THE NON-VANISHING OF THE TRACE OF T_3

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Abstract. A generalized Lehmer conjecture predicts that, for every positive integer n, the trace of the Hecke operator T_n in level one does not vanish, unless the space of cusp forms acted upon is trivial. So far, this has only been established for $n = 2$. In this paper, we use p-adic methods to prove the statement for $n = 3$.

1. Introduction

Consider the Delta function

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n, \quad (q = \exp(2\pi i z))$$

which is the unique normalized cusp form of weight 12 for the full modular group. A famous open question, due to Lehmer [Leh47], asks whether there exists an integer $n \geq 1$ such that $\tau(n)$ is zero. It is widely believed not only that Δ has no vanishing Fourier coefficients, but also that the same phenomenon occurs whenever the space of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ is one-dimensional, i.e., the weight $2k$ is in the set $\{12, 16, 18, 20, 22, 26\}$. For these values, the n-th Fourier coefficient of the corresponding normalized cusp form coincides with the trace of the Hecke operator T_n acting on the space of cusp forms of weight $2k$ and level one, which will be denoted by $\mathrm{Tr}\ T_n(2k)$.

Motivated by the above, Rouse [Rou06] proposed a "Generalized Lehmer Conjecture", which posits that the trace $\mathrm{Tr}\ T_n(2k)$ does not vanish for *all* even weights $2k \geq 16$ or $2k = 12$. In fact, [Rou06, Conjecture 1.5] predicts the non-vanishing of the trace of T_n, with $n \geq 1$ not a square, in every level N coprime to n. As evidence, Rouse proved the case $n = 2$ using a computational algorithm.

In this paper, we make further progress on the Generalized Lehmer Conjecture by proving it for $n = 3$ and level one.

Theorem 1. *Suppose that* $2k \geq 16$ *or* $2k = 12$. *Then* $\mathrm{Tr}\ T_3(2k) \neq 0$.

We mention that the analogous statement for $n = 2$ also follows from a recent paper of Chiriac and Jorza [CJ22], where a stronger result was obtained—namely that $\mathrm{Tr}\ T_2(2k)$ takes no repeated values, except for 0, which occurs only when the space is trivial. Their proof combines 2-adic results from [CJ21] and a new application of classical bounds on linear forms in logarithms in the context of exponential sums with more than two terms. The case $n = 3$ appears to be more difficult because of the presence of additional terms in the exponential sums defining the trace.

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We take our lead from [CJ22] and use the Eichler-Selberg trace formula to express the trace of T_3 in terms of linear recurrence sequences. This reduces the problem to proving the non-vanishing of three subsequences, according to some congruence classes of k (see Proposition 3). To this end, we employ a general practical method, based on p-adic arguments, developed by Mignotte and Tzanakis [MT91]. This method has found applications in the study of ternary sequences, particularly Berstel's sequence [MT93]. The idea is that given an equation of the form $u_n = c$ and a conjectured set $\mathcal{M}$ of solutions $n \in \mathbb{Z}$, a suitable choice of primes possessing certain properties can guarantee that $\mathcal{M}$ does indeed include all solutions; we elaborate on the specifics in Section 4.

2. Background

In this section, we briefly review some basic facts from the theory of modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$, that is, of level one. For a more comprehensive account, the reader is referred to [Kil15].

Let $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half-plane. A modular form of even weight $2k$ and level one is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ with the following properties:

(i) $f(z+1) = f(z)$ and $f(-1/z) = z^{2k} f(z)$ for all $z \in \mathfrak{h}$;
(ii) $f(z)$ is bounded as $\mathrm{Im}(z) \to \infty$.

Every modular form f admits a unique Fourier expansion $f(z) = \sum_{n\geq 0} a(n)q^n$, and if $a(0) = 0$ we call f a cusp form. In addition, a cusp form f is normalized if $a(1) = 1$.

Denote by $\mathcal{S}_{2k}$ the set of cusp forms of weight $2k$ and level one. This is a vector space over $\mathbb{C}$ of finite dimension d, namely

$$d = \begin{cases} \lfloor k/6 \rfloor - 1 \text{ if } k \equiv 1 \pmod 6 \\ \lfloor k/6 \rfloor \text{ otherwise.} \end{cases}$$

The space $\mathcal{S}_{2k}$ is endowed with the action of certain linear transformations, called Hecke operators. These can be defined, for all positive integer m, as linear maps $T_m : \mathcal{S}_{2k} \to \mathcal{S}_{2k}$ given by

$$T_m\left(\sum_{n\geq 1} a(n)q^n\right) = \sum_{n\geq 1}\left(\sum_{d\mid \gcd(m,n)} d^{2k-1} a(mn/d^2)\right) q^n.$$

If p is a prime number, the effect of T_p on $f(z) = \sum_{n\geq 1} a(n)q^n$ can be described as

$$T_p f = \sum_{n\geq 1} \left(a(pn) + p^{2k-1} a(n/p)\right) q^n,$$

with the understanding that $a(n/p) = 0$ whenever $p \nmid n$. It turns out that the characteristic polynomial of T_m has integer coefficients. In particular, its trace $\mathrm{Tr}\ T_m(2k)$ is also an integer.

A common way to compute traces of Hecke operators is using the Eichler-Selberg trace formula. This involves the Hurwitz class number $H(n)$, which counts the weighted number of equivalence classes of positive definite binary quadratic forms of discriminant $-n$. More precisely, the class containing x^2+y^2 is weighted by $1/2$, and the class containing x^2+xy+y^2 is weighted by $1/3$. For instance, $H(3) = 1/3$ and $H(12) = 4/3$, whereas $H(8) = H(11) = 1$. We also set $H(n) = 0$ if $n \equiv 1$ or $2 \pmod 4$, and $H(0) = -1/12$.

Following Zagier's Appendix to [Lan76], we recall the following version of the Eichler-Selberg trace formula on $\mathrm{SL}_2(\mathbb{Z})$: for all integers $m \geq 1$ and $k \geq 2$ we have

$$\mathrm{Tr}\ T_m(2k) = -\frac{1}{2} \sum_{|t| \leq 2\sqrt{m}} P_{2k}(t,m) H(4m - t^2) - \frac{1}{2} \sum_{dd'=m} \min(d, d')^{2k-1},$$

where $P_{2k}(t, m)$ is the coefficient of x^{2k-2} in the power series expansion of $(1-tx+mx^2)^{-1}$. It is not hard to verify (see, for example, the proof of [CJ22, Lemma 3]) that $P_{2k}(t, m)$ satisfies the following combinatorial formula:

$$P_{2k}(t,m) = \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} m^j t^{2k-2-2j}. \tag{1}$$

Since our main interest is in the case $m = 3$, we compute

$$\begin{aligned} \mathrm{Tr}\ T_3(2k) &= -\frac{1}{2} P_{2k}(0,3)H(12) - P_{2k}(1,3)H(11) - P_{2k}(2,3)H(8) - P_{2k}(3,3)H(3) - 1 \\ &= -\frac{2}{3} P_{2k}(0,3) - P_{2k}(1,3) - P_{2k}(2,3) - \frac{1}{3} P_{2k}(3,3) - 1, \end{aligned}$$

which combined with (1) gives

$$\mathrm{Tr}\ T_3(2k) = -1 - 2 \cdot (-3)^{k-2} - \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^j (1 + 2^{2k-2-2j} + 3^{2k-3-2j}). \tag{2}$$

3. The trace as a sum of recurrent sequences

Our immediate goal is to manipulate identity (2) in order to obtain certain recurrent sequences. The following lemma summarizes the calculations needed.

Lemma 2. *For any $u, v \in \mathbb{R} \setminus \{0\}$, consider the sequence $\{\alpha_n\}_{n \geq 0}$ defined as*

$$\alpha_n := \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} u^j v^{n-j}.$$

Then for every integer $n \geq 2$ we have the recurrence relation

$$\alpha_n = (v - 2u)\alpha_{n-1} - u^2 \alpha_{n-2},$$

where $\alpha_0 = 1$ and $\alpha_1 = v - u$.

Proof. The generating function of the sum given by the right-hand side is

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} u^j v^{n-j} x^n.$$

Setting $m = n - j$ and changing variables gives us

$$\sum_{m=0}^{\infty}\sum_{j=0}^{n}(-1)^j\binom{2m+j}{j}u^jv^mx^jx^m = \sum_{m=0}^{\infty}(vx)^m\sum_{j=0}^{\infty}(-1)^j\binom{(2m+1)+j-1}{j}(ux)^j$$

$$= \sum_{m=0}^{\infty}(vx)^m\frac{1}{(1+ux)^{2m+1}} \tag{3}$$

$$= \frac{1}{1+ux}\sum_{m=0}^{\infty}\left(\frac{vx}{(1+ux)^2}\right)^m$$

$$= \frac{1}{1+ux}\left(1-\frac{vx}{(1+ux)^2}\right)^{-1} \tag{4}$$

$$= \frac{1}{1+ux}\left(\frac{1+2ux+u^2x^2-vx}{(1+ux)^2}\right)^{-1}$$

$$= \frac{1+ux}{1+(2u-v)x+u^2x^2}$$

where in (3) we have used the negative binomial series

$$(1+x)^{-d} = \sum_{j=0}^{\infty}(-1)^j\binom{d+j-1}{j}x^j,$$

and in (4) used the formula for the sum of a geometric series. Now, let $G(x) = \sum_{n\geq 0}\alpha_n x^n$ be the generating function of the sequence $\{\alpha_n\}_{n\geq 0}$. As

$$\alpha_n + (2u-v)\alpha_{n-1} + u^2\alpha_{n-2} = 0$$

for $n \geq 2$, it follows that $G(x)$ satisfies

$$(G(x) - 1 + (u-v)x) + (2u-v)x(G(x)-1) + u^2x^2G(x) = 0.$$

This further shows that

$$G(x) = \frac{1+ux}{1+(2u-v)x+u^2x^2}.$$

The generating functions of both sequences are the same, so the statement of the lemma follows.

□

Using Lemma 2, we are now prepared to prove the main result of this section.

Proposition 3. *Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be the sequences given by the recurrences*

$$a_n = -5a_{n-1} - 9a_{n-2} \text{ for } n \geq 2, \quad a_0 = 1, \quad a_1 = -2$$

and

$$b_n = -2b_{n-1} - 9b_{n-2} \text{ for } n \geq 2, \quad b_0 = 1, \quad b_1 = 1,$$

respectively. Then for all integers $k \geq 2$, we have that

$$\text{Tr } T_3(2k) = \begin{cases} -1 - a_{k-1} - b_{k-1} & \text{if } k \equiv 2 \ (mod\ 3) \\ -1 - a_{k-1} - b_{k-1} - 3^{k-1} & \text{if } k \equiv 1, 3 \ (mod\ 6) \\ -1 - a_{k-1} - b_{k-1} + 3^{k-1} & \text{if } k \equiv 0, 4 \ (mod\ 6) \end{cases}.$$

Proof. In view of identity (2), we can write Tr $T_3(2k) = -1 - S_1 - S_2 - S_3$, where

$$S_1 = \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^j$$

$$S_2 = \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^j 2^{2k-2j-2}$$

$$S_3 = 2 \cdot (-3)^{k-2} + \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^{2k-3-j}.$$

Setting $u = 3$ and $v = 1$ in Lemma 2, we see that S_1 is equal to a_{k-1}. Similarly, $u = 3$ and $v = 4$ give that S_2 is equal to b_{k-1}. It remains to show that

$$S_3 = \begin{cases} 0 & \text{if } k \equiv 2 \pmod 3 \\ -3^{k-1} & \text{if } k \equiv 1, 3 \pmod 6 \\ 3^{k-1} & \text{if } k \equiv 0, 4 \pmod 6. \end{cases}$$

Equivalently, for every $n \geq 1$, we must prove that

$$2(-3)^{n-1} - \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{2n-j-1} = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 3 \\ -3^n & \text{if } n \equiv 0, 2 \pmod 6 \\ 3^n & \text{if } n \equiv 3, 5 \pmod 6 \end{cases}.$$

To do this, we introduce an auxiliary sequence defined as

$$d_n := \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j}.$$

Applying Lemma 2 with $u = 1$ and $v = 3$, we see that $\{d_n\}$ is a sequence with initial values $d_0 = 1$ and $d_1 = 2$, and $d_n = d_{n-1} - d_{n-2}$ for $n \geq 2$. Induction verifies that $\{d_n\}$ is a periodic sequence of period 6, and

$$\sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j} = \begin{cases} 2 & \text{if } n \equiv 1 \pmod 6 \\ -2 & \text{if } n \equiv 4 \pmod 6 \\ 1 & \text{if } n \equiv 0, 2 \pmod 6 \\ -1 & \text{if } n \equiv 3, 5 \pmod 6. \end{cases}$$

Adding $2(-1)^n$ to both sides yields

$$2(-1)^n + \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j} = \begin{cases} 0 & \text{if } n \equiv 1, 4 \pmod 6 \\ 3 & \text{if } n \equiv 0, 2 \pmod 6 \\ -3 & \text{if } n \equiv 3, 5 \pmod 6 \end{cases},$$

and multiplying by -3^{n-1} gives the desired result.

□

As Tr T_3 is the sum of recurrent sequences, it is also a recurrent sequence. The following corollary makes this observation explicit.

Corollary 4. *Let $\{t_k\}_{k\geq 2}$ be the sequence defined as $t_k := \text{Tr } T_3(2k)$. Then $\{t_k\}$ has initial values given by the table below*

k	2	3	4	5	6	7	8	9
t_k	0	0	0	0	252	0	-3348	-4284

and for every $k \geq 10$ it satisfies the recurrence relation

$$t_k = -6t_{k-1} - 21t_{k-2} - 62t_{k-3} - 180t_{k-4} - 486t_{k-5} - 945t_{k-6} - 486_{k-7} + 2187t_{k-8}.$$

Proof. It is easy to compute the initial values directly from Proposition 3. The characteristic polynomial of $\{t_k\}$, denoted by $t(x)$, is the product of the characteristic polynomials of the individual sequences that comprise Tr $T_3(2k)$.

Obviously, one can regard the constant -1 as a trivial recurrent sequence with characteristic polynomial $x-1$. The characteristic polynomials of $\{a_n\}$ and $\{b_n\}$ are x^2+5x+9 and x^2+2x+9, respectively. Finally, let $\{c_n\}$ be the sequence satisfying $c_n = -27c_{n-3}$ for $n \geq 3$ with the initial conditions $c_0 = -1, c_1 = 0$, and $c_2 = -9$; its characteristic polynomial is x^3+27. Using induction, one can show that

$$c_n = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 3 \\ -3^n & \text{if } n \equiv 0, 2 \pmod 6 \\ 3^n & \text{if } n \equiv 3, 5 \pmod 6 \end{cases}$$

for all $n \geq 0$. We obtain that

$$\begin{aligned} t(x) &= (x-1)(x^2+5x+9)(x^2+2x+9)(x^3+27) \\ &= x^8 + 6x^7 + 21x^6 + 62x^5 + 180x^4 + 486x^3 + 945x^2 + 486x - 2187 \end{aligned}$$

and the conclusion follows.

□

4. The non-vanishing of the trace

With a_n and b_n as they appear in Proposition 3, let $u_n = a_n + b_n$. We know that

$$\text{Tr } T_3(2k) = \begin{cases} -1 - u_{k-1} & \text{if } k \equiv 2 \pmod 3 \\ -1 - u_{k-1} - 3^{k-1} & \text{if } k \equiv 1, 3 \pmod 6 \\ -1 - u_{k-1} + 3^{k-1} & \text{if } k \equiv 0, 4 \pmod 6 \end{cases}.$$

To establish Theorem 1, we must prove that Tr $T_3(2k) = 0$ if and only if $k \in \{2, 3, 4, 5, 7\}$. In fact, we will completely determine when the sequences $\{u_n\}$ and $\{u_n \pm 3^n\}$ take the value -1. Our main tool is the previously mentioned result of Mignotte and Tzanakis. Their setup works for any k-th degree recurrence $\{u_n\}$ with integer coefficients, as long as its characteristic polynomial $g(x)$ has k *distinct* complex roots $\omega_1, \ldots, \omega_k$.

Given a fixed integer c, to solve the equation $u_n = c$ (for n), we choose an odd prime p, not dividing the discriminant or any of the coefficients of $g(x)$, such that all the roots ω_i are p-adic units. We then search for positive integers S such that all the numbers ω_i^S are congruent to some common integer A modulo p. If A has the same order R both modulo p and modulo p^2, then the following result [MT91, Theorem 1] holds[1].

[1]There is a typo in the statement of [MT91, Theorem 1], namely the part "if $n \in \mathcal{P}$" in (ii) is omitted. The correct version appears in [MT93, Theorem 1].

Proposition 5 (Mignotte and Tzanakis)**.** *Suppose that $\mathcal{M}$ is a finite set of solutions $m \in \mathbb{Z}$ to the equation $u_m = c$, where either $c \not\equiv 0 \pmod p$ or $c = 0$. Let $\mathcal{P}$ be a complete system of residues modulo S such that $\mathcal{M} \subseteq \mathcal{P}$ and which satisfies the following conditions:*

(i) $u_m = c$ for each $m \in \mathcal{M}$;
(ii) if $n \in \mathcal{P}$ and $u_n \equiv cA^r \pmod p$ for some $r \in \{0, 1, \ldots, R-1\}$, then $n \in \mathcal{M}$;
(iii) $u_{m+S} \not\equiv Au_m \pmod{p^2}$ for every $m \in \mathcal{M}$.

Then $u_n = c$ implies $n \in \mathcal{M}$.

It is important to emphasize that $\mathcal{M}$ contains *all* integer solutions, not just positive integers. To extend the definition of u_n to negative integers is straightforward. Indeed, since the characteristic polynomial of $\{u_n\}$ is assumed to have distinct roots, the general term is of the form

$$u_n = \alpha_1 \omega_1^n + \ldots + \alpha_k \omega_k^n,$$

with $\alpha_i \in \mathbb{Q}(\omega_1, \ldots, \omega_k)$. Therefore, it makes sense to talk about u_n for every integer n. This will play a role in the proof of Proposition 6 below.

While directly tackling the sequence from Corollary 4 with this method is certainly possible, we take a more gradual approach. This has the advantage of better illustrating the "dead-ends" one can run into when searching for appropriate choices of p and S. To perform this search, we used a combination of SageMath and Pari/GP.

Proposition 6. *Let $u_n = a_n + b_n$ where $\{a_n\}$ and $\{b_n\}$ are the sequences given in Proposition 3. Extend the definition of $\{u_n\}$ to all $n \in \mathbb{Z}$ as described above. Then*

(a) $u_n = -1$ if and only if $n \in \{1, 4\}$;
(b) $u_n + 3^n = -1$ if and only if $n \in \{2, 6\}$;
(c) $u_n - 3^n = -1$ if and only if $n \in \{-1, 3\}$.

Proof. We begin by including a table of the first few values of the relevant sequences, with the occurrences of the value -1 circled:

n	-1	0	1	2	3	4	5	6
u_n	$-2/3$	2	(-1)	-10	26	(-1)	-10	-730
$u_n + 3^n$	$-1/3$	3	2	(-1)	53	80	233	(-1)
$u_n - 3^n$	(-1)	1	-4	-19	(-1)	-82	-253	-1459

We also note that the characteristic polynomial of the sequence $\{u_n\}$ is

$$g(x) = (x^2 + 2x + 9)(x^2 + 5x + 9) = x^4 + 7x^3 + 28x^2 + 63x + 81$$

and its discriminant is $2^5 \cdot 3^8 \cdot 11$.

(a) We choose $p = 59$. The roots ω_1 and ω_2 of $x^2 + 2x + 9$, written 59-adically, are

$$12 + 43 \cdot 59 + 28 \cdot 59^2 + O(59^3)$$

and

$$45 + 15 \cdot 59 + 30 \cdot 59^2 + O(59^3).$$

Since $\left(\frac{12}{59}\right) = \left(\frac{45}{59}\right) = 1$, we see that $\omega_1^{29} \equiv \omega_2^{29} \equiv 1 \pmod{59}$.

Similarly, the roots ω_3 and ω_4 of x^2+5x+9, written 59-adically, are $5 + 55 \cdot 59 + 57 \cdot 59^2 + O(59^3)$ and $49 + 3 \cdot 59 + 59^2 + O(59^3)$. As before, $\omega_3^{29} \equiv \omega_4^{29} \equiv 1 \pmod{59}$. Thus, all the roots of $g(x)$ satisfy $\omega_i^{29} \equiv 1 \pmod{59}$.

We now apply Proposition 5 with $p = 59$, $S = 29$, $A = 1$ (so $R = 1$), $c = -1$, $\mathcal{M} = \{1, 4\}$ and $\mathcal{P} = \{0, \ldots, 28\}$. Condition (i) is clear. Next, a simple computer check shows that the only n in the range $0 \le n \le 28$ for which $u_n \equiv -1 \pmod{59}$ are $n = 1$ and $n = 4$.

For requirement (iii), we compute

$$u_{1+S} \equiv 707 \not\equiv u_1 \pmod{59^2}$$
$$u_{4+S} \equiv 766 \not\equiv u_4 \pmod{59^2}.$$

In conclusion, the elements of $\mathcal{M}$ are the only integers n such that $u_n = -1$.

(b) For convenience, let $u'_n := u_n + 3^n$. We claim that the set of solutions to $u'_n = -1$ is $\mathcal{M} = \{2, 6\}$.

The characteristic polynomial of u'_n is $g(x)(x-3) = x^5 + 4x^4 + 7x^3 - 21x^2 - 108x - 243$, which has discriminant $2^{11} \cdot 3^{12} \cdot 11^3$. While $3^{29} \equiv 1 \pmod{59}$, we note that the choice from part (a): $(p, S, A) = (59, 29, 1)$, will not work. Indeed, requirement (ii) is not satisfied, for

$$u'_{24} = 326954692403 \equiv -1 \pmod{59}$$

even though $24 \notin \mathcal{M}$.

Fortunately, there are other relatively small values of p which will work. More precisely, we take $p = 251$. The roots of $g(x)(x-3)$ reduced modulo 251 are 3, 45, 68, 181, and 201, and

$$3^{125} \equiv 45^{125} \equiv 68^{125} \equiv 181^{125} \equiv 201^{125} \equiv 1 \pmod{251}.$$

Therefore $(p, S, A) = (251, 125, 1)$ is a valid triple. Once again, one can use software to verify that requirement (ii) is met for the choice $\mathcal{P} = \{0, \ldots, 124\}$. For (iii), we find that

$$u'_{2+S} \equiv 24597 \not\equiv u'_2 \pmod{251^2}$$
$$u'_{6+S} \equiv 34386 \not\equiv u'_6 \pmod{251^2}.$$

Proposition 5 tells us that $u'_n = u_n + 3^n = -1$ only when $n \in \{2, 6\}$.

(c) Now let $u''_n := u_n - 3^n$. This case is different from the previous ones because it is the first time that we encounter a negative solution, namely

$$u''_{-1} = u_{-1} - 3^{-1} = (-2/3) - (1/3) = -1.$$

As a result, we take $\mathcal{M} = \{-1, 3\}$. To accommodate for the negative value in $\mathcal{M}$, we let $\mathcal{P} = \{-1, \ldots, 27\}$.

The characteristic polynomial of u''_n is also $g(x)(x-3)$, so the choice $(p, S, A) = (59, 29, 1)$ passes the root requirement. As before, software verifies requirement (ii), and we see that

$$u''_{-1+S} \equiv 2418 \not\equiv u''_{-1} \pmod{59^2}$$
$$u''_{3+S} \equiv 3303 \not\equiv u''_3 \pmod{59^2}.$$

Applying Proposition 5, we obtain that $u''_n = -1$ only when $n \in \{-1, 3\}$.

□

5. Concluding Remarks

The methods used in this paper are amenable to generalization for larger values of n, as well as other congruence subgroups. For instance, one can similarly establish the non-vanishing of the trace of T_3 in level 2, denoted by $\mathrm{Tr}\ T_3(2k, \Gamma_0(2))$. Indeed, work by Frechette, Ono, and Papanikolas [FOP04, Theorem 2.3] gives that for all $k \geq 2$

$$\mathrm{Tr}\ T_3(2k, \Gamma_0(2)) = -2 - b_{k-1} - (-3)^{k-1},$$

where $\{b_n\}$ is the sequence from Proposition 3, namely $b_0 = b_1 = 1$ and $b_n = -2b_{n-1} - 9b_{n-2}$ for $n \geq 2$. Applying Proposition 5 with $u_n = b_n \pm 3^n$ and $(p, S, A) = (11, 5, 1)$ we find that the only zeros occur for $2k \in \{4, 6\}$, which is precisely when the space of weight-$2k$ cusp forms on $\Gamma_0(2)$ (of dimension $\lfloor k/2 \rfloor - 1$) is trivial.

We also remark that in the case of level 4 or level 8, the situation is even easier, for

$$\mathrm{Tr}\ T_3(2k, \Gamma_0(4)) = -3 - (-3)^{k-1}$$

and

$$\mathrm{Tr}\ T_3(2k, \Gamma_0(8)) = -4,$$

as can be seen from [FOP04, Proposition 2.1].

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