Appendix C

IVPs: practice with a simple oscillator

C.1 overview

Cycles emerge in natural systems for many reasons. In some cases, for example seasonal temperature variations, the cycle is imposed by an external forcing (in this case, the inclination of Earth's spin axis together with Earth's orbit around the Sun). In other cases, for example predator-prey population dynamics, cycles arise internally, due to non-linearities in the relationships among components of a system. In still other cases, such as the El Niño Southern Oscillation (ENSO) or Pliocene and Pleistocene glaciations, an internal cycle may be excited by an external forcing and as a result have characteristics of both.

An **oscillation** is the repetitive variation of some quantity about a central value. The variation may be between two extreme states, as is the case with the swinging of a pendulum, or may have more complicated structure, as is the case with ENSO. The simplest oscillators simply respond to an initial forcing according to a law like Robert Hooke's (1635 to 1703) *Ut tensio, sic vis.* In the absence of a damping force, the sum of forces acting on the pendulum mass is always the same (put another way, the potential and kinetic energies always sum to the same total) and the oscillation persists without end.

For a pendulum weight subject to an initial displacement from the equilibrium position $\theta = \theta_o$ conservation of momentum (or of energy) yields

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta = 0 \tag{C.1}$$

in which g represents the acceleration due to gravity and L represents the length of the pendulum string. The equation may be linearized by making the small angle approximation $\sin\theta \approx \theta$.

The model may be improved by the addition of a damping force. This force is likely to go with the angular velocity ω of the pendulum

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

coefficient	value	units
g	9.81	m s^{-1}
L	100	m
c	0.02	
F_o	0.01	
f	1/50	

according to some constant so that

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{L}\sin\theta = 0 \tag{C.2}$$

Finally, we may wish to consider a driving force, a term that varies with the independent variable and may or may not go with the dependent variable. If the driving force is periodic, we may write

$$F_{driving}(t) = F_o \cos(2\pi f t)$$

in which F_o is the amplitude and f represents the frequency of the forcing. All together, the equation of motion becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta}{\mathrm{d}t} + \frac{g}{L}\sin\theta = F_{driving}(t) \tag{C.3}$$

in which we have taken care to be mindful of sign conventions. For our pendulum, $F_{driving}$ may be a torque applied at its pivot. In a climate problem, a term like $F_{driving}$ might represent a daily, seasonal, or longer cycle.

C.2 exercises

1. Suppose you wish to simulate the undamped oscillation of a pendulum with an initial displacement of 90°. Write equation C.1 so that it can be integrated numerically using the Euler single step scheme. Write an implementation algorithm and then create the corresponding model in Matlab. Use the values supplied in Table 1.

2. Modify your model to include a damping term and a driving term. Complete the third column in Table 1. Please create this model in a new Matlab script, that is, do not add it to the script you wrote for the first two questions. Make a figure showing both the forcing function and the oscillation of the pendulum. What happens to the frequency of the oscillation over time?

3. Modify the script you wrote in part 2 to simulate *two* oscillators, with initial values 180° out of phase. What happens over time?

C.3 concluding thoughts

Simple oscillator models constructed from coupled reduced-order equations have been used to explore a number of climate phenomena. For example, Battisti and Hirst (1989) derived a *delayed* *oscillator* equation for sea surface temperature in the eastern tropical Pacific, with the delay representing the travel time for Rossby and Kelvin waves across the basin. Ashkenazy and Tziperman (2004) used a simple model driven by feedbacks among sea ice cover, precipitation, and ice sheet size to generate glacial cycles excited by orbital (Milankovitch) forcing.

C.4 references

Battisti, D.S. and A. C. Hirst, 1989, Interannual Variability in a Tropical AtmosphereOcean Model: Influence of the Basic State, Ocean Geometry and Nonlinearity, *Journal of Atmospheric Science*, 46, 1687-1712.

Ashkenazy, Y. and E. Tziperman, 2004, Are the 41 kyr glacial oscillations a linear response to Milankovitch forcing?, Quaternary Science Reviews, 23, 1879-1890.