VAR and VEC

Using R

VAR: Vector Autoregression

$$\mathbf{y}_t = \mathbf{v} + \sum_{j=1}^{p} \mathbf{A}_j \mathbf{y}_{t-j} + \mathbf{u}_t$$

$$t = 1, \ldots, T$$

- Assumptions:
  - $\mathbf{y}_t$: Stationary or cointegrated K-variable vector
  - $\mathbf{v}$: K constant parameters vector
  - $\mathbf{A}_j$: K by K parameters matrix, j=1,…,p
  - $\mathbf{u}_t$: i.i.d.($\mathbf{0}$,Σ)
- Trend may be included: δt, where δ is K by 1
- Exogenous variables **X** may be added

Example

- C: Log Personal Consumption Expenditure
- Y: Log Disposable Personal Income
- C ~ I(1), Y ~ I(1)
- Consumption-Income Relationship:
  $C_t = v_c + a_{cc1}C_{t-1} + a_{cy1}Y_{t-1} + \ldots (+ \delta_c t) + \varepsilon_c$
  $Y_t = v_y + a_{yc1}C_{t-1} + a_{yy1}Y_{t-1} + \ldots (+ \delta_y t) + \varepsilon_y$
- Model Specification
  - How many lags?
  - Parameter restrictions?

Example

(Ct, Yt): 1947Q1 – 2015Q4

- VAR(6) without parameter restrictions:

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \sum_{j=1}^{6} \begin{bmatrix} a_{ccj} & a_{cyj} \\ a_{ycj} & a_{yyj} \end{bmatrix} \begin{bmatrix} C_{t-j} \\ Y_{t-j} \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0.006 \\ 0.047^{**} \end{bmatrix} + \begin{bmatrix} 0.951^{**} & 0.149^{**} \\ 0.311^{**} & 0.764^{**} \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} 0.335^{**} & -0.047 \\ -0.151 & 0.099 \end{bmatrix} \begin{bmatrix} C_{t-2} \\ Y_{t-2} \end{bmatrix}$$

$$+ \begin{bmatrix} -0.180^{*} & -0.235^{**} \\ 0.203^{*} & -0.135^{*} \end{bmatrix} \begin{bmatrix} C_{t-3} \\ Y_{t-3} \end{bmatrix} + \begin{bmatrix} -0.214^{**} & 0.108 \\ -0.064 & -0.065 \end{bmatrix} \begin{bmatrix} C_{t-4} \\ Y_{t-4} \end{bmatrix}$$

$$+ \begin{bmatrix} 0.055 & -0.046 \\ -0.249^{**} & 0.138^{*} \end{bmatrix} \begin{bmatrix} C_{t-5} \\ Y_{t-5} \end{bmatrix} + \begin{bmatrix} 0.012 & 0.112^{**} \\ -0.012 & 0.157^{**} \end{bmatrix} \begin{bmatrix} C_{t-6} \\ Y_{t-6} \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

Example

VAR(6)

- VAR(6) with parameter restrictions:

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0.047^{**} \end{bmatrix} + \begin{bmatrix} 0.946^{**} & 0.133^{**} \\ 0.335^{**} & 0.759^{**} \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \begin{bmatrix} 0.325^{**} & 0 \\ -0.183 & 0.107 \end{bmatrix} \begin{bmatrix} C_{t-2} \\ Y_{t-2} \end{bmatrix}$$

$$+ \begin{bmatrix} -0.169^{*} & -0.265^{**} \\ 0.175^{*} & -0.167^{*} \end{bmatrix} \begin{bmatrix} C_{t-3} \\ Y_{t-3} \end{bmatrix} + \begin{bmatrix} -0.156^{**} & 0.079 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{t-4} \\ Y_{t-4} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ -0.289^{**} & 0.109^{*} \end{bmatrix} \begin{bmatrix} C_{t-5} \\ Y_{t-5} \end{bmatrix} + \begin{bmatrix} 0 & 0.107^{**} \\ 0 & 0.150^{**} \end{bmatrix} \begin{bmatrix} C_{t-6} \\ Y_{t-6} \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

$$Var\left(\begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix} \right) = \begin{bmatrix} 5.210\text{e}-05 & 2.475\text{e}-05 \\ 2.475\text{e}-05 & 7.709\text{e}-05 \end{bmatrix}$$

VAR and VEC

- If $\mathbf{y}_t$ is not stationary, VAR or VEC can only be applied for cointegrated $\mathbf{y}_t$ system:
  - VAR (Vector Autoregression)

$$\mathbf{y}_t = \mathbf{v} + \sum_{j=1}^{p} \mathbf{A}_j \mathbf{y}_{t-j} (+ \boldsymbol{\delta} t) + \mathbf{u}_t$$

$$\Downarrow$$

$$\Delta \mathbf{y}_t = \mathbf{v} + \boldsymbol{\Pi} \mathbf{y}_{t-1} + \sum_{j=1}^{p-1} \boldsymbol{\Pi}_j \Delta \mathbf{y}_{t-j} (+ \boldsymbol{\delta} t) + \mathbf{u}_t$$

  - VEC (Vector Error Correction)

VEC: Vector Error Correction

- If there is no trend in $\mathbf{y}_t$, let $\mathbf{\Pi} = \alpha\beta'$ ($\mathbf{\Pi}$ is K by K, α is K by r, β is K by r, r is the rank of $\mathbf{\Pi}$, 0<r<K):

$$\Delta \mathbf{y}_t = \mathbf{v} + \mathbf{\Pi}\mathbf{y}_{t-1} + \sum_{j=1}^{p-1} \mathbf{\Pi}_j \Delta \mathbf{y}_{t-j} + \mathbf{u}_t$$

$$\Downarrow \quad \mathbf{v} = \boldsymbol{\alpha}\boldsymbol{\mu} + \boldsymbol{\gamma},\ \boldsymbol{\gamma}'(\boldsymbol{\alpha}\boldsymbol{\mu}) = 0$$

$$\Delta \mathbf{y}_t = \boldsymbol{\alpha}\left(\boldsymbol{\beta}'\mathbf{y}_{t-1} + \boldsymbol{\mu}\right) + \sum_{j=1}^{p-1} \mathbf{\Pi}_j \Delta \mathbf{y}_{t-j} + \boldsymbol{\gamma} + \mathbf{u}_t$$

VEC: Vector Error Correction

$$\Delta\mathbf{y}_t = \boldsymbol{\alpha}\left(\boldsymbol{\beta}'\mathbf{y}_{t-1} + \boldsymbol{\mu}\right) + \sum_{j=1}^{p-1}\boldsymbol{\Pi}_j\Delta\mathbf{y}_{t-j} + \boldsymbol{\gamma} + \mathbf{u}_t$$

- No-constant or No-drift Model: **v** = 0
  $\boldsymbol{\gamma} = 0$, $\boldsymbol{\mu} = 0$
- Restricted-constant Model: **v** = $\boldsymbol{\alpha\mu}$
  $\boldsymbol{\gamma} = 0$, $\boldsymbol{\mu} \neq 0$
- Constant or Drift Model:
  $\boldsymbol{\gamma} \neq 0$, $\boldsymbol{\mu} \neq 0$

VEC: Vector Error Correction

- If there is trend in $\mathbf{y}_t$

$$\mathbf{y}_t = \mathbf{v} + \sum_{j=1}^{p} \mathbf{A}_j \mathbf{y}_{t-j} + \boldsymbol{\delta} t + \mathbf{u}_t$$

$$\Downarrow$$

$$\Delta \mathbf{y}_t = \mathbf{v} + \boldsymbol{\Pi} \mathbf{y}_{t-1} + \sum_{j=1}^{p-1} \boldsymbol{\Pi}_j \Delta \mathbf{y}_{t-j} + \boldsymbol{\delta} t + \mathbf{u}_t$$

$$\Downarrow \qquad \mathbf{v} = \boldsymbol{\alpha\mu} + \boldsymbol{\gamma},\ \boldsymbol{\gamma}^{'}(\boldsymbol{\alpha\mu}) = 0$$

$$\boldsymbol{\delta} t = (\boldsymbol{\alpha\rho} + \boldsymbol{\tau}) t,\ \boldsymbol{\tau}^{'}(\boldsymbol{\alpha\rho}) = 0$$

$$\Delta \mathbf{y}_t = \boldsymbol{\alpha}\left(\boldsymbol{\beta}^{'} \mathbf{y}_{t-1} + \boldsymbol{\mu} + \boldsymbol{\rho} t\right) + \sum_{j=1}^{p-1} \boldsymbol{\Pi}_j \Delta \mathbf{y}_{t-j} + \boldsymbol{\gamma} + \boldsymbol{\tau} t + \mathbf{u}_t$$

VEC: Vector Error Correction

$$\Delta \mathbf{y}_t = \boldsymbol{\alpha}\left(\boldsymbol{\beta}'\mathbf{y}_{t-1} + \boldsymbol{\mu} + \boldsymbol{\rho}t\right) + \sum_{j=1}^{p-1} \boldsymbol{\Pi}_j \Delta \mathbf{y}_{t-j} + \boldsymbol{\gamma} + \boldsymbol{\tau}t + \mathbf{u}_t$$

- No-drift No-trend Model: $\mathbf{v} = 0$, $\delta = 0$
  $\gamma = 0$, $\mu = 0$, $\tau = 0$, $\rho = 0$
- Restricted-constant Model: $\mathbf{v} = \alpha\mu$, $\delta = 0$
  $\gamma = 0$, $\mu \neq 0$, $\tau = 0$, $\rho = 0$
- Constant or Drift Model: $\delta = 0$
  $\gamma \neq 0$, $\mu \neq 0$, $\tau = 0$, $\rho = 0$
- Restricted-trend Model: $\delta = \alpha\rho$
  $\gamma \neq 0$, $\mu \neq 0$, $\tau = 0$, $\rho \neq 0$
- Trend Model:
  $\gamma \neq 0$, $\mu \neq 0$, $\tau \neq 0$, $\rho \neq 0$

Example Continued

Johansen Test with constant models and 6 lags

- VAR:

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \sum_{j=1}^{6} \begin{bmatrix} a_{ccj} & a_{cyj} \\ a_{ycj} & a_{yyj} \end{bmatrix} \begin{bmatrix} C_{t-j} \\ Y_{t-j} \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

- VEC:

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \begin{bmatrix} \pi_{cc} & \pi_{cy} \\ \pi_{yc} & \pi_{yy} \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

- Rank 1: $\Pi = \alpha\beta'$

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \begin{bmatrix} \alpha_c \\ \alpha_y \end{bmatrix} \begin{bmatrix} \beta_c & \beta_y \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

Example Continued

Johansen Test with constant models and 6 lags

- Constant model: v = $\alpha\mu + \gamma$ ($\alpha \neq 0$, $\gamma \neq 0$)

$$\begin{bmatrix}\Delta C_t \\ \Delta Y_t\end{bmatrix} = \begin{bmatrix}\alpha_c \\ \alpha_y\end{bmatrix}\left(\begin{bmatrix}\beta_c & \beta_y\end{bmatrix}\begin{bmatrix}C_{t-1} \\ Y_{t-1}\end{bmatrix} + \mu\right) + \sum_{j=1}^{5}\begin{bmatrix}\pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj}\end{bmatrix}\begin{bmatrix}\Delta C_{t-j} \\ \Delta Y_{t-j}\end{bmatrix} + \begin{bmatrix}\gamma_c \\ \gamma_y\end{bmatrix} + \begin{bmatrix}u_{ct} \\ u_{yt}\end{bmatrix}$$

- Restricted-constant model: v = $\alpha\mu$ ($\alpha \neq 0$)

$$\begin{bmatrix}\Delta C_t \\ \Delta Y_t\end{bmatrix} = \begin{bmatrix}\alpha_c \\ \alpha_y\end{bmatrix}\left(\begin{bmatrix}\beta_c & \beta_y\end{bmatrix}\begin{bmatrix}C_{t-1} \\ Y_{t-1}\end{bmatrix} + \mu\right) + \sum_{j=1}^{5}\begin{bmatrix}\pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj}\end{bmatrix}\begin{bmatrix}\Delta C_{t-j} \\ \Delta Y_{t-j}\end{bmatrix} + \begin{bmatrix}u_{ct} \\ u_{yt}\end{bmatrix}$$

- No-constant model: v = 0

$$\begin{bmatrix}\Delta C_t \\ \Delta Y_t\end{bmatrix} = \begin{bmatrix}\alpha_c \\ \alpha_y\end{bmatrix}\begin{bmatrix}\beta_c & \beta_y\end{bmatrix}\begin{bmatrix}C_{t-1} \\ Y_{t-1}\end{bmatrix} + \sum_{j=1}^{5}\begin{bmatrix}\pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj}\end{bmatrix}\begin{bmatrix}\Delta C_{t-j} \\ \Delta Y_{t-j}\end{bmatrix} + \begin{bmatrix}u_{ct} \\ u_{yt}\end{bmatrix}$$

Example Continued

Johansen Test with trend models and 6 lags

- VAR:

$$\begin{bmatrix} C_t \\ Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \sum_{j=1}^{6} \begin{bmatrix} a_{ccj} & a_{cyj} \\ a_{ycj} & a_{yyj} \end{bmatrix} \begin{bmatrix} C_{t-j} \\ Y_{t-j} \end{bmatrix} + \begin{bmatrix} \delta_c t \\ \delta_y t \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

- VEC:

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \begin{bmatrix} \pi_{cc} & \pi_{cy} \\ \pi_{yc} & \pi_{yy} \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} \delta_c t \\ \delta_y t \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

- Rank 1: $\Pi = \alpha\beta'$

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} v_c \\ v_y \end{bmatrix} + \begin{bmatrix} \alpha_c \\ \alpha_y \end{bmatrix} \begin{bmatrix} \beta_c & \beta_y \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} \delta_c t \\ \delta_y t \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

Example Continued

Johansen Test with trend models and 6 lags

- Trend model: $\delta = \alpha\rho+\tau$
  $\gamma \neq 0, \mu \neq 0, \tau \neq 0, \rho \neq 0$

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} \alpha_c \\ \alpha_y \end{bmatrix} \left(\begin{bmatrix} \beta_c & \beta_y \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \rho t + \mu \right) + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} \tau_c t \\ \tau_y t \end{bmatrix} + \begin{bmatrix} \gamma_c \\ \gamma_y \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

- Restricted trend model: $\delta = \alpha\rho$
  $\boldsymbol{\gamma} \neq 0, \boldsymbol{\mu} \neq 0, \boldsymbol{\tau} = 0, \boldsymbol{\rho} \neq 0$

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} \alpha_c \\ \alpha_y \end{bmatrix} \left(\begin{bmatrix} \beta_c & \beta_y \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \rho t + \mu \right) + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} \gamma_c \\ \gamma_y \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$

- No trend or constant model: $\delta = 0$
  $\boldsymbol{\gamma} \neq 0, \boldsymbol{\mu} \neq 0, \boldsymbol{\tau} = 0, \boldsymbol{\rho} = 0$

$$\begin{bmatrix} \Delta C_t \\ \Delta Y_t \end{bmatrix} = \begin{bmatrix} \alpha_c \\ \alpha_y \end{bmatrix} \left(\begin{bmatrix} \beta_c & \beta_y \end{bmatrix} \begin{bmatrix} C_{t-1} \\ Y_{t-1} \end{bmatrix} + \mu \right) + \sum_{j=1}^{5} \begin{bmatrix} \pi_{ccj} & \pi_{cyj} \\ \pi_{ycj} & \pi_{yyj} \end{bmatrix} \begin{bmatrix} \Delta C_{t-j} \\ \Delta Y_{t-j} \end{bmatrix} + \begin{bmatrix} \gamma_c \\ \gamma_y \end{bmatrix} + \begin{bmatrix} u_{ct} \\ u_{yt} \end{bmatrix}$$