Distributed Lag Models

Lags in Econometric Models

Autoregressive model vs. distributed lag model

Dynamic behavior: long-run vs. short-run

Problems:

What is the optimal length of lags?

The more lags, the less degree of freedom for estimation.

Multicolinearity issue: need non-sample information.

General Distributed Lag Models

Consider a model with one explanatory variable X:

$$Y_i = \alpha + \beta_0 X_i + \beta_1 X_{i-1} + \cdots + \beta_\ell X_{i-\ell} + \cdots + \varepsilon_i$$

β_0 = Short - run (impact) coefficient of X;

$\beta_1, \beta_2, \cdots \beta_\ell, \cdots$ = Delay coefficients of X;

$$\beta = \sum_{\ell=0}^{\infty} \beta_\ell = \text{Long – run (equilibrium) coefficient of X.}$$

Assume $\beta < \infty$.

Let $\varpi_\ell = \frac{\beta_\ell}{\beta}$, $(\ell = 0,1,2,\ldots)$, then

ϖ_ℓ is the ℓ - th **lag weight**, and

$$Y = \alpha + \beta \sum_{\ell=0}^{\infty} \varpi_\ell X_{i-\ell} + \varepsilon_i$$

Median lag is the periods required for half of total effect β: that is,

find T such that $\sum_{\ell=0}^{T-1} \varpi_\ell = 0.5$.

If all the lag weights have same sign, then **Mean (average) lag** is defined as the

lag weighted average $\sum_{\ell=0}^{\infty} \ell \varpi_\ell$.

Estimation of A Distributed Lag Model

An infinite lags model can not be estimated directly. A prior restriction on lags is required to transform the model into an autoregressive model:

Geometric Lag Model

$\varpi_\ell = (1-\lambda)\lambda^\ell, \quad \ell = 0,1,2,\ldots$

where $\quad 0 < \lambda < 1$

Gamma Lag Model

$\varpi_\ell = (1+\ell)^\gamma \lambda^\ell, \quad \ell = 0,1,2,\ldots$

where $\quad 0 < \lambda < 1, \quad 0 < \gamma < 1$

Although a truncated infinite distributed lag or a finite distributed lag model can be estimated directly, the problem of multicolinearity with large number of lags may prevent a meaningful model to be estimated.

Polynomial Lag Model

Geometric Distributed Lag Model

Geometric (Koyck) Lag Model

Consider a model with one explanatory variable X:

$$Y_i = \alpha + \beta_0 X_i + \beta_1 X_{i-1} + \cdots + \beta_q X_{i-q} + \cdots + \varepsilon_i$$

In terms of lag weights expression,

$$Y_i = \alpha + \beta \sum_{\ell=0}^{\infty} \varpi_\ell X_{i-\ell} + \varepsilon_i$$

where β is the **long-run coefficient**.

Assume the lag weight ϖ_ℓ follows the Koyck restriction on lags:

$$\varpi_\ell = (1-\lambda)\lambda^\ell, \quad \ell = 0,1,2,\ldots$$

where $0 < \lambda < 1$ is the rate of decline, or $0 < (1-\lambda) < 1$ is the rate of adjustment.

Clearly, $\sum_{\ell=0}^{\infty} \varpi_\ell = 1$ (since $\sum_{\ell=0}^{\infty} \lambda^\ell = \frac{1}{1-\lambda}$)

Notes:

(1) The **short-run coefficient** is

$$\beta_0 = \beta(1-\lambda)$$

(or $\beta = \frac{\beta_0}{1-\lambda}$)

(2) **Median lag**:

Find the period T from

$$\sum_{\ell=0}^{T-1} \varpi_\ell = (1-\lambda)\sum_{\ell=0}^{T-1} \lambda^\ell = 0.5$$

or $1 - \lambda^T = 0.5$

That is, $T = \frac{\ell n(0.5)}{\ell n(\lambda)} = -\frac{\ell n(2)}{\ell n(\lambda)}$

(3) **Mean lag**:

Lag-weighted average

$$\sum_{\ell=0}^{\infty} \ell \varpi_\ell = (1-\lambda)\sum_{\ell=0}^{\infty} \ell \lambda^\ell$$

$$= (1-\lambda)\lambda\left(\sum_{\ell=0}^{\infty} \lambda^\ell\right)^2 = \frac{\lambda}{1-\lambda}$$

By substituting the geometric (Koyck) lag structure in the model:

$$\begin{cases} Y_i = \alpha + \beta(1-\lambda)\sum_{\ell=0}^{\infty} \lambda^{\ell} X_{i-\ell} + \varepsilon_i \\ \lambda Y_{i-1} = \lambda\alpha + \beta(1-\lambda)\sum_{\ell=0}^{\infty} \lambda^{\ell+1} X_{i-1-\ell} + \lambda\varepsilon_{i-1} \end{cases}$$

$$\Downarrow$$

$$Y_i - \lambda Y_{i-1} = (1-\lambda)\alpha + \beta(1-\lambda)X_i + (\varepsilon_i - \lambda\varepsilon_{i-1})$$

or

$$Y_i = (1-\lambda)\alpha + \beta_0 X_i + \lambda Y_{i-1} + \nu_i$$

This is an autoregressive model and $\nu_i = \varepsilon_i - \lambda\varepsilon_{i-1}$ will be serially correlated.

Estimation of Geometric Lag Model

The model suffers from the problems of random regressors and autocorrelation. Ordinary least squares estimators will be biased, inconsistent, and inefficient.

Lagged Dependent Variable Model

- ***Instrumental Variable Estimation***
- ***Autocorrelation Correction***

Examples

Adaptive Expectation Model

Partial Adjustment Model

Error Correction Model

Geometric Distributed Lag Model

Adaptive Expectation Model

Consider a model with one explanatory variable X:

$$Y_i = \alpha + \beta X_i^* + \varepsilon_i \quad (i = 1,2,\ldots N)$$

where X_i^* is an unobservable equilibrium or expected value of X_i satisfying the following:

Adaptive Expectation Hypothesis

$$X_i^* - X_{i-1}^* = (1-\lambda)(X_i - X_{i-1}^*) \quad \text{or}$$

$$X_i^* = (1-\lambda)X_i + \lambda X_{i-1}^*$$

$$= (1-\lambda)\left[X_i + \lambda X_{i-1} + \lambda^2 X_{i-2} + \cdots\right]$$

where $0 < (1 - \lambda) < 1$ is the coefficient of expectation, and the expectation tends to be imperfect.

Combining the model with adaptive expectation hypothesis

$$Y_i = \alpha + \beta\left[(1-\lambda)X_i + \lambda X_{i-1}^*\right] + \varepsilon_i$$

or

$$Y_i = \alpha + \beta(1-\lambda)\left[X_i + \lambda X_{i-1} + \lambda^2 X_{i-2} + \cdots\right] + \varepsilon_i$$

$$= \alpha + \beta \sum_{\ell=0}^{\infty} \varpi_\ell X_{i-\ell} + \varepsilon_i$$

with $\varpi_\ell = (1-\lambda)\lambda^\ell \quad (\ell = 0,1,2,\ldots)$

Estimation of Adaptive Expectation Model

This follows the geometric (Koyck) model estimation and interpretation:

$$Y_i = \alpha(1-\lambda) + \beta(1-\lambda)X_i + \lambda Y_{i-1} + \nu_i$$

where

$$\nu_i = \varepsilon_i - \lambda\varepsilon_{i-1}$$

$\beta(1-\lambda) = \beta_0 = \text{Short-run coefficient of } X_i$;

$\beta = \text{Long-run coefficient of } X_i$;

$$\text{Median Lag} = \frac{\ell n(0.5)}{\ell n(\lambda)};$$

$$\text{Mean Lag} = \frac{\lambda}{1-\lambda}$$

Geometric Distributed Lag Model

Partial Adjustment Model

Consider a model with one explanatory variable X:

$$Y_i{}^* = \alpha + \beta X_i + \varepsilon_i \quad (i = 1,2,\ldots N)$$

where $Y_i{}^*$ is an unobservable equilibrium or long-run desired value of Y_i satisfying the following:

Partial Adjustment Hypothesis

$$Y_i - Y_{i-1} = (1-\lambda)(Y_i{}^* - Y_{i-1}) \quad \text{or}$$

$$\begin{aligned}
Y_i &= (1-\lambda)Y_i{}^* + \lambda Y_{i-1} \\
&= (1-\lambda)Y_i{}^* + \lambda\left[(1-\lambda)Y_{i-1}{}^* + \lambda Y_{i-2}\right] \\
&= (1-\lambda)Y_i{}^* + \lambda\left[(1-\lambda)Y_{i-1}{}^* + \lambda\left[(1-\lambda)Y_{i-2}{}^* + \lambda Y_{i-2}\right]\right] \\
&= \cdots \\
&= (1-\lambda)\left[Y_i{}^* + \lambda Y_{i-1}{}^* + \lambda^2 Y_{i-2}{}^* + \cdots\right]
\end{aligned}$$

where $0 < (1-\lambda) < 1$ is the coefficient of adjustment, and the adjustment tends to be imperfect.

Combining the model with partial adjustment hypothesis

$$\begin{aligned}
Y_i = (1-\lambda)\Big\{&\alpha\left[1 + \lambda + \lambda^2 + \cdots\right] \\
&+ \beta\left[X_i + \lambda X_{i-1} + \lambda^2 X_{i-2} + \cdots\right] \\
&+ \left[\varepsilon_i + \lambda\varepsilon_{i-1} + \lambda^2\varepsilon_{i-2} + \cdots\right]\Big\}
\end{aligned}$$

or

$$Y_i = \alpha + \beta\sum_{\ell=0}^{\infty} \varpi_\ell X_{i-\ell} + \nu_i$$

where $\varpi_\ell = (1-\lambda)\lambda^\ell \quad (\ell = 0,1,2,\ldots)$

and $\nu_i = \sum_{\ell=0}^{\infty} \varpi_\ell \varepsilon_{i-\ell}$

Estimation of Partial Adjustment Model

This follows the geometric (Koyck) model estimation and interpretation:

$$Y_i = \alpha(1-\lambda) + \beta(1-\lambda)X_i + \lambda Y_{i-1} + u_i$$

where

$$\begin{aligned} u_i &= \nu_i - \lambda\nu_{i-1} \\ &= \sum_{\ell=0}^{\infty} \varpi_\ell \varepsilon_{i-\ell} - \lambda \sum_{\ell=0}^{\infty} \varpi_\ell \varepsilon_{i-\ell-1} \\ &= (1-\lambda)\left(\sum_{\ell=0}^{\infty} \lambda^\ell \varepsilon_{i-\ell} - \sum_{\ell=0}^{\infty} \lambda^{\ell+1} \varepsilon_{i-\ell-1}\right) \\ &= (1-\lambda)\varepsilon_i \end{aligned}$$

$\beta(1-\lambda) = \beta_0 =$ Short – run coefficient of X_i;

$\beta =$ Long – run coefficient of X_i;

$$\text{Median Lag} = \frac{\ell n(0.5)}{\ell n(\lambda)};$$

$$\text{Mean Lag} = \frac{\lambda}{1-\lambda}$$

Geometric Distributed Lag Model

Error Correction Model (ECM)

Consider a model with one explanatory variable X:

$$Y_i{*} = \alpha + \beta X_i + \varepsilon_i \quad (i = 1,2,\ldots N)$$

where $Y_i{*}$ is an unobservable equilibrium or long-run desired value of Y_i satisfying the following:

Error Correction Hypothesis

$$Y_i - Y_{i-1} = (1-\gamma)(Y_i{*} - Y_{i-1}{*}) + (1-\lambda)(Y_{i-1}{*} - Y_{i-1})$$

where

$Y_i{*} - Y_{i-1}{*}$ = change in the desired values;

$Y_{i-1}{*} - Y_{i-1}$ = previous disequilibrium;

$0 < 1\text{-}\gamma < 1$

$0 < 1\text{-}\lambda < 1$

If $\gamma = \lambda$, it is the partial adjustment hypothesis:

$$Y_i - Y_{i-1} = (1-\lambda)(Y_i{*} - Y_{i-1})$$

Combining the model with error correction hypothesis

$$\begin{aligned} Y_i - Y_{i-1} &= (1-\gamma)\big(\beta(X_i - X_{i-1}) + (\varepsilon_i - \varepsilon_{i-1})\big) \\ &\quad + (1-\lambda)\big(\alpha + \beta X_{i-1} + \varepsilon_{i-1} - Y_{i-1}\big) \end{aligned}$$

or

$$Y_i = (1-\lambda)\alpha + (1-\gamma)\beta X_i + (\gamma-\lambda)\beta X_{i-1} + \lambda Y_{i-1} + v_i$$

where

$$v_i = (1-\gamma)\varepsilon_i + (\gamma-\lambda)\varepsilon_{i-1}$$

Estimation of Error Correction Model

The model can be estimated in two forms:

$$Y_i = (1-\lambda)\alpha + (1-\gamma)\beta X_i + (\gamma-\lambda)\beta X_{i-1} + \lambda Y_{i-1} + \nu_i$$

or

$$\Delta Y_i = (1-\gamma)\beta\Delta X_i - (1-\lambda)\left(Y_{i-1} - \alpha - \beta X_{i-1}\right) + \nu_i$$

where

$\nu_i = (1-\gamma)\varepsilon_i + (\gamma-\lambda)\varepsilon_{i-1}$ is serial correlated.

ΔY_i depends on ΔX_i and an error correction $Y_{i-1} - \alpha - \beta X_{i-1}$.

Error correction model is closely related to the concept of cointegration.

Short – run coefficient of X_i

$= \beta(1-\gamma) + \beta(\gamma-\lambda) = \beta(1-\lambda)$

Long – run coefficient of $X_i = \beta$

$$\text{Median Lag} = \frac{\ell n(0.5)}{\ell n(\lambda)};$$

$$\text{Mean Lag} = \frac{\lambda}{1-\lambda}$$

Polynomial Distributed Lag Model

Polynomial (Almon) Lag Model

Consider a model with one explanatory variable X:

$$Y_i = \alpha + \beta_0 X_i + \beta_1 X_{i-1} + \cdots + \beta_q X_{i-q} + \varepsilon_i$$

Assume the delay coefficient β_ℓ is a p - th order polynomial function for $\ell = 0,1,2,\ldots,q$:

$$\beta_\ell = a_0 + a_1 \ell + a_2 \ell^2 + \cdots + a_p \ell^p$$
$$= \sum_{j=0}^{p} a_j \ell^j$$

Note: p - th polynominal has p - 1 turning points!

To specify q lags, it requires that $0 < p \leq q$.

For example: $q = 4,\ p = 3$

$$\begin{cases} \beta_0 = a_0 \\ \beta_1 = a_0 + a_1 + a_2 + a_3 \\ \beta_2 = a_0 + 2a_1 + 4a_2 + 8a_3 \\ \beta_3 = a_0 + 3a_1 + 9a_2 + 27a_3 \\ \beta_4 = a_0 + 4a_1 + 16a_2 + 64a_3 \end{cases}$$

From the model

$$Y_i = \alpha + \sum_{\ell=0}^{q} \beta_\ell X_{i-\ell} + \varepsilon_i$$
$$= \alpha + \sum_{j=0}^{p} a_j \sum_{\ell=0}^{q} \ell^j X_{i-\ell} + \varepsilon_i$$
$$= \alpha + \sum_{j=0}^{p} a_j Z_j + \varepsilon_i$$

where

$$Z_j = \sum_{\ell=0}^{q} \ell^j X_{i-\ell}, \quad j = 0,1,2,\ldots,p$$

Therefore estimating the model

$$Y_i = \alpha + \sum_{\ell=0}^{q} \beta_\ell X_{i-\ell} + \varepsilon_i \quad \text{with}$$

$$\beta_\ell = \sum_{j=0}^{p} a_j \ell^j \quad (\ell = 0,1,2,\ldots,q)$$

is identical to estimating

$$Y_i = \alpha + \sum_{j=0}^{p} a_j Z_j + \varepsilon_i \quad \text{with}$$

$$Z_j = \sum_{\ell=0}^{q} \ell^j X_{i-\ell}, \quad (j = 0,1,2,\ldots,p)$$

Estimation of Polynomial Lag Model

For a p-th order polynomial q lags model:

$$\beta_\ell = \sum_{j=0}^{p} a_j \ell^j, \quad \ell = 0,1,2,\ldots,q; \quad p \le q$$

That is,

$$\underset{(q+1)x1}{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_q \end{bmatrix}} = \underset{(q+1)x(p+1)}{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 4 & \cdots & 2^p \\ & & \vdots & & \\ 1 & q & q^2 & \cdots & q^p \end{bmatrix}} \underset{(p+1)x1}{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}}$$

or

$$\boldsymbol{\beta} = \mathbf{Ha}$$

End-Points Restrictions

Additional end-point (tie-down) restrictions can be imposed.

Left - end restriction:

$\beta_{-1} = a_0 - a_1 + a_2 - \cdots = 0$

Right - end restriction:

$\beta_{q+1} = a_0 + a_1(q+1) + a_2(q+1)^2 + \cdots = 0$

Both – end restriction: $\beta_{-1} = 0$ and $\beta_{q+1} = 0$

Least Squares Estimation

From the model

$$Y_i = \alpha + \sum_{\ell=0}^{q} \beta_\ell X_{i-\ell}, \quad i = 1,2,\ldots,N$$

or

$$\begin{aligned}\mathbf{Y} &= \alpha + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \\ &= \alpha + \mathbf{XHa} + \boldsymbol{\varepsilon} \\ &= \alpha + \mathbf{Za} + \boldsymbol{\varepsilon} \quad (\text{let } \mathbf{Z} = \mathbf{XH}) \\ &= \begin{bmatrix} 1 & \mathbf{Z} \end{bmatrix} \begin{bmatrix} \alpha \\ \mathbf{a} \end{bmatrix} + \boldsymbol{\varepsilon} \\ &= \overline{\mathbf{Z}}\overline{\mathbf{a}} + \boldsymbol{\varepsilon}\end{aligned}$$

Least Squares Estimators:

$\hat{\bar{\mathbf{a}}} = (\overline{\mathbf{Z}}'\overline{\mathbf{Z}})^{-1}\overline{\mathbf{Z}}'\mathbf{Y}$

$\mathrm{Var}(\hat{\bar{\mathbf{a}}}) = \hat{\sigma}^2(\overline{\mathbf{Z}}'\overline{\mathbf{Z}})^{-1}$

where

$$\hat{\sigma}^2 = \frac{(\mathbf{Y} - \overline{\mathbf{Z}}\hat{\bar{\mathbf{a}}})'(\mathbf{Y} - \overline{\mathbf{Z}}\hat{\bar{\mathbf{a}}})}{N - (p+2)}$$

Note: Only one explanatory variable X with p - th order polynomialof q lags is considered here.

The original coefficients $\beta_0, \beta_1, \beta_2, \ldots, \beta_q$ are obtained from: $\hat{\boldsymbol{\beta}} = \mathbf{H}\hat{\mathbf{a}}$ and

$\mathrm{Var}(\hat{\boldsymbol{\beta}}) = \mathbf{H}\mathrm{Var}(\hat{\mathbf{a}})\mathbf{H}' = \hat{\sigma}^2\mathbf{H}(\overline{\mathbf{Z}}'\overline{\mathbf{Z}})^{-1}\mathbf{H}'$

Granger Causality

Granger Causality Test

A combination of autoregressive and distributed lag models can be used to test the causality in Granger sense.

Does X Granger cause Y or Y Granger cause X?

Let X and Y are expressed in deviation forms, and denote:

$X \to Y$	X Granger cause Y.
$Y \to X$	Y Granger cause X.
$Y \leftrightarrow X$	X Granger cause Y and Y Granger cause X. (Feedback)

	Does X Cause Y?	*Does Y Cause X?*
The Model	$Y_i = \sum_{\ell=1}^{m} \alpha_\ell Y_{i-\ell} + \sum_{\ell=1}^{n} \beta_\ell X_{i-\ell} + \varepsilon_i$	$X_i = \sum_{\ell=1}^{m} a_\ell Y_{i-\ell} + \sum_{\ell=1}^{n} b_\ell X_{i-\ell} + e_i$
Hypothesis	$\begin{cases} H_0: \text{X does not cause Y} \\ H_1: \text{X causes Y} \end{cases}$	$\begin{cases} H_0: \text{Y does not cause X} \\ H_1: \text{Y causes X} \end{cases}$
Restricted Model	$Y_i = \sum_{\ell=1}^{m} \alpha_\ell Y_{i-\ell} + \varepsilon_i$	$X_i = \sum_{\ell=1}^{n} b_\ell X_{i-\ell} + e_i$
Test Statistic	$F = \frac{(RSS_R - RSS_{UR})/n}{RSS_{UR}/(N-m-n)} \sim F(n, N-m-n)$	$F = \frac{(RSS_R - RSS_{UR})/m}{RSS_{UR}/(N-m-n)} \sim F(m, N-m-n)$
Granger Causality Test	If $F \geq F_c(n, N-m-n)$ then reject H_0. That is, X does cause Y. Otherwise, X does not cause Y.	If $F \geq F_c(m, N-m-n)$ then reject H_0. That is, Y does cause X. Otherwise, Y does not cause X.

Conclusion

$X \to Y$	Reject H_0	Not reject H_0
$Y \to X$	Not reject H_0	Reject H_0
$Y \leftrightarrow X$	Reject H_0	Reject H_0