# THE AVERAGE NUMBER OF DIVISORS IN CERTAIN ARITHMETIC SEQUENCES

LIUBOMIR CHIRIAC

Abstract. In this paper we study the sum $\sum_{p\leq x}\tau(n_p)$, where $\tau(n)$ denotes the number of divisors of $n$, and $\{n_p\}$ is a sequence of integers indexed by primes. Under certain assumptions we show that the aforementioned sum is $\ll x$ as $x\to\infty$. As an application, we consider the case where the sequence is given by the Fourier coefficients of a modular form.

*Keywords and phrases*: divisor function, Fourier coefficients, modular forms.

## 1. Introduction

Starting with the early work of Bellman and Shapiro, Erdős, and Hooley in the 1950s, there has been an interest in estimating the average number of divisors over polynomial values. Given an irreducible polynomial $F(x)\in\mathbb{Z}[x]$, a classical result of Erdős [Erd52] asserts that

$$x\log x \ll \sum_{n\leq x}\tau(F(n)) \ll x\log x, \qquad \text{as } x\to\infty. \tag{1}$$

In the case of a quadratic polynomial $F$, this can be strengthened to give an asymptotic formula of the form $\sum_{n\leq x}\tau(F(n))\sim\lambda x\log x$ for some constant $\lambda$ depending on $F$ (see [McK95] and [McK99] for an expression of $\lambda$ in terms of Hurwitz class numbers). No such results have been shown for polynomials of higher degree.

Another averaging result mentioned in [Erd52], this time over primes $p$, is that

$$\sum_{p\leq x}\tau(F(p)) \ll x. \tag{2}$$

When $F(x)=x+a$, $a\neq 0$, we obtain the Titchmarsh divisor problem, which is concerned with the average number of divisors over shifted primes. As it is well known, one has again an asymptotic formula $\sum_{p\leq x}\tau(p+a)\sim Cx$, for some explicit constant $C$ depending on $a$.

Motivated by the Titchmarsh divisor problem, one can study the behavior of

$$\sum_{p\leq x}\tau(n_p), \tag{3}$$

where the $n_p$'s are quantities of arithmetic significance, such as natural invariants associated to objects arising from arithmetic geometry. In this direction, Akbary and Ghioca [AG12] examined a version of geometric flavor, in the context of abelian varieties.

Another notable result is due to Pollack [Pol16b], who investigated the average number of divisors of $|E(\mathbb{F}_p)|$, for elliptic curves $E$ over $\mathbb{Q}$. Pollack's strategy is based on a refinement of Erdős's original idea from [Erd52], which is also outlined in Elsholtz and Tao [ET13, Section 7]. A remark in [Pol16b] suggests that Erdős's method is amenable to the sum in (3)

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when the $n_p$'s are given by the Fourier coefficients of a cuspidal eigenform without complex multiplication (non-CM).

The purpose of this note is to supply a proof of the above remark. In fact, we consider a somewhat more general setting that includes sequences $\{n_p\}$ of integers indexed by prime numbers, subject to two assumptions:

- (H1) There exists a fixed positive integer $k$ such that $|n_p| \leq p^k$ for all primes $p$.
- (H2) There exists a fixed number $c \in (0,1)$ such that

$$\#\{p \leq x \mid n_p \neq 0 \text{ and } n_p \equiv 0 \pmod{m}\} \ll \frac{\pi(x)}{\varphi(m)},$$

holds uniformly for all positive integers $m \leq x^c$, and the implied constant depends only on the sequence. As usual, $\pi(x) = \sum_{p \leq x} 1$ and $\varphi$ is Euler's totient function.

**Theorem 1.1.** *Let $\{n_p\}$ be a sequence of integers indexed by prime numbers, for which (H1) and (H2) hold. Then*

$$\sum_{\substack{p \leq x \\ n_p \neq 0}} \tau(n_p) \ll x \quad \text{as } x \to \infty.$$

It is important to recognize that another paper of Pollack [Pol16a] gives, under some assumptions, an upper bound for sums of the form $\sum_{n \leq x} a_n \tau_r(n)$, where $\{a_n\}$ is a sequence of nonnegative real numbers and $\tau_r(n)$ is the $r$-fold divisor function. In principle, it should be possible to recover Theorem 1.1 by following the line of reasoning from [Pol16b], with some modifications. Nonetheless, we have included an argument in order to give a complete proof of the following result, which is our main focus.

**Corollary 1.2.** *Let $f = \sum_{n \geq 1} a_n q^n$ be a non-CM newform of weight $k \geq 2$ with integer Fourier coefficients. Then under GRH we have that*

$$\sum_{\substack{p \leq x \\ a_p \neq 0}} \tau(a_p) \ll x \quad \text{as } x \to \infty. \tag{4}$$

Corollary (1.2) improves an estimate of Gun and Murty [GM14], who showed that

$$x \ll \sum_{\substack{p \leq x \\ a_p \neq 0}} \tau(a_p) \ll x(\log x)^A, \tag{5}$$

for some absolute constant $A$ depending on $f$. The assumption on the Generalized Riemann Hypothesis (GRH) is required, as in [GM14], for the use of an effective version of the Chebotarev density theorem, which allows for the verification of (H2). Combining (4) with the lower bound from (5) it follows that, conditional on GRH, the order of magnitude for the divisor sum over Fourier coefficients is $x$. In the spirit of the Titchmarsh divisor problem, it would not be unreasonable to expect that this sum is in fact asymptotic to $Cx$, for some constant $C$ depending on $f$. However, establishing this asymptotic relation may be beyond the capabilities of the current methods, even under GRH.

## 2. Preliminaries

In this section, we briefly recall some key facts about smooth numbers, which are then used to establish two technical results that will play a role in the proof of Theorem 1.1.

For $X \geq Y \geq 2$, denote by $S(X,Y)$ the set of all positive integers not exceeding $X$ and free of any prime divisors larger than $Y$. These numbers are sometimes referred to as being $Y$-smooth, or $Y$-friable. Also denote by $\Psi(X,Y)$ the cardinality of $S(X,Y)$, and by $u$ the ratio $\log X/\log Y$.

In the range

$$1 \leq u \leq (1-\varepsilon)\frac{\log x}{\log\log x}$$

with $\varepsilon > 0$ being fixed, it is known that

$$\Psi(X,Y) \ll X \exp\left(\frac{-u\log u}{2}\right). \tag{6}$$

This follows, for example, from the Corollary in [CEP83, Page 15].

Another standard result [Mor13, Page 790] is that

$$\Psi(X,(\log X)^{\alpha}) = X^{1-1/\alpha+O(1/\log\log X)} \tag{7}$$

for any fixed $\alpha > 1$.

We now use the estimates (6) and (7) to sum over certain smooth numbers the functions $1/n$ and $1/\varphi(n)$, respectively.

**Lemma 2.1.** *Assume $c \in (0,1)$ is a constant. If $r \geq 1/c$ is a large enough integer in the range $x^{1/r} > (\log x)^2$, then letting $x \to \infty$ we have*

$$\sum_{d\in S(x^c,x^{1/r}),\ d\geq x^{c/4}} \frac{1}{d} \ll (\log x)\exp\left(-\frac{cr\log r}{8}\right).$$

*If $x^{1/r} \leq (\log x)^2$ then the above sum is $\ll x^{-\delta}$ for some $\delta > 0$.*

*Proof.* Note that

$$\begin{aligned}\sum_{d\in S(x^c,x^{1/r}),\ d\geq x^{c/4}} \frac{1}{d} &= \int_{x^{c/4}}^{x^c} \frac{d\Psi(t,x^{1/r})}{t} \\ &= \frac{\Psi(x^c,x^{1/r})}{x^c} - \frac{\Psi(x^{c/4},x^{1/r})}{x^{c/4}} + \int_{x^{c/4}}^{x^c} \frac{\Psi(t,x^{1/r})}{t^2}dt.\end{aligned}$$

If $x^{1/r} > (\log x)^2$ then by (6) we get

$$\Psi(t,x^{1/r}) \ll t\exp\left(-\frac{cr\log r}{8}\right),$$

and, as a result, the required sum of $1/d$ is bounded by $(\log x)\exp(-cr\log r/8)$.

Now assume that $x^{1/r} \leq (\log x)^2$. We also suppose that $x$ is large enough so that $\log x \geq (4/c)^4$. Then $(\log t)^{8/3} \geq (\log x)^2$ whenever $t \geq x^{c/4}$. Using (7) we get

$$\Psi(t,x^{1/r}) \leq \Psi(t,(\log t)^{8/3}) = t^{5/8+O(1/\log\log t)},$$

which is eventually bounded by $t^{3/4}$, as $t$ tends to infinity (with $x$). It follows that the required sum of $1/d$ is bounded by some negative power of $x$.

□

**Lemma 2.2.** *With the notation from Lemma 2.1, we have*

$$\sum_{d\in S(x^c,x^{1/r}),\ d\geq x^{c/4}} \frac{1}{\varphi(d)} \ll (\log x)\exp\left(-\frac{cr\log r}{16}\right)$$

*if* $x^{1/r} > (\log x)^2$. *Otherwise, the sum is* $\ll x^{-\delta}(\log\log x)$ *for some* $\delta > 0$.

*Proof.* For the first statement, we use the Cauchy-Schwarz inequality

$$\left(\sum_{d\in S(x^c,x^{1/r}),\ d\geq x^{c/4}} \frac{1}{\phi(d)}\right)^2 \ll \left(\sum_{d\leq x^c} \frac{d}{\phi(d)^2}\right)\left(\sum_{d\in S(x^c,x^{1/r}),\ d\geq x^{c/4}} \frac{1}{d}\right).$$

Since $\sum_{d\leq X}\left(\frac{d}{\phi(d)}\right)^2 \ll X$, summation by parts gives

$$\sum_{d\leq X}\frac{d}{\phi(d)^2} = \sum_{d\leq X}\left(\frac{d}{\phi(d)}\right)^2\frac{1}{d} \ll \log X.$$

The conclusion now follows from Lemma 2.1. The second statement is also a consequence of Lemma 2.1 and the inequality

$$\frac{1}{\varphi(d)} \ll \frac{\log\log d}{d}.$$

□

## 3. Proof of the Theorem

Our goal is to show that

$$\sum_{\substack{p\leq x\\ n_p\neq 0}} \tau(n_p) \tag{8}$$

is $\ll x$ as $x\to\infty$. To this end, we will employ the method developed originally by Erdős in [Erd52], and refined by Elsholtz and Tao [ET13]. We follow closely an adaptation of this method due to Pollack [Pol16b] (see also [Pol16a]), who applied it in the context of elliptic curves.

For every nonzero term of the sequence $n_p$ we consider its prime factorization:

$$|n_p| = p_1\dots p_J, \tag{9}$$

where the prime factors are arranged in nondecreasing order, and repetitions are allowed. Pick the largest index $j\leq J$ such that

$$p_1\dots p_j \leq x^c. \tag{10}$$

If no such index exists (i.e., $j=0$), then all prime divisors of $n_p$ are greater than $x^c$, and so $\tau(n_p)=O(1)$ by the assumption (H1). The contribution of these terms $n_p$ towards (8) is trivially $\ll x$. Hence, without loss of generality, we may assume that $j\geq 1$.

Next we consider the terms $n_p$ for which the corresponding quantities $J$ and $j$ are close, that is $J - j = O(1)$. In this case, the submultiplicative property of the divisor function $\tau$, in combination with (10), implies that

$$\begin{aligned}\tau(n_p) &\leq \tau(p_1 \dots p_j)\tau(p_{j+1}) \dots \tau(p_J) \\ &\leq \tau(p_1 \dots p_j) 2^{O(1)} \\ &\ll \sum_{\substack{d \leq x^c \\ n_p \equiv 0 \pmod{d}}} 1.\end{aligned}$$

It follows that the contribution of all the primes of this type towards (8) is at most

$$\sum_{d \leq x^c} \#\{p \leq x \mid n_p \neq 0 \text{ and } n_p \equiv 0 \pmod{d}\},$$

which by (H2) is bounded above by

$$\pi(x) \sum_{d \leq x^c} \frac{1}{\phi(d)}. \tag{11}$$

It is known that $\sum_{d \leq X} 1/\phi(d) \ll \log X$; in fact, the following more precise estimate holds (see [SR85]):

$$\sum_{d \leq x} \frac{1}{\phi(d)} = \frac{315\zeta(3)}{2\pi^4} \left( \log x + \gamma - \sum_p \frac{\log p}{p^2 - p + 1} \right) + O\left( \frac{(\log x)^{2/3}}{x} \right),$$

where $\gamma$ is the Euler-Mascheroni constant. Therefore, the sum in (11) is $\ll x$, as desired.

We now turn to estimating the contribution of the coefficients $a_p$ for which the corresponding difference $J - j$ is not bounded above by an absolute constant. To be more specific, we will assume that $J - j \geq (2k+1)/c$.

If $p_{j+1} \geq x^{c/2}$, then the fact that the prime factors are non-decreasing implies

$$\begin{aligned}|n_p| &= (p_1 \dots p_j)(p_{j+1} \dots p_J) \\ &\geq p_{j+1} \dots p_J \\ &\geq x^{(J-j)c/2} \geq x^{(2k+1)/2},\end{aligned}$$

which contradicts (H1): $|n_p| \leq p^k \leq x^k$. Thus

$$p_{j+1} < x^{c/2}$$

and

$$p_1 \dots p_j > x^{c/2}, \tag{12}$$

for otherwise $p_1 \dots p_{j+1} < x^c$, so $j$ would no longer be the largest index in (10).

Define $r$ to be the positive integer such that

$$x^{1/(r+1)} \leq p_j < x^{1/r}. \tag{13}$$

Equation (13) shows that the quantity $p_1 \dots p_j$ is $x^{1/r}$-smooth, i.e., all its prime factors are at most $x^{1/r}$. This is where the results from Section 2 will come into play. As seen there, it will be convenient to treat two cases, depending on the range for $r$.

**Case 1:** $x^{1/r} > (\log x)^2$.

The prime factors $p_{j+1}, \dots p_J$ are all at least $x^{1/(r+1)}$, so their product is at least $x^{(J-j)/(r+1)}$. At the same time, this product does not exceed $|n_p| \le x^k$ (again, by (H1)). Therefore $J - j \le (r+1)k$, which gives

$$\tau(p_{j+1} \dots p_J) \le 2^{J-j} \le 2^{(r+1)k} \ll 2^{rk}$$

and as a result

$$\begin{aligned} \tau(n_p) &\ll \tau(p_1, \dots p_j) 2^{rk} \\ &= 2^{rk} \sum_{d | p_1, \dots p_j} 1. \end{aligned}$$

Using the square root trick, we can restrict this sum to divisors $d$ that are at least $(p_1 \dots p_j)^{1/2}$, which by (12) is at least $x^{c/4}$. Hence

$$\tau(n_p) \ll 2^{rk} \sum_{\substack{d \in S(x^c, x^{1/r}),\ d \ge x^{c/4} \\ n_p \equiv 0 \pmod d}} 1. \tag{14}$$

We obtain that the contribution towards (8) from these primes is

$$\ll \sum_r 2^{rk} \sum_{d \in S(x^c, x^{1/r}),\ d \ge x^{c/4}} \#\{p \le x \mid n_p \ne 0 \text{ and } n_p \equiv 0 \pmod d\}.$$

Assumption (H2) together with Lemma 2.2 shows that the previous sum is

$$\begin{aligned} &\ll \pi(x) \sum_r 2^{rk} \sum_{d \in S(x^c, x^{1/r}),\ d \ge x^{c/4}} \frac{1}{\phi(d)} \\ &\ll x \sum_r 2^{rk} \exp\left(-\frac{cr \log r}{16}\right). \end{aligned}$$

Since $\sum_r 2^{rk} \exp(-cr \log r/16)$ converges, we get that the expression above is $\ll x$.

**Case 2**: $x^{1/r} \le (\log x)^2$.

We start with the inequality

$$\tau(n) \ll \exp\left(\frac{\log n}{\log \log n}\right)$$

valid for all $n \ge 3$. This follows, for example, from the explicit estimation ([NR83])

$$\max_{n \ge 2} \frac{\log \tau(n) \log \log n}{\log 2 \log n} \approx 1.5379.$$

Now instead of (14), we obtain

$$\tau(n_p) \ll \exp\left(\frac{\log x}{\log \log x}\right) \sum_{\substack{d \in S(x^c, x^{1/r}),\ d \ge x^{c/4} \\ n_p \equiv 0 \pmod d}} 1.$$

Using Lemma 2.2 again, we see that the contribution towards (8) is

$$\ll \pi(x) \exp\left(\frac{\log x}{\log\log x}\right) x^{-\delta} \log\log(x) \sum_r 1,$$

for some $\delta > 0$. The fact that this expression is $\ll x$ follows immediately from (13), which ensures that $r \leq \log(x)/\log(2)$, and thus $\sum_r 1 \ll \log(x)$.

The proof of Theorem 1.1 is now complete.

## 4. Proof of the Corollary

We shall check that the two necessary assumptions of Theorem 1.1 are verified. Since it is clear that (H1) is implied by the Ramanujan bound: $|a_p| \leq 2p^{(k-1)/2}$, we only need to establish (H2). As we explain below, this is a consequence of an effective version of the Chebotarev density theorem.

Fix an integer $m \geq 1$. Let

$$f(z) = \sum_{n\geq 1} a_n q^n \quad (q = e^{2\pi i z})$$

be a non-CM newform of weight $k \geq 2$, level $N$ and character $\chi$, with integer Fourier coefficients. Associated to $f$ and $m$ is a Galois representation

$$\rho_m : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$$

with the property that if $p$ is a prime not dividing $mN$ then

$$\operatorname{tr} \rho_m(\mathrm{Frob}_p) \equiv a_p \pmod{m}$$

and

$$\det \rho_m(\mathrm{Frob}_p) \equiv \chi(p) p^{k-1} \pmod{m},$$

where $\mathrm{Frob}_p$ denotes a Frobenius element at $p$.

Let $K_m$ the subfield of $\overline{\mathbb{Q}}$ fixed by the kernel of $\rho_m$. Define

$$C_m := \{g \in Im(\rho_m) : \operatorname{tr}(g) \equiv 0 \pmod{m},$$

and put

$$\delta(m) = \frac{|C_m|}{|\operatorname{Gal}(K_m/\mathbb{Q})|}.$$

Note that $C_m$ is nonempty because it contains the image of complex conjugation. Moreover, by the Chebotarev density theorem

$$\#\{p \leq x : a_p \equiv 0 \pmod{m}\} \sim \delta(m)\pi(x).$$

We are interested in the subset where $a_p \neq 0$, namely

$$\pi(x, m) := \#\{p \leq x : a_p \neq 0 \text{ and } a_p \equiv 0 \pmod{m}\}.$$

Assuming GRH, an effective version of the Chebotarev Theorem (see [MM84, Lemma 5.3]) gives that for $x \geq 2$ we have

$$\pi(x, m) = \delta(m)\pi(x) + O(m^3 x^{1/2} \log(mNx)) + O(x^{3/4}).$$

As explained in [GM14, Page 235], for an upper bound on $\delta(m)$ one can use the quantity

$$\prod_{\ell^n \| m} \frac{\ell}{\ell^{n-1}(\ell^2-1)} \leq \frac{1}{m} \prod_{\ell | m} \frac{\ell}{\ell - 1} = \frac{1}{\varphi(m)}.$$

Therefore, under GRH, we obtain that the inequality

$$\pi(x, m) \ll \frac{\pi(x)}{\varphi(m)}$$

holds uniformly for all $m \leq x^{1/9}$, so $c = 1/9$ satisfies (H2).

The conclusion of the corollary is now clear from Theorem 1.1.

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Portland State University, Fariborz Maseeh Department of Mathematics and Statistics, Portland, OR 97201

*Email address*: `chiriac@pdx.edu`