ARITHMETIC ASPECTS OF THE TRACE OF T_2

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Abstract. We study arithmetic structure in the traces of the Hecke operator T_2. Our main results establish rigidity for arithmetic progressions in the trace sequence and classify the traces having the smallest possible prime support. The proofs combine local information with global Diophantine estimates to obtain effective classifications in both problems. Numerical evidence suggests that these results are part of a broader phenomenon linking exceptional additive relations among Hecke traces with unusually restricted multiplicative structure.

1. Introduction

Denote by t_k the trace of Hecke operator T_2 acting on the space S_{2k} of cusp forms of weight $2k \geq 12$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The purpose of this paper is to investigate additive and multiplicative properties of the sequence $(t_k)_{k\geq 6}$.

Since there are no repetitions among these traces, as proved in [CJ22], a natural next question is whether they contain arithmetic progressions. As a consequence of the trace formula, the sequence (t_k) satisfies a linear recurrence, placing the problem within the broader framework of linear equations in recurrence sequences initiated by Schlickewei and Schmidt [SS93] and further developed, in the context of arithmetic progressions, by Pinter and Ziegler [PZ12]. While these results show that infinite families of arithmetic progressions are highly exceptional, we pursue a complementary and effective direction. Specifically, we seek an explicit classification in the first anchored case, namely progressions of length at least three beginning with the first nonzero trace t_6. Our first main result establishes that only one such progression exists.

Theorem 1. *There is only one arithmetic progression among the traces $(t_k)_{k\geq 6}$, whose first term is t_6. It is given by*

$$(t_6, t_8, t_{10}) = (-24, 216, 456).$$

Our proof shows that this concrete question leads to a substantial Diophantine problem. Although the trace sequence (t_k) satisfies a sixth-order linear recurrence, its special structure allows us to relate it to a Lucas sequence of negative discriminant. This connection provides access to several powerful results, each addressing a different obstruction. We first use a theorem of Szikszai and Ziegler [SZ19] on three-term arithmetic progressions in Lucas sequences to eliminate an exceptional residue class that is not accessible through the initial local analysis alone. Effective lower bounds for Lucas sequences due to Hajdu and Tijdeman [HT25], building on work of Laurent [Lau08], then show that the gap between the variable indices is at most quadratic-logarithmic in the larger index. Next, an explicit estimate of Chim [Chi25] for linear forms in two p-adic logarithms yields an absolute upper

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bound for that index. Although this bound is substantial and sits near the limit where direction verification is practical, classical best-approximation properties of continued fractions reduce the index difference to only eight possibilities. Seven are excluded by a carefully chosen congruence argument, while the final case is resolved using a p-adic method developed by Mignotte and Tzanakis [MT93]. In conclusion, the resulting proof is genuinely local–global: global estimates successively reduce the range of the indices, while local information is essential both in the initial structural reduction and in the final elimination of the remaining case. Beyond the particular progression in Theorem 1, the argument illustrates how these complementary forms of arithmetic information can be combined to study additive relations among Hecke traces.

Numerical experiments suggest that Theorem 1 is part of a much more rigid pattern. Remarkably, the three arithmetic progressions that appear experimentally involve exactly the traces attached to the one-dimensional spaces of level-one cusp forms, together with $t_7 = 0$, corresponding to the exceptional weight $2k = 14$ for which the space is empty. More precisely, every nonzero term occurring in these progressions comes from one of the six one-dimensional spaces

$$\{S_{12}, S_{16}, S_{18}, S_{20}, S_{22}, S_{26}\}.$$

This observation suggests that the additve structure of the trace sequence is concentrated at the lowest-dimensional part of the level-one spectrum, and leads to the following conjecture.

Conjecture 2. *The only arithmetic progressions in the sequence $(t_k)_{k\geq 6}$ are given by*

$$(t_6, t_8, t_{10}) = (-24, 216, 456);$$
$$(t_9, t_{11}, t_{13}) = (-528, -288, -48);$$
$$(t_{13}, t_6, t_7) = (-48, -24, 0).$$

In particular, each has length exactly three.

The conjecture exhibits a further arithmetic feature. In each of the three progressions, two of the terms have exactly two distinct prime divisors. Since every trace t_k is divisible by 24, this is the smallest possible prime support. Thus, our conjecture indicates that the exceptional additive relations above are accompanied by unusually restrictive multiplicative structure. This motivates our second main result, which completely classifies the traces of minimal prime support.

Theorem 3. *Let $k \geq 6$. The only solutions to the equation $|t_k| = 2^a 3^b$ are given by the following table.*

k	t_k	a	b
6	-24	3	1
8	216	3	3
11	-288	5	2
13	-48	4	1

The analogous problem for the Berstel sequence was treated in [MT93] (see also [MT91]). That sequence is distinguished by its extremal zero set among ternary linear recurrences. We follow a strategy similar to that of [MT93, Theorem 2]. The method is local in nature but delicate to implement: one studies the sequence simultaneously modulo several prime powers and identifies residue classes in which divisibility by different primes is linked. Such relations impose constraints on the possible exponents of the primes dividing t_k, and their repeated use reduces the problem to a small number of cases. The remaining large-index cases are then eliminated using the Primitive Divisor Theorem for Lucas sequences [BHV01].

Although Theorem 1 is our principal result, we first prove Theorem 3. This order reflects the arithmetic structure of the trace sequence; once its decomposition in terms of Lucas sequences has been established, the classification of traces with minimal prime support becomes the natural first application. The argument also develops the local congruence and primitive-divisor techniques that reappear in the study of arithmetic progressions. The proof of Theorem 1, which requires additional Archimedean and p-adic Diophantine tools, is carried out in the subsequent section.

2. From the trace formula to Lucas sequences

The starting input for both main results is the explicit consequence of the trace formula obtained in [CJ22], which shows that, although the sequence (t_k) itself satisfies a sixth-order recurrence, its arithmetic is essentially governed by a Lucas sequence. More precisely, [CJ22, Proposition 2], gives

$$t_k = -1 - a_{k-1} + \begin{cases} 0 & \text{if } k \equiv 2,3 \pmod 4 \\ 2^{k-1} & \text{if } k \equiv 0 \pmod 4 \\ -2^{k-1} & \text{if } k \equiv 1 \pmod 4 \end{cases}$$

where $a_0 = 1, a_1 = -1$, and $a_n = -3a_{n-1} - 4a_{n-2}$ for $n \geq 2$.

Let

$$\omega = \frac{1-\sqrt{-7}}{2} \text{ and } \overline{\omega} = \frac{1+\sqrt{-7}}{2}$$

be the roots of $x^2 - x + 2 = 0$. The associated Lucas sequence

$$U_n = \frac{\omega^n - \overline{\omega}^n}{\omega - \overline{\omega}},$$

satisfies $U_0 = 0, U_1 = 1$, and the recurrence $U_{n+2} = U_{n+1} - 2U_n$. The companion sequence

$$V_n = \omega^n + \overline{\omega}^n$$

satisfies the same recurrence relation with initial values $V_0 = 2$ and $V_1 = 1$.

A direct comparison reveals that (a_n) tracks exactly the odd-indexed U-terms:

$$a_n = U_{2n+1}.$$

We first record the local propagation law for U_n that will be used repeatedly in the congruence arguments below.

Lemma 4. *Let $p \geq 3$ be a prime.*

(1) *If* $\left(\frac{-7}{p}\right) = 1$, *then* $U_{n+(p-1)} \equiv U_n \pmod p$.

(2) *If* $\left(\frac{-7}{p}\right) = -1$, *then* $U_{n+(p+1)} \equiv 2U_n \pmod p$.

Proof. This follows immediately by combining the addition formula

$$U_{m+n} = U_{m+1}U_n - 2U_mU_{n-1}$$

with the classic congruences $U_p \equiv \left(\frac{-7}{p}\right) \pmod p$ and $p \mid U_{p-\left(\frac{-7}{p}\right)}$. Additional background material on Lucas sequences can be found in [BW23, Chapter 2]. □

We will also require the Primitive Divisor Theorem for Lucas sequences [BHV01], which asserts that U_n admits a primitive prime divisor for all $n > 30$. Recall that a prime q is a primitive divisor of U_n if $q \mid U_n$ but q does not divide $(\omega-\overline{\omega})^2 U_1 \dots U_{n-1}$. Since the Primitive Divisor Theorem leaves only the indices $n \leq 30$ to be trated separately, we record in Table 1 the corresponding values and factorizations of U_n.

TABLE 1. The first 30 terms of U_n and their prime factorizations.

n	U_n	Factorization
1	1	1
2	1	1
3	-1	-1
4	-3	-3
5	-1	-1
6	5	5
7	7	7
8	-3	-3
9	-17	-17
10	-11	-11
11	23	23
12	45	$3^2 \cdot 5$
13	-1	-1
14	-91	$-7 \cdot 13$
15	-89	-89

n	U_n	Factorization
16	93	$3 \cdot 31$
17	271	271
18	85	$5 \cdot 17$
19	-457	-457
20	-627	$-3 \cdot 11 \cdot 19$
21	287	$7 \cdot 41$
22	1541	$23 \cdot 67$
23	967	967
24	-2115	$-3^2 \cdot 5 \cdot 47$
25	-4049	-4049
26	181	181
27	8279	$17 \cdot 487$
28	7917	$3 \cdot 7 \cdot 13 \cdot 29$
29	-8641	-8641
30	-24475	$-5^2 \cdot 11 \cdot 89$

3. Traces with minimal prime support

We begin with two preliminary results that will be used throughout the proof of Theorem 3: a description of the 2-adic valuation of t_k in several congruence classes of k, and a periodicity statement for the traces.

Lemma 5. *Suppose $k \geq 6$. The 2-adic valuation of t_k satisfies: $v_2(t_k) = 3$ for even k, and $v_2(t_k) = 4$ if $k \equiv 1 \pmod 4$. Moreover, when $k \equiv 3 \pmod 4$, we have:*

- $v_2(t_k) = 5$ *if* $k \equiv 3, 11 \pmod{16}$;
- $v_2(t_k) = 6$ *if* $k \equiv 15 \pmod{16}$;
- $v_2(t_k) \geq 7$ *if* $k \equiv 7 \pmod{16}$.

Proof. Recall the initial expression of t_k in terms of the sequence (a_n) via the relation

$$t_k = -1 - a_{k-1} + \delta_k,$$

where $\delta_k = 0$ when $k \equiv 2, 3 \pmod 4$, $\delta_k = 2^{k-1}$ if $k \equiv 0 \pmod 4$, and $\delta_k = -2^{k-1}$ if $k \equiv 1 \pmod 4$. Since $a_{n+2} = -3a_{n+1} - 4a_n$, we see that $a_{n+4} = a_{n+2} - 16a_n \equiv a_{n+2} - 16 \pmod{32}$. Therefore, using that all terms of (a_n) are odd we get

$$a_{n+4} \equiv a_n \pmod{32},$$

and since $a_5 = 23$, $a_6 = -1$, $a_7 = -89$, $a_8 = 271$, the first statement follows.

Next, starting with the identity $a_{2(n+2)} = a_{2(n+1)} - 16a_{2n}$, and the initial values $a_4 = -17$ and $a_6 = -1$, we obtain by induction that

$$a_{2n} \equiv 16n - 49 \pmod{256}.$$

Therefore, for $k = 4m + 3$, we get

$$t_k = -1 - a_{2(2m+1)} \equiv 2^5(1-m) \pmod{256},$$

from which the second statement follows by considering m modulo 4.

□

Lemma 6. *For every $k \geq 6$, we have*

$$t_{k+12} \equiv t_k \pmod{2^5 \cdot 3^2 \cdot 5}.$$

Furthermore, if $k \equiv 3 \pmod 4$, then

$$t_{k+12} \equiv t_k \pmod{2^5 \cdot 3^3 \cdot 5}.$$

Proof. We first work modulo 45. Since $U_{24} = -2115 \equiv 0 \pmod{45}$ and $U_{25} = -4049 \equiv 1 \pmod{45}$, the formula $U_{r+24} = U_{25}U_r - 2U_{24}U_{r-1}$ gives $U_{r+24} \equiv U_r \pmod{45}$. Also, $2^{12} \equiv 1 \pmod{45}$, which leads to $t_{k+12} \equiv t_k \pmod{45}$.

At the same time, the congruence $a_{n+4} \equiv a_n \pmod{32}$ (from Lemma 5) implies that $t_{k+4} \equiv t_k \pmod{32}$ for $k \geq 6$. Thus, in fact,

$$t_{k+12} \equiv t_k \pmod{1440}.$$

For the second statement, if $k \equiv 3 \pmod 4$ then $t_{k+12} - t_k = U_{2k-1} - U_{2k+23}$. As before, $U_{r+24} \equiv U_r - 90U_{r-1} \pmod{135}$, so $t_{k+12} - t_k \equiv 90U_{2k-2} \pmod{135}$. Note that $2k - 2 \equiv 0 \pmod 4$ and $U_4 = -3$, so $3 \mid U_{2k-2}$ for $k \geq 3$. This gives the additional power of 3.

□

With these preparatory results in place, we restate Theorem 3 and proceed to its proof.

Theorem 3. *Let $k \geq 6$. The only solutions to the equation $|t_k| = 2^a 3^b$ are given by the following table.*

k	t_k	a	b
6	-24	3	1
8	216	3	3
11	-288	5	2
13	-48	4	1

Proof. It will be convenient to employ a factorization of the shifted traces in terms of the Lucas sequence U_n and its companion V_n. Namely, since

$$U_{k-1}V_k = a_{k-1} - 2^{k-1} \text{ and } U_kV_{k-1} = a_{k-1} + 2^{k-1},$$

it follows that

$$t_k + 1 = \begin{cases} -U_{2k-1} & k \equiv 2, 3 \pmod 4 \\ -U_{k-1}V_k & k \equiv 0 \pmod 4 \\ -U_kV_{k-1} & k \equiv 1 \pmod 4. \end{cases}$$

We now consider separately the four cases arising from each congruence class of k modulo 4.

Case 1. Let $k \equiv 0 \pmod 4$. By Lemma 5, we know that $a = 3$.

Suppose $t_k = -8\cdot 3^b$. On the one hand, it clear that

$$-8\cdot 3^b \equiv 3^{b+1} \equiv 1,3,4,5,9 \pmod{11}.$$

On the other hand, since $U_{n+10} \equiv U_n \pmod{11}$ by Lemma 4, and $2^{4m+19} \equiv 2^{4m-1} \pmod{11}$, we get

$$t_{4m+20} \equiv t_{4m} \pmod{11}.$$

Considering $t_8, t_{12}, t_{16}, t_{20}$ and t_{24} modulo 11, we obtain the remainders $7,2,8,0,0$, respectively. This set of remainders is disjoint from that corresponding to $-8\cdot 3^b$, a contradiction.

Now suppose $t_k = 8\cdot 3^b$. Writing $t_{4m} = -1 - U_{8m-1} + 2^{4m-1}$ we get

$$16\cdot 3^b = 2\cdot t_{4m} = 16^m - 2U_{8m-1} - 2,$$

and thus

$$3^b \equiv 4 - 2U_{8m-1} \pmod{5}.$$

From Lemma 4: $U_{n+24} \equiv U_n \pmod{5}$, so

$$4 - 2U_{8m-1} \equiv \begin{cases} 0 \pmod{5} \text{ if } m \equiv 0,1 \pmod{3} \\ 2 \pmod{5} \text{ if } m \equiv 2 \pmod{3}. \end{cases}$$

The only possible option is $m \equiv 2 \pmod{3}$, yielding $k \equiv 8 \pmod{12}$. Equivalently

$$k \equiv 8, 20, 32 \pmod{36}.$$

Consider first $k \equiv 20 \pmod{36}$. Then $3^b \equiv 2 \pmod{5}$ and so $b \equiv 3 \pmod{4}$; in particular b is odd. However, by Lemma 4: $U_{n+36} \equiv U_n \pmod{37}$, while $2^{36} \equiv 1 \pmod{37}$. Thus $t_{4m+36} \equiv t_{4m} \pmod{37}$, and so

$$t_k \equiv t_{20} \equiv 35 \pmod{37}.$$

Hence $8\cdot 3^b \equiv 35 \pmod{37}$, or $3^b \equiv 3^2 \pmod{37}$, giving $b \equiv 2 \pmod{18}$, and contradicting the oddness of b.

Finally, consider the remaining two cases. As in the proof of Lemma 6, one can see that

$$t_{4m+36} \equiv t_{4m} \pmod{81}.$$

Since $t_8 \equiv 54 \pmod{81}$ and $t_{32} \equiv 27 \pmod{81}$, we conclude that $v_3(t_k) = 3$ for $k \equiv 8, 32 \pmod{36}$. Consequently

$$t_k = 216.$$

Then U_{k-1} divides $1 + t_k = 217 = 7\cdot 31$. For $k \geq 32$, the Primitive Divisor Theorem would give a prime divisor p of U_{k-1}, and so $p \in \{7, 31\}$. However, neither is primitive because $U_7 = 7$ and $U_{16} = 3\cdot 31$. For $k < 32$, only $k = 8$ needs to be checked, and this is indeed a solution: $t_8 = 216 = 2^3\cdot 3^3$.

Case 2. Let $k \equiv 1 \pmod{4}$. Then $a = 4$ by Lemma 5. To determine the exponent b, we observe from the table below that t_k is divisible by 9 if and only if it is divisible by 5.

$k \pmod{12}$	1	5	9
$t_k \pmod{9}$	6	0	3
$t_k \pmod{5}$	2	0	2

We conclude that $b = 1$, restricting our search to $t_k \in \{-48, 48\}$. In the range $6 \leq k \leq 30$, a direct check reveals only one solution: $t_{13} = -48$. No other value of k satisfies $t_k = -48$ because, by [CJ22], the sequence $(t_k)_{k\geq 6}$ has no repetitions.

Assume now that $k > 30$ and $t_k = 48$. Then $U_k V_{k-1} = t_k + 1 = 49$, forcing $U_k \mid 7^2$. Since $U_7 = 7$, this contradicts the Primitive Divisor Theorem.

Case 3. Let $k \equiv 2 \pmod 4$. Lemma 5 gives $a = 3$. As before, we track the sequence modulo 5 and 9:

$k \pmod{12}$	2	6	10
$t_k \pmod 9$	0	3	6
$t_k \pmod 5$	0	1	1

As in previous case, the condition $9 \mid t_k$ implies $5 \mid t_k$. This leaves $b = 1$ as the only valid exponent, and restricts the targets to $t_k \in \{-24, 24\}$. For $6 \le k \le 16$, only one solution emerges: $t_6 = -24$. Thus $k = 6$ is the only solution to $t_k = -24$, by [CJ22].

For $k \ge 16$ and $t_k = 24$, we again invoke the Primitive Divisor Theorem. Indeed, the only prime divisor of $U_{2k-1} = t_k - 1 = -5^2$ already appears in $U_6 = 5$.

Case 4. Let $k \equiv 3 \pmod 4$. Note that $t_7 = 0$, $t_{11} = -288$ and $t_{15} = 8640$. Thus $t_k \equiv 0 \pmod 9$, by Lemma 6. In fact, we can go a step further and track the sequence modulo 27 and 5:

$k \pmod{12}$	3	7	11
$t_k \pmod{27}$	0	0	9
$t_k \pmod 5$	0	0	2

Since $27 \mid t_k$ iff $5 \mid t_k$, we conclude that $b = 2$. In this case we have

$$\pm 2^a \cdot 9 = t_k = -1 - U_{2k-1},$$

with $2k - 1 \equiv 5 \pmod 8$. Note that

$$U_{n+16} = U_n U_{17} - 2U_{16} U_{n-1} \equiv -8U_n \pmod{31},$$

implying that

$$U_{8m+5} \equiv -(-8)^{\lfloor m/2 \rfloor} \pmod{31}.$$

Since $\{\pm 1, \pm 2, \pm 4, \pm 8, \pm 15\}$ is the set of residues of $(-8)^n \pmod{31}$, the same set represents $U_{8m+5} \pmod{31}$. Hence, the set of remainders of $-1 - U_{8m+5}$ modulo 31 is

$$\mathcal{R} = \{0, 1, 3, 7, 14, 15, 22, 26, 28, 29\}.$$

At the same time, the set of remainders of $2^a \cdot 9$ modulo 31 is

$$\{5, 9, 10, 18, 20\}.$$

The above two sets are disjoint, showing that $2^a \cdot 9 = t_k$ has no solutions.

Now assume that $-2^a \cdot 9 = t_k$. Since $-2^a \cdot 9$ modulo 31 is $\{11, 13, 21, 22, 26\}$, we see that it intersects $\mathcal{R}$ at $\{22, 26\}$. This forces $a \equiv 0, 2 \pmod 5$ and $\lfloor m/2 \rfloor \equiv 1, 9 \pmod{10}$; in particular, $\lfloor m/2 \rfloor$ is odd.

If $a \ge 7$ then $v_2(t_k) \ge 7$, which implies that $k \equiv 7 \pmod{16}$. Writing $k = 16j + 7$ leads to $m = 4j + 1$. Then $\lfloor m/2 \rfloor = 2j$, which contradicts the finding above. Hence $a < 7$, leaving $a = 5$ as the only option.

Now, we must find all k such $t_k = -2^5 \cdot 3^2 = -288$, or equivalently $U_{2k-1} = 287 = 7 \cdot 41$. If $2k - 1 > 30$, then U_{2k-1} has a primitive prime divisor. But neither 7 nor 41 are primitive because $U_{21} = 7 \cdot 41$. For $2k - 1 < 30$ the only solution is indeed $2k - 1 = 21$, i.e., $k = 11$. $\square$

4. Arithmetic progressions among traces

The starting point of our proof of Theorem 1 is to show that (m, n) lies in a unique pair of residue classes modulo 12. Although the proof of the following Proposition is brief, it uses a substantial global input: a result of Szikszai and Ziegler [SZ19] on three-term arithmetic progressions in Lucas sequences. In the form relevant here, [SZ19, Theorem 3] asserts that if $u_{r+2} = Au_{r+1} + Bu_r$, with $(A, B) \neq (-1, -2)$, $|B| \leq 10^4$, and $u_k < u_m < u_n$ form a three-term arithmetic progression, then

$$\max\{k, m, n\} \leq 26.$$

Interestingly, their computation also identify $(A, B) = (1, -2)$ as an extremal case for the number of three-term arithmetic progressions, suggesting that our Lucas sequence U_n is especially rich in additive structure.

Proposition 7. *Suppose that $2t_m = t_n - 24$ for some $m, n > 6$. Then*

$$(m, n) \equiv (8, 10) \pmod{12}.$$

Proof. From Lemma 6, we know that the period of t_k modulo 1440 is 12; more precisely

$k \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11
$t_k \pmod{1440}$	1080	−48	360	0	1080	720	−24	0	216	−528	456	−288

The congruence modulo 1440 shows that

$$(m, n) \equiv (6, 6) \quad \text{or} \quad (8, 10) \pmod{12}.$$

Suppose that $(m, n) \equiv (6, 6) \pmod{12}$. Then

$$t_m = -1 - U_{2m-1}, \qquad t_n = -1 - U_{2n-1},$$

and the equation $2t_m = t_n - 24$ becomes

$$2U_{2m-1} = U_{11} + U_{2n-1}.$$

The term U_{2m-1} lies strictly between U_{11} and U_{2n-1}. After possibly reversing the two endpoints, [SZ19, Theorem 3] gives

$$\max\{2m - 1, 2n - 1\} \leq 26.$$

On the other hand, $m, n > 6$ and $m, n \equiv 6 \pmod{12}$ imply $2m - 1, 2n - 1 \geq 35$, a contradiction. Therefore, the only valid possibility is $(m, n) \equiv (8, 10) \pmod{12}$. ∎

The next step is to control the separation between the two indices in terms of their size. The main input is the effective lower bound for terms of Lucas sequences established in [HT25], building on Laurent's estimates for linear forms in two logarithms [Lau08]. Comparing this lower bound with the elementary upper estimates arising from the equation $2t_m = t_n - 24$, we obtain a polylogarithmic bound for $|m - n|$ in terms of

$$R := \max\{M, N\},$$

where $M := m - 1$ and $N := n - 1$. At this stage the bound is not absolute, but it already shows that the two indices must be comparatively close whenever R is large.

Proposition 8. *Suppose that* $2t_m = t_n - 24$ *for some* $m, n > 6$. *Let* $R = \max\{m-1, n-1\}$. *If* $R > 5 \cdot 10^8$ *then*

$$|m-n| < 748 \log^2(R).$$

Proof. By Proposition 7, we know that $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$. Letting $M = m-1 \geq 7$ and $N = n-1 \geq 9$, we can write

$$t_m = -1 - a_M + 2^M \text{ and } t_n = -1 - a_N,$$

subject to the relation

$$a_N + 23 = 2(a_M - 2^M). \tag{1}$$

First, consider the case $M \leq N$. Then, using (1)

$$|a_N| = |2(a_M - 2^M) - 23| \leq 2|a_M - 2^M| + 23.$$

Letting $c = \sqrt{8/7}$, we have $|a_j| \leq c2^j$ and so $|a_j - 2^j| \leq (1+c)2^j$. Since $M \geq 7$, this implies that

$$|a_N| \leq \left(2(1+c) + \frac{23}{2^M}\right) 2^M \leq \left(2(1+c) + \frac{23}{2^7}\right) 2^M,$$

so

$$\log|a_N| \leq M \log 2 + 1.4628. \tag{2}$$

Applying [HT25, Theorem 2.1], we have for $r > 5 \cdot 10^8$

$$|U_r| \geq \frac{1}{4} e^{-250(\log r)^2} (\sqrt{2})^{r-2},$$

which implies that for $2r + 1 > 5 \cdot 10^8$

$$\log|a_r| = \log|U_{2r+1}| \geq r \log 2 - \frac{5}{2}\log 2 - 250 \log^2(2r+1). \tag{3}$$

Combining (2) and (3) gives

$$(N-M)\log 2 \leq 250 \log^2(2N+1) + 3.1957 < 268 \log^2(N),$$

and thus

$$N - M < 387 \log^2(N).$$

Similarly, if $N < M$ we have

$$|a_M - 2^M| \leq \frac{1}{2}\left(c + \frac{23}{512}\right) 2^N. \tag{4}$$

To derive a lower bound, note that $a_r - 2^r = U_r V_{r+1}$ and $U_{2j} = U_j V_j$. Clearly $|U_j| \leq \frac{2}{\sqrt{7}}(\sqrt{2})^j$, so for $2j > 5 \cdot 10^8$:

$$|V_j| = \frac{|U_{2j}|}{|U_j|} \geq \frac{\sqrt{7}}{8} e^{-250 \log^2(2j)} (\sqrt{2})^{j-2}.$$

Hence, for $r > 5 \cdot 10^8$ we have

$$|a_r - 2^r| = |U_r V_{r+1}| \geq \frac{\sqrt{7}}{64\sqrt{2}} 2^r e^{-250[\log^2 r + \log^2(2r+2)]},$$

yielding

$$\log|a_r - 2^r| \geq r\log 2 - 250\left(\log^2 r + \log^2(2r+2)\right) - \log\left(\frac{64\sqrt{2}}{\sqrt{7}}\right). \tag{5}$$

Then from (4) and (5) we get

$$(M-N)\log 2 \leq 250\left(\log^2 M + \log^2(2M+2)\right) + 2.9473 < 518\log^2(M),$$

which implies that $M - N < 748\log^2(M)$. □

We now obtain an absolute upper bound for R. To this end, we apply a specialization of a recent result of Chim [Chi25] on linear forms in two p-adic logarithms. Its main use is to derive a polylogarithmic upper bound for M in terms of R. Combined with the preceding bound for $|M-N|$, this yields an inequality of the form

$$R \ll (\log R)^3$$

and hence an effective absolute bound for R. This will subsequently turn the relative estimate for the index difference into an absolute one.

We note that the following estimate is not a direct quotation of Chim's main theorem [Chi25, Theorem 2.1], but rather one of its variants from the end of that article. It gives an upper bound for the p-adic valuation $v_p(\Lambda)$ for

$$\Lambda = \alpha_1^{b_1} - \alpha_2^{b_2},$$

where α_1, α_2 are multiplicatively independent algebraic numbers and b_1, b_2 are positive integers. For $p = 2$, assuming that $\mathbb{Q}_2(\alpha_1,\alpha_2) = \mathbb{Q}_2$, $[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}] = 2$ and $v_2(\alpha_1) = v_2(\alpha_2) = 0$, [Chi25, Table 1] provides a list of allowed values for the variables (x_1, x_2, x_3, C_2) such that

$$v_2(\Lambda) < C_2 \cdot \frac{2^5}{(\log 2)^3}(x_3 + 2\log 2)H\log(A_1)\log(A_2) \tag{6}$$

where

$$H = \max\{\log b' + \log\log 2, \frac{x_1\log 2}{2}, x_2(x_3 + 2\log 2)\log 2\}, \tag{7}$$

and $A_1, A_2 > 1$ are real numbers such that

$$\log A_i \geq \max\{h(\alpha_i), \frac{\log 2}{2}\}$$

and

$$b' = \frac{b_1}{2\log(A_2)} + \frac{b_2}{2\log(A_1)}.$$

As usual, $h(\alpha)$ denotes the absolute logarithmic height of the algebraic number α, i.e., assuming $a\prod_{i=1}^{d}(x - \alpha^{(i)})$ is the minimal polynomial of α over $\mathbb{Q}$ we have

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d}\log\max(1, |\alpha^{(i)}|)\right).$$

Proposition 9. *Suppose that $2t_m = t_n - 24$ for some $m, n > 6$. Then*

$$\max\{m, n\} < 6.5 \cdot 10^{13}.$$

Proof. Assume that $\max\{m,n\} \geq 6.5 \cdot 10^{13}$. As before, let $M = m-1$, $N = n-1$ and $R = \max\{M,N\}$. By Proposition 8, we have that

$$|M-N| < 748 \log^2(R).$$

We claim that $M < 2N$; this is immediate if $M \leq N$. If $M > N$, then $R = M > 6.5 \cdot 10^{13} - 1$ and thus $M - N < 748(\log M)^2 < M/2$, as claimed.

Now, let $\omega = \frac{1-\sqrt{-7}}{2}$, so that

$$a_j = \frac{\omega^{2j+1} - \overline{\omega}^{2j+1}}{\omega - \overline{\omega}}.$$

The equation $2t_m = t_n - 24$ becomes

$$-2^{M+1} - \frac{\omega^{2N+1} - 2\omega^{2M+1}}{\omega - \overline{\omega}} = 23 - \frac{\overline{\omega}^{2N+1} - 2\overline{\omega}^{2M+1}}{\omega - \overline{\omega}}.$$

Denote

$$\mathcal{R}_{N,M} := 23 - \frac{\overline{\omega}^{2N+1} - 2\overline{\omega}^{2M+1}}{\omega - \overline{\omega}}.$$

Then

$$\mathcal{R}_{N,M} = -2^{M+1}\left(1 + \frac{\omega^{2N+1}}{(\omega-\overline{\omega})2^{M+1}} - \frac{\omega^{2M+1}}{(\omega-\overline{\omega})2^{M}}\right).$$

Since $v_2(\omega) = 1$, $v_2(\overline{\omega}) = 0$ and $2N - M > 0$, it follows that

$$v_2(\mathcal{R}_{N,M}) = M + 1.$$

At the same time,

$$\mathcal{R}_{N,M} = \frac{\overline{\omega}}{\omega - \overline{\omega}}\left(2 - \overline{\omega}^{2(N-M)}\right)\left(\overline{\omega}^{2M} + \frac{23(\omega - \overline{\omega})}{\overline{\omega}(2 - \overline{\omega}^{2(N-M)})}\right),$$

which shows that

$$v_2(\mathcal{R}_{N,M}) = v_2\left(\overline{\omega}^{2M} + \frac{23(\omega - \overline{\omega})}{\overline{\omega}(2 - \overline{\omega}^{2(N-M)})}\right).$$

Letting

$$\alpha_1 = \overline{\omega} \text{ and } \alpha_2 = -\frac{23(\omega - \overline{\omega})}{\overline{\omega}(2 - \overline{\omega}^{2(N-M)})},$$

as well as

$$b_1 = 2M \text{ and } b_2 = 1$$

we obtain

$$M + 1 = v_2(\mathcal{R}_{N,M}) = v_2(\alpha_1^{b_1} - \alpha_2^{b_2}). \tag{8}$$

Because $\sqrt{-7} \in \mathbb{Q}_2$, both α_1 and α_2 belong to $\mathbb{Q}_2$, so $\mathbb{Q}_2(\alpha_1, \alpha_2) = \mathbb{Q}_2$. It is also clear that both α_1 and α_2 are 2-adic units, and that

$$[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{-7}) : \mathbb{Q}] = 2.$$

We claim that they are also multiplicatively independent. Indeed, in the ring of integers of $K = \mathbb{Q}(\sqrt{-7})$, the ideal generated by 7 ramifies as $(7) = \mathfrak{q}^2$, where $\mathfrak{q} = (\sqrt{-7})$. Then

$$\overline{\omega} = \frac{1+\sqrt{-7}}{2} \equiv \frac{1+0}{2} \equiv 4 \pmod{\mathfrak{q}}.$$

Since $N-M=n-m\equiv 2 \pmod 3$ (by Proposition 7) and 4 has order 3 in $\mathbb{F}_7^\times$,

$$2-\overline{\omega}^{2(N-M)}\equiv 2-4^{2(N-M)}\equiv 2-4\not\equiv 0 \pmod{\mathfrak{q}}.$$

It follows that $v_\mathfrak{q}(\alpha_1)=0$ and $v_\mathfrak{q}(\alpha_2)=1$. Any relation $\alpha_1^r\alpha_2^s=1$ implies that

$$r\cdot v_\mathfrak{q}(\alpha_1)+s\cdot v_\mathfrak{q}(\alpha_2)=0,$$

which forces $s=0$. Taking norms then gives $2^r=1$, and hence $r=0$.

Next, we turn to heights estimations. It is easy to see that $h(\alpha_1)=\frac{1}{2}\log 2$. Using the standard height inequalities gives

$$h(\alpha_2)\le \log 23+\frac{1}{2}\log 7+\frac{1}{2}\log 2+h(2-\overline{\omega}^{2(N-M)})\le |N-M|\log 2+C_h,$$

where $C_h=\log 23+\frac{1}{2}\log 7+\frac{5}{2}\log 2=5.841317\ldots$.

We choose $A_1=\sqrt{2}$ and $A_2>1$ so that

$$\log(A_1)=\frac{1}{2}\log 2,\qquad \log A_2=|N-M|\log 2+C_h.$$

By Proposition 8

$$\log A_2<748\log 2(\log R)^2+C_h<518.49(\log R)^2. \tag{9}$$

Using the variant corresponding to

$$(x_1,x_2,x_3,C_2)=(250,25,4.93,5360)$$

in Table 1 of [Chi25, pp. 346–348], the estimate (6) becomes

$$v_2\left(\alpha_1^{b_1}-\alpha_2^{b_2}\right)<C_2\cdot\frac{2^5}{(\log 2)^3}(x_3+2\log 2)H\log(A_1)\log(A_2)=\mathcal{C}H\log(A_2),$$

where

$$\mathcal{C}:=5360\frac{2^4}{(\log 2)^2}(4.93+2\log(2))=1127447.198\ldots<1127448,$$

and according to (7)

$$H=\max\{\log(b')+\log\log(2),109.4531\}$$

and since $\log(A_2)>1$.

$$b'=\frac{b_1}{2\log(A_2)}+\frac{b_2}{2\log(A_1)}=\frac{M}{\log(A_2)}+\frac{1}{\log(2)}<R+2.$$

By assumption, $R>6.5\cdot 10^{13}-1$ so $109.4531<3.45\log(R)$ and also

$$\log(b')+\log\log(2)<\log(R+2)<3.45\log(R),$$

thus

$$H<3.45\log(R).$$

Combining (8) and (9), it follows that

$$M+1<\mathcal{C}_1(\log R)^3,$$

where

$$\mathcal{C}_1=1127448\cdot 3.45\cdot 518.49=2016768271.644.$$

Since

$$R=\max\{M,N\}\le M+|N-M|,$$

Proposition 8 gives

$$R < \mathcal{C}_1(\log R)^3 + 748\log^2(R),$$

which is impossible for $R > 6.5 \cdot 10^{13} - 1$. □

Having obtained an absolute upper bound for R, we now return to the separation between the two indices and reduce it to a small absolute range. The main tool is the theory of continued fractions, in particular the classical best-approximation properties of convergents. Using the closed-form expression for the Lucas sequence, the defining equation produces an exponentially small trigonometric quantity, which can be compared with the continued-fraction approximations to an irrational number that arises naturally from the definition of U_n. We treat the cases $N > M$ and $M > N$ separately. Although the two arguments follow the same general principle, the latter is somewhat more delicate, since it leads to a product of two trigonometric factors rather than a single homogeneous approximation.

Proposition 10. *Suppose that $2t_m = t_n - 24$ for some $m, n > 6$. Then*

$$|m - n| < 50.$$

Proof. First, we consider the case $N > M$. Let $d = N - M > 0$. Substituting the closed-form formula

$$a_j = U_{2j+1} = c2^j \sin\left((2j+1)\theta\right),$$

where $c = \sqrt{8/7}$ and $\theta = \arccos\left(\frac{1}{2\sqrt{2}}\right)$, into the equation $a_N + 23 = 2(a_M - 2^M)$ yields

$$c2^N \sin((2N+1)\theta) + 23 = 2\left(c2^M \sin((2M+1)\theta) - 2^M\right).$$

Rearranging terms, we obtain:

$$|\sin((2N+1)\theta)| \leq 2^{-d}\left(2 + \frac{2}{c}\right) + \frac{23}{c2^N}.$$

Since $N = M + d$, we can write $\frac{23}{c2^N} = \frac{23}{c2^M}2^{-d}$. Because $M \geq 7$, this term is bounded by $\frac{23}{128c}2^{-d} \approx 0.168 \cdot 2^{-d}$. Since $2 + \frac{2}{c} = 2 + \sqrt{7/2} \approx 3.871$, we conclude

$$|\sin((2N+1)\theta)| < 4.039 \cdot 2^{-d}.$$

Let

$$\alpha = \frac{\theta}{\pi},$$

and note that α is irrational. Indeed, if $\alpha = u/v$ for $u, v \in \mathbb{Z}$, then $\cos(\pi u/v) = \frac{1}{2\sqrt{2}}$ would imply that $\frac{1}{\sqrt{2}} = 2\cos(\pi u/v) = e^{i\pi u/v} + e^{-i\pi u/v}$ is an algebraic integer, which is a contradiction. For any $x \in \mathbb{R}$, let $||x||$ denote the distance from x to the nearest integer. Choose $k_0 \in \mathbb{Z}$ such that $||(2N+1)\alpha|| = |(2N+1)\alpha - k_0|$. Defining

$$\theta_1 = |(2N+1)\theta - k_0\pi| = \pi||(2N+1)\alpha||,$$

we have $\theta_1 \in [0, \pi/2]$. Then $\sin(\theta_1) \geq \frac{2}{\pi}\theta_1$, and so

$$|\sin((2N+1)\theta)| = \sin(\theta_1) \geq \frac{2}{\pi}\left(\pi||(2N+1)\alpha||\right) = 2||(2N+1)\alpha||.$$

Combining this with our upper bound yields:

$$||(2N+1)\alpha|| < 2.02 \cdot 2^{-d}. \tag{10}$$

Because α is irrational, we may consider its convergents p_j/q_j. It is a fundamental property of continued fractions that $||q_j\alpha|| \leq ||q\alpha||$ for all $1 \leq q < q_{j+1}$, and furthermore, $||q_j\alpha|| > \frac{1}{q_j+q_{j+1}}$. A direct computation of the convergents of α reveals that

$$q_{31} = 1,422,145,792,362 \text{ and } q_{32} = 382,763,762,442,793.$$

In particular

$$q_{31} < 1.3 \cdot 10^{14} < q_{32} \text{ and } q_{31} + q_{32} < 10^{15}/2.6.$$

Since $N < 6.5 \cdot 10^{13}$ by Proposition 9, it follows that $2N + 1 \leq 1.3 \cdot 10^{14} < q_{32}$. Therefore, q_{31} is the best approximation denominator in this range, giving:

$$\frac{2.6}{10^{15}} < \frac{1}{q_{31} + q_{32}} < ||q_{31}\alpha|| \leq ||(2N + 1)\alpha||.$$

Together with (10), we find:

$$2^d < \frac{2.02}{2.6} \cdot 10^{15},$$

so we must have $d < (15 \log(10) + \log(2.02/2.6))/\log(2) < 50$. Since $d \equiv 2 \pmod{12}$, the only possibilities are

$$d \in \{2, 14, 26, 38\}.$$

Now, we turn to the case $M > N$. This time, we write

$$\begin{aligned} c2^N \sin((2N+1)\theta) + 23 &= c2^{M+1}\left(\sin((2M+1)\theta) - \frac{1}{c}\right) \\ &= c2^{M+1}\left(\sin((2M+1)\theta) - \sin\theta\right) \\ &= c2^{M+2} \sin(M\theta)\cos((M+1)\theta) \end{aligned}$$

Consequently, setting $d' = M - N > 0$ we get

$$|\sin(M\theta)\cos((M+1)\theta)| < 2^{-d'}\left(\frac{1}{4} + \frac{23}{c2^{N+2}}\right) < 0.2606 \cdot 2^{-d'}.$$

The formula

$$\cos\theta = \cos((M+1)\theta - M\theta) = \cos((M+1)\theta)\cos(M\theta) + \sin((M+1)\theta)\sin(M\theta),$$

implies that

$$\frac{1}{2\sqrt{2}} = |\cos\theta| \leq |\cos((M+1)\theta)| + |\sin(M\theta)|.$$

Since the sum of these two non-negative terms is at least $\frac{1}{2\sqrt{2}}$, their maximum must be at least $\frac{1}{4\sqrt{2}}$. Accordingly, the smaller of the two factors is bounded above by

$$4\sqrt{2}|\sin(M\theta)\cos((M+1)\theta)| < 4\sqrt{2}(0.2606 \cdot 2^{-d'}) < 1.475 \cdot 2^{-d'}.$$

This leaves us with two cases, depending on which factor attains the minimum.

- If $|\sin(M\theta)| < 1.475 \cdot 2^{-d'}$, by the same reasoning used in the $N > M$ case, we write $\theta = \pi\alpha$ to obtain

  $$2\frac{2.6}{10^{15}} < 2||M\alpha|| \leq |\sin(M\theta)| < 1.475 \cdot 2^{-d'},$$

  giving $d' < \big(15\log(10) + \log(0.7375/2.6)\big)/\log(2) < 49$.

- If $|\cos((M+1)\theta)| < 1.475\cdot 2^{-d'}$ note that
$$\cos((M+1)\theta) = \sin\left((M+1)\theta - \frac{\pi}{2}\right) = \sin\big(\pi((M+1)\alpha - 1/2)\big).$$
Let k_1 be the nearest integer to $(M+1)\alpha - 1/2$, so that $||(M+1)\alpha - 1/2|| = |(M+1)\alpha - 1/2 - k_1|$. Because this distance lies in $[0, 1/2]$, we have
$$|\cos((M+1)\theta)| = \sin\big(\pi||(M+1)\alpha - 1/2||\big) \geq 2||(M+1)\alpha - 1/2||.$$
Thus
$$||2(M+1)\alpha|| \leq 2||(M+1)\alpha - 1/2|| < 1.475\cdot 2^{-d'}.$$
As before, we get $\frac{2.6}{10^{15}} < 1.475\cdot 2^{-d'}$, which in turn implies $d' < 50$.

□

We now complete the proof by treating the eight possible values of $n-m$ left by the preceding Proposition. A congruence argument eliminates seven of them, leaving only
$$n - m = 2.$$
To handle this final case, we use a method of Mignotte and Tzanakis [MT93], described below. Other applications of this method, in the context of Hecke traces, can be found in [CW25] and [CKW24].

Let (u_n) be a linear recurrence with integer coefficients and distinct characteristic roots ω_i, and let $c \not\equiv 0 \pmod{p}$. Suppose that there exists an odd prime p, dividing neither the discriminant nor any coefficients of the characteristic polynomial of u_n, and some integer $S > 0$ such that all ω_i are p-adic units and satisfy $\omega_i^S \equiv 1 \pmod{p}$. Let $\mathcal{M}$ be a finite set of known integer solutions of $u_n = c$, and let $\mathcal{P}$ be a complete residue system modulo S containing $\mathcal{M}$. Assume the following:

- if $n \in \mathcal{P}$ and $u_n \equiv c \pmod{p}$ then $n \in \mathcal{M}$;
- $u_{m+S} \not\equiv u_m \pmod{p^2}$ for every $m \in \mathcal{M}$.

Then [MT93, Theorem 1] asserts that $\mathcal{M}$ is the complete set of integer solutions of $u_n = c$.

Theorem 1. *There is only one arithmetic progression among the traces $(t_k)_{k\geq 6}$, whose first term is t_6. It is given by*
$$(t_6, t_8, t_{10}) = (-24, 216, 456).$$

Proof. Since $|n-m| < 50$ by Proposition 10, and $(m, n) \equiv (8, 10) \pmod{12}$ by Proposition 7, it follows that $n-m$ lies in the set
$$\mathcal{S} := \{-46, -34, -22, -10, 2, 14, 26, 38\}.$$

We have $N \equiv 9 \pmod{12}$, $M \equiv 7 \pmod{12}$, and
$$a_N \equiv 2a_M - 2^{M+1} - 1 \pmod{11}.$$

Note that
$$a_{n+5} = -11a_{n+1} + 12a_n \equiv a_n \pmod{11}.$$
Thus, looking at a_0 through a_4 it follows that

$j \pmod{5}$	0	1	2	3	4
$a_j \pmod{11}$	1	-1	-1	7	5

The condition $M \equiv 7 \pmod{12}$ restricts $M \pmod{60}$ to five possibilities: $7, 19, 31, 43$, or 55. At the same time, we see that

$$2a_M - 2^{M+1} - 1 \equiv \begin{cases} 5 \pmod{11} \text{ if } M \equiv 7 \pmod{60} \\ 8 \pmod{11} \text{ if } M \equiv 19, 43 \pmod{60} \\ 4 \pmod{11} \text{ if } M \equiv 31 \pmod{60} \\ 3 \pmod{11} \text{ if } M \equiv 55 \pmod{60}. \end{cases}$$

This forces $M \equiv 7 \pmod{60}$ and $a_N \equiv 5 \pmod{11}$. As a result $N \equiv 4 \pmod 5$, and so $N - M \equiv 2 \pmod 5$. The only element of the set $\mathcal{S}$ with this property is 2, and therefore

$$n - m = 2.$$

We now consider the sequence given for $n \geq 9$ by

$$v_n = t_n - 2t_{n-2},$$

with initial values $(v_3, v_4, v_5, v_6, v_7, v_8) = (6, 0, 0, -24, 0, 264)$. In particular,

$$v_{10} = t_{10} - 2t_8 = 456 - 2 \cdot 216 = 24.$$

We will show that this is the unique solution to the equation $v_n = 24$ by using the result described above.

It is easy to see that (v_n) and (t_n) have the same characteristic polynomial, namely

$$\begin{aligned} g(x) &= (x-1)(x^2+3x+4)(x+2)(x^2+4) \\ &= x^6 + 4x^5 + 9x^4 + 14x^3 + 12x^2 - 8x - 32, \end{aligned}$$

whose discriminant is $2^{22} \cdot 3^6 \cdot 5^2 \cdot 7$. Thus

$$v_{n+6} = -4v_{n+5} - 9v_{n+4} - 14v_{n+3} - 12v_{n+2} + 8v_{n+1} + 32v_n. \tag{11}$$

In the setup described above we choose

$$(p, S, A) = (37, 36, 1), \quad \mathcal{M} = \{10\}, \text{ and } \mathcal{P} = \{3, 4, \ldots, 38\} \pmod{36}.$$

This is a valid choice because 37 divides neither the discriminant nor any coefficients of g. Moreover

$$g(x) \equiv (x-1)(x+2)(x-7)(x+10)(x-12)(x+12) \pmod{37},$$

so all six roots of g reduce to distinct nonzero elements of $\mathbb{F}_{37}$, and hence are 37-adic units. A direct verification shows that for $3 \leq n \leq 38$ only $n = 10$ satisfies $v_n \equiv 10 \pmod{39}$. In addition, $v_{10+36} \equiv 320 \pmod{37^2}$, so $v_{46} \not\equiv v_{10} \pmod{37^2}$. Since all three conditions are verified, we conclude that $t_n - 2t_{n-2} = 24$ has a unique solution: $n = 10$.

It remains to prove that the progression $(t_6, t_8, t_{10}) = (-24, 216, 456)$ cannot be extended to a four-term. The existence of a fourth term, say t_r, would imply $t_r = 456 + 240 = 696$ for some $r > 6$. However, the congruence from the proof of Proposition 7 shows that 696 (mod 1440) is not a valid remainder. Therefore, the unique arithmetic progression that starts with t_6 has length exactly three.

□

References

[BHV01] Yu. Bilu, G. Hanrot, and P. M. Voutier. Existence of primitive divisors of Lucas and Lehmer numbers. *J. Reine Angew. Math.*, 539:75–122, 2001. With an appendix by M. Mignotte.

[BW23] Christian J.-C. Ballot and Hugh C. Williams. *The Lucas sequences—theory and applications*, volume 8 of *CMS/CAIMS Books in Mathematics*. Springer, Cham, 2023.

[Chi25] Kwok Chi Chim. Lower bounds for linear forms in two p-adic logarithms. *J. Number Theory*, 266:295–349, 2025.

[CJ22] Liubomir Chiriac and Andrei Jorza. The trace of T_2 takes no repeated values. *Indag. Math. (N.S.)*, 33(5):936–945, 2022.

[CKW24] Liubomir Chiriac, Daphne Kurzenhauser, and Erin Williams. The nonvanishing of the trace of T_3. *Involve*, 17(2):263–272, 2024.

[CW25] Liubomir Chiriac and Erin Williams. A generalized Lehmer conjecture for the trace of T_3. *Ramanujan J.*, 68(3):Paper No. 91, 16, 2025.

[HT25] Lajos Hajdu and Rob Tijdeman. Integers represented by Lucas sequences. *Ramanujan J.*, 66(4):Paper No. 74, 27, 2025.

[Lau08] Michel Laurent. Linear forms in two logarithms and interpolation determinants. II. *Acta Arith.*, 133(4):325–348, 2008.

[MT91] M. Mignotte and N. Tzanakis. Arithmetical study of recurrence sequences. *Acta Arith.*, 57(4):357–364, 1991.

[MT93] M. Mignotte and N. Tzanakis. Arithmetical study of a certain ternary recurrence sequence and related questions. *Math. Comp.*, 61(204):901–913, 1993.

[PZ12] Ákos Pintér and Volker Ziegler. On arithmetic progressions in recurrences—a new characterization of the Fibonacci sequence. *J. Number Theory*, 132(8):1686–1706, 2012.

[SS93] H. P. Schlickewei and W. M. Schmidt. Linear equations in members of recurrence sequences. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 20(2):219–246, 1993.

[SZ19] Márton Szikszai and Volker Ziegler. On arithmetic progressions in Lucas sequences—II. *J. Number Theory*, 201:354–376, 2019.

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