72·974	190·094	461
Ohio	1611·899	157·978	259·916	363
Pennsylvania	617·579	34·324	98·152	233
Texas	527·413	22·736	109·728	308
Virginia	174·394	7·173	31·301	85
Washington	636·948	30·807	87·963	179
West Virginia	22·700	1·543	4·063	15
Wisconsin	349·711	22·001	52·818	142

* Nonlinear aggregation considerations—see Zellner ([9], p. 111 ff.)—point to the desirability of working with variables on a per establishment basis. Thus, for each state, we used $V = V_s/n$, $K = K_s/n$ and $L = L_s/n$ in our calculations.

† Net capital stock is defined as " gross book value on December 31, 1957 " minus " accumulated depreciation and depletion up to December 31, 1956," minus " depreciation and depletion charged in 1957." Capital service flow is defined as depreciation and depletion charged in 1957 plus 0·06 times the net capital stock plus the sum of insurance premiums, rental payments and property taxes paid.

‡ These figures refer to production workers.

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