THE VARIATION OF ZEROS OF THE MILLER BASIS

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Abstract. We exhibit a connection between the variation of zeros in the Miller basis of modular forms $q^m + O(q^{\ell+1})$ and a logarithmic version $\mathcal{S}_\delta$ of the Szegő curve, where $\delta = m/\ell$. When $\delta < 0.6194$ we show that all the zeros are on the unit arc for $k \gg 0$, while if δ is asymptotically close to 1, we show that all the zeros lie on $\mathcal{S}_\delta$. In general, we posit that for all δ, the zeros are located on the union of the unit arc and the log Szegő curve, obtaining a partial result, and find conjectural thresholds for m/ℓ with all zeros on the unit arc, and no zeros on the arc. Finally, we enumerate all algebraic zeros of Miller forms up to $\ell - m \leq 25$.

1. Introduction

Denote by M_k the space of all holomorphic modular forms of even weight $k \geq 4$ and level one on the upper half-plane $\mathfrak{H}$. For any element of M_k we consider its zeros in the standard fundamental domain $\mathcal{F}$. The study of the location of these zeros was revolutionized by Rankin and Swinnerton-Dyer [RSD70], who proved that all non-elliptic zeros, i.e., distinct from i and $\rho = e^{2\pi i/3}$, of the Eisenstein series

$$E_k(z) = \frac{1}{2} \sum_{(c,d)=1} \frac{1}{(cz+d)^k}$$

in $\mathcal{F}$ lie on the arc $\mathcal{A}$ of the unit circle joining i and ρ. Their elegant approach, which involves approximating E_k by an elementary function on $\mathcal{A}$, continues to serve as a blueprint for most subsequent results on the subject.

Building on this foundational idea, Duke and Jenkins [DJ08] incorporated the circle method as a crucial new component to show that the zeros of certain weakly holomorphic modular forms also lie on the unit circle. Raveh [Rav25] recently adapted their proof to study the zeros of certain elements of the Miller basis of M_k denoted by

$$g_{k,m} = q^m + O\left(q^{\ell+1}\right), \text{ where } \ell + 1 = \dim M_k, 0 \leq m \leq \ell. \tag{1.1}$$

He showed that for fixed $m \geq 1$, once ℓ exceeds a certain linear bound in m, the zeros of $g_{k,m}$ lie on $\mathcal{A}$, a property that always holds true for $m = 1$. In contrast, when the difference $D = \ell - m$ is fixed, Rudnick [Rud24] used Faber polynomials to show that the zeros of $g_{k,m}$ cluster near vertical lines as $k \to \infty$. Subsequently, Zilka [Zil25] studied the moments of the zeros of the Faber polynomials to extend the range of the elements in the Miller basis for which at least one of the zeros is not on the arc; in particular, if $m/\ell > 30/31$ then, asymptotically, at least one zero of $g_{k,m}$ does not lie on $\mathcal{A}$. Related questions regarding the clustering of zeros of Eisenstein series for congruence subgroups are considered in [CSCR25].

One of the main motivations for this paper is to provide a framework that encompasses the entire family of Miller basis. Our first result provides a lower bound for the proportion of non-elliptic zeros on the arc depending on the ratio $\delta = m/\ell$.

Theorem A. *(Theorem 5) As $k \to \infty$ we have that*

$$\frac{1}{D}\left|\left\{\begin{matrix}\textit{non-elliptic zeros}\\ \textit{of } g_{k,m} \textit{ on } \mathcal{A}\end{matrix}\right\}\right| \geq \begin{cases}1 & \delta < 0.6194\\ 1-2.9832(\delta-0.6194) & 0.6194 \leq \delta < 0.9546\\ 0 & \delta \geq 0.9546.\end{cases}$$

In particular, for sufficiently large weights k, all the zeros of $g_{k,m}$ in the fundamental domain lie on the arc $\mathcal{A}$ whenever $\delta < 61.94\%$, and $g_{k,m}$ has at least one zero on $\mathcal{A}$ as long as $\delta < 95.46\%$. These theoretical thresholds align closely with our numerical experiments, which we ran for many k up to $600,000$. A sample of the data is shown in the following table.

k	ℓ	Not all roots on $\mathcal{A}$	No roots on $\mathcal{A}$
18000	1500	$m \geq 936$	$m \geq 1433$
18002	1500	$m \geq 947$	$m \geq 1421$
18004	1500	$m \geq 942$	$m \geq 1434$
18006	1500	$m \geq 937$	$m \geq 1421$
18008	1500	$m \geq 946$	$m \geq 1434$
18010	1500	$m \geq 943$	$m \geq 1421$

In Section 4 we derive a parallel result for the Miller basis of the space of weakly modular forms, complementing the work of Duke and Jenkins.

A central insight of Rudnick's result, as discussed above, is that one can estimate the coefficients of the Faber polynomials associated to the Miller basis when D is fixed. In section 5, we extend the method by allowing $D \to \infty$, where the analysis becomes substantially more delicate and requires the growth of D to remain sufficiently slow relative to k. This refinement reveals a new phenomenon in the asymptotic behavior of the zeros: they converge toward a logarithmic analogue of the classical Szegő curve

$$\mathcal{S} = \{z \in \mathbb{C} : |ze^{1-z}| = 1\}.$$

Its x-intercepts occur at $-W(e^{-1}) \approx -0.27846$, where W is the Lambert W-function, and 1, while its y-intercepts at $\pm e^{-1}$.

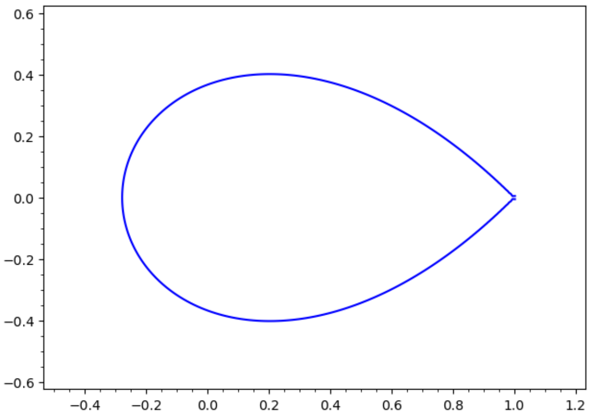


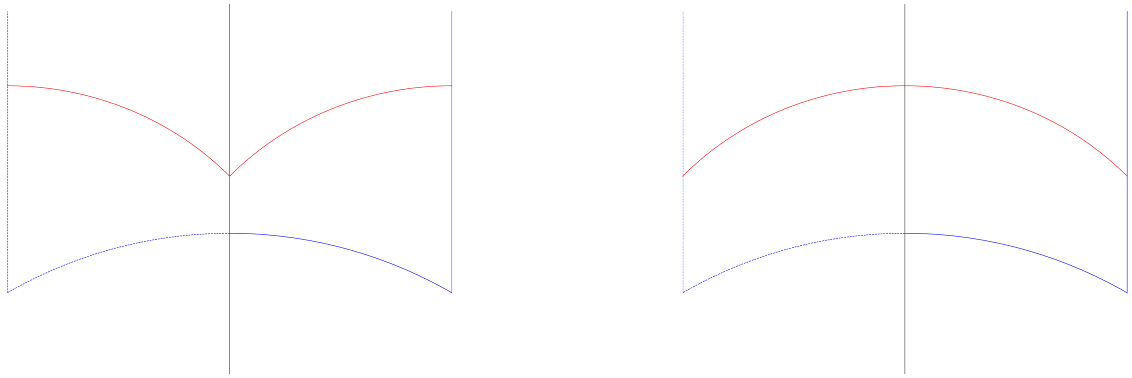
FIGURE 1. The Szegő curve

The logarithmic Szegő curve $\mathcal{L}_\pm = \{\tau \in \mathfrak{H} : |\operatorname{Re}\tau| \leq 1/2, \pm 24e^{2\pi i\tau} \in \mathcal{S}\}$ is the graph of the function $g_\pm : [-0.5, 0.5] \to \mathfrak{H}$ given by

$$g_\pm(x) = \frac{\ln 24 - \ln u^{-1}(\pm\cos(2\pi x))}{2\pi},$$

where $u : [W(e^{-1}), 1] \to [-1, 1]$ is the bijection $u(x) = (1 + \ln x)/x$.

Theorem B. *(Theorem 14) Suppose $D \to \infty$ such that $D < \frac{\alpha \log|k|}{\log\log|k|}$ for some $\alpha \in (0, 1)$. Then the zeros of $g_{k,m}$ asymptotically approach $\mathcal{S}_\delta = \mathcal{L}_\pm - \frac{1}{2\pi}\log|1-\delta|$, where $\delta = \frac{m}{\ell}$ and $\pm$ is the sign of k.*



(A) $\mathcal{S}_\delta$ when $\delta = 0.98$

(B) $\mathcal{S}_\delta$ when $\delta = 1.02$

FIGURE 2. Asymptotic location of zeros as $\delta \to 1$

The Szegő curve arises naturally as the limiting locus of the zeros of truncated exponential polynomials under an appropriate normalization. A noteworthy aspect of our theorem is that the relevant truncation threshold is governed solely by the ratio m/ℓ, as in the preceding result. This observation leads us to formulate the following general conjecture.

Conjecture C. *(Conjecture 15) For each $\delta > 0$, the zeros of $g_{k,m}$ asymptotically approach the curve $\mathcal{C}_\delta$, defined as the upper hull of the union of $\mathcal{A}$ and $\mathcal{S}_\delta$.*

The cutoffs of 0.6194 and 0.9546 in Theorem A, as well as 1.1026 and 1.1598 for weakly holomorphic modular forms from Section 4, can be recontextualized in light of Conjecture C, as approximations of the conjectural cutoffs (see Section 6, particularly Remark 2):

$$\delta_{\mathcal{A}}^+ = 1 - \frac{24}{W(e^{-1})e^{\sqrt{3}\pi}} = 0.6265\ldots \qquad \delta_{\mathcal{A}}^- = 1 + \frac{24}{W(e^{-1})e^{2\pi}} = 1.1609\ldots$$

$$\delta_{\mathcal{S}}^+ = 1 - \frac{24}{e^{2\pi}} = 0.9551\ldots \qquad \delta_{\mathcal{S}}^- = 1 + \frac{24}{e^{\sqrt{3}\pi}} = 1.1040\ldots$$

Theorems A and B prove parts of Conjecture C. In general, we have the following partial result:

Theorem D (Theorem 16)**.** *If $k \to \infty$ then all the zeros of the Miller modular forms $g_{k,m}$ lie in the region $\{z \in \mathcal{F} \mid \operatorname{Im} z < \frac{2\log k}{1-c} + O(1)\}$, where $c = \frac{e^{2\pi}}{1728}$.*

Beyond their geometric location, the arithmetic nature of these zeros is also of significant interest. Kohnen [Koh03] proved that all non-elliptic zeros of Eisenstein series are transcendental. Alongside a theorem of Schneider and classical results from the theory of complex multiplication, the fact that all such zeros lie on the arc $\mathcal{A}$ plays a decisive role in his analysis. In Section 7, we conclude this paper by contrasting this phenomenon with the Miller basis, exhibiting forms $g_{k,m}$ that possess algebraic zeros other than ρ or i. More precisely, we determine all such forms in the range $\ell - m \leq 25$.

2. Approximating the Miller forms

If $k \geq 4$ is an even weight and $\ell = \dim M_k - 1$, we can write $k = 12\ell + k'$ for some

$$k' \in \{0, 4, 6, 8, 10, 14\}.$$

Consider the Miller basis of M_k given by $g_{k,m} = q^m + O(q^{\ell+1})$. Since each $g_{k,m}$ has integer Fourier coefficients, it is clear that $\overline{g_{k,m}(z)} = g_{k,m}(-\overline{z})$. Setting $z = e^{i\theta}$ we obtain

$$\overline{g_{k,m}(e^{i\theta})} = g_{k,m}(-e^{-i\theta}) = g_{k,m}(\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right) e^{i\theta}) = e^{ik\theta} g_{k,m}(e^{i\theta}).$$

Therefore $g_{k,m}$ can be rescaled along the arc to obtain the real-valued function $e^{ik\theta/2} g_{k,m}(e^{i\theta})$. We denote

$$\overline{g}_{k,m}(e^{i\theta}) = e^{ik\theta/2 + 2\pi m \sin\theta} g_{k,m}(e^{i\theta}).$$

One can show (see, for example, [Rav25, Lemma 3.7]) that for each $R > 0$ there exists an $A > 1$ such that for all $z \in \mathcal{F}$ with $|j(z)| < R$ we have

$$g_{k,m}(z) = \int_{-\frac{1}{2}+iA}^{\frac{1}{2}+iA} G(z,\tau)\, d\tau,$$

where

$$G(z,\tau) = \frac{\Delta^{\ell}(z) E_{k'}(z) E_{14-k'}(\tau)}{\Delta^{\ell+1}(\tau)(j(\tau) - j(z))} e^{2\pi i m\tau}.$$

Since we are primarily interested in roots on $\mathcal{A}$, we may take $R > 1728$ and a corresponding A, so that the above integral formula holds. Duke-Jenkins and Raveh use the circle method to find good estimates of the above integral. Abstracting their process, for each $B > 0$, we will integrate on the following contour

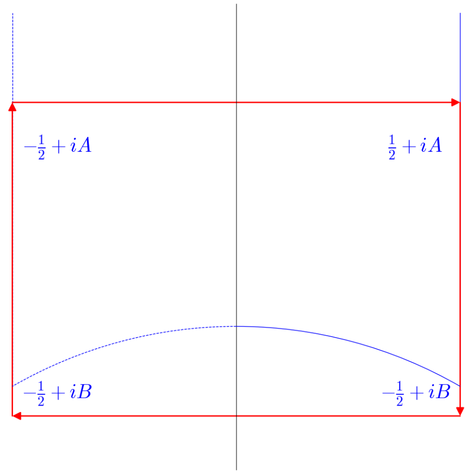


FIGURE 3. Contour of integration

This yields

$$g_{k,m}(e^{i\theta}) = \int_{-\frac{1}{2}+iB}^{\frac{1}{2}+iB} G(\tau, e^{i\theta})d\tau - 2\pi i \sum_{\gamma} \operatorname{Res}_{\tau=\gamma e^{i\theta}} G(\tau, e^{i\theta}), \tag{2.1}$$

where the sum is over all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\operatorname{Re}\gamma e^{i\theta} = -\frac{1}{2}$ and $\operatorname{Im}\gamma e^{i\theta} > B$, as well as the identity matrix I_2 and $S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. The main idea is that the residues at I_2 and S control the size of $\overline{g}_{k,m}(e^{i\theta})$, the other terms being errors that one can estimate.

Lemma 1. *For any $e^{i\theta} \in \mathcal{F}$, the function $j(\tau) - j(e^{i\theta})$ has finitely many roots inside the contour above. At any root $\gamma e^{i\theta}$, which corresponds to a simple pole of $G(\tau, e^{i\theta})$, we have*

$$\operatorname{Res}_{\tau=\gamma e^{i\theta}} G(\tau, e^{i\theta}) = -\frac{1}{2\pi i}(ce^{i\theta}+d)^{-k}e^{2\pi i m(\gamma e^{i\theta})},$$

giving

$$|-2\pi i \operatorname{Res}_{\tau=\gamma e^{i\theta}} G(\tau, e^{i\theta})| = |ce^{i\theta}+d|^{-k}e^{-2\pi m\sin\theta|ce^{i\theta}+d|^{-2}}.$$

Proof. Indeed,

$$\begin{aligned}
\operatorname{Res}_{\tau=\gamma e^{i\theta}} G(\tau, e^{i\theta}) &= \frac{\Delta^{\ell}(e^{i\theta})E_{k'}(e^{i\theta})E_{14-k'}(\gamma e^{i\theta})}{\Delta^{\ell+1}(\gamma e^{i\theta})\frac{dj}{d\tau}(\gamma e^{i\theta})}e^{2\pi i m\gamma e^{i\theta}} \\
&= (ce^{i\theta}+d)^{-k}\frac{\Delta^{\ell}(\gamma e^{i\theta})E_{k'}(\gamma e^{i\theta})E_{14-k'}(\gamma e^{i\theta})}{\Delta^{\ell+1}(\gamma e^{i\theta})\frac{dj}{d\tau}(\gamma e^{i\theta})}e^{2\pi i m\gamma e^{i\theta}} \\
&= -\frac{1}{2\pi i}(ce^{i\theta}+d)^{-k}e^{2\pi i m(\gamma e^{i\theta})},
\end{aligned}$$

as $E_{k'}(\tau)E_{14-k'}(\tau) = E_{14}(\tau)$ and $\frac{dj}{d\tau} = -2\pi i\frac{E_{14}(\tau)}{\Delta(\tau)}$. □

In particular, the residues at z and $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} z$ have a combined contribution of

$$\begin{aligned}
-2\pi i\left(\operatorname{Res}_{\tau=e^{i\theta}} + \operatorname{Res}_{\tau=-e^{-i\theta}}\right) &= e^{2\pi i m e^{i\theta}} + e^{-ik\theta}e^{-2\pi i m e^{-i\theta}} \\
&= e^{-2\pi m\sin\theta - ik\theta/2}2\cos\left(2\pi m\cos(\theta) + k\theta/2\right).
\end{aligned}$$

The technical core of this section, and of the zeros counting argument in the next, relies on estimating each of the remaining terms in (2.1). First, it is clear that the error satisfies

$$\begin{aligned} E &= |\overline{g}_{k,m}(e^{i\theta}) - 2\cos\left(2\pi m \cos(\theta) + k\theta/2\right)| \\ &\leq e^{2\pi m \sin\theta} \int_{-\frac{1}{2}+iB}^{\frac{1}{2}+iB} |G(\tau, e^{i\theta})| d\tau + e^{2\pi m \sin\theta} \sum_{\gamma \neq I_2, S} |ce^{i\theta} + d|^{-k} e^{-2\pi m \sin\theta |ce^{i\theta}+d|^{-2}}. \end{aligned}$$

Lemma 2. *Let* $\frac{\pi}{2} \leq \alpha \leq \beta \leq \frac{2\pi}{3}$*. For each* $B > 0$ *such that* $\operatorname{Im} \gamma e^{i\theta} \neq B$ *for any* γ *as above and all* $\theta \in [\alpha, \beta]$*, there is a constant* $h_{\alpha,\beta,B}$ *such that*

$$\int_{-\frac{1}{2}+iB}^{\frac{1}{2}+iB} |G(\tau, e^{i\theta})| d\tau \leq e^{-2\pi m B + h_{\alpha,\beta,B}} \left(\frac{|\Delta(e^{i\theta})|}{|\Delta(iB)|} \right)^{\ell}.$$

Proof. We follow Duke-Jenkins in estimating the integral:

$$\int_{-\frac{1}{2}+iB}^{\frac{1}{2}+iB} |G(\tau, e^{i\theta})| d\tau \leq e^{-2\pi m B} \max_{|x| \leq \frac{1}{2}} \left(\frac{|\Delta(e^{i\theta})|}{|\Delta(x+iB)|} \right)^{\ell} \int_{-\frac{1}{2}+iB}^{\frac{1}{2}+iB} \frac{|E_{k'}(e^{i\theta}) E_{14-k'}(\tau)|}{|\Delta(\tau)(j(\tau) - j(e^{i\theta}))|} d\tau.$$

Remark that $\min_{|x| \leq \frac{1}{2}} |\Delta(x + iB)| = |\Delta(iB)|$. Indeed, the q-expansion of $\frac{q}{\Delta} = \sum c_n q^n$ has nonnegative coefficients $c_n \geq 0$. Plugging in $z = x + iB$, so $|q| = e^{-2\pi B}$, we get:

$$\frac{e^{-2\pi B}}{|\Delta(x + iB)|} \leq \sum c_n e^{-2\pi n B} = \frac{e^{-2\pi B}}{|\Delta(iB)|}.$$

Finally, the function $f(\theta, \tau) = \frac{|E_{k'}(e^{i\theta}) E_{14-k'}(\tau)|}{|\Delta(\tau)(j(\tau) - j(e^{i\theta}))|}$ is analytic on the compact set $[\sigma, \beta] \times \{x + iB \mid x \in [-\frac{1}{2}, \frac{1}{2}]\}$ and thus achieves a maximum $e^{h_{\alpha,\beta,B}}$. ☐

Lemma 3. *Let* α, β *as above and* $B \in (0, \sin\alpha)$*. There exists a constant* $I(\alpha, \beta, B)$ *such that for all* $\theta \in [\alpha, \beta]$ *we have*

$$\overline{g}_{k,m}(e^{i\theta}) = 2\cos\left(2\pi m \cos(\theta) + k\theta/2\right) + o(1),$$

as $k \to \infty$ *and* $\frac{m}{\ell} < I(\alpha, \beta, B)$.

Proof. Let $\mathcal{R}$ be the finite set of elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\operatorname{Im} \gamma e^{i\theta} \geq B$ for some $\theta \in [\alpha, \beta]$, excluding I_2 and S. From Lemma 2 we conclude that

$$E \leq e^{2\pi m(\sin\theta - B) + h_B - \ell(\ln|\Delta(iB)| - \ln|\Delta(e^{i\theta})|)} + \sum_{\gamma \in \mathcal{R}} e^{2\pi m \sin\theta(1 - |ce^{i\theta}+d|^{-2}) - k \ln|ce^{i\theta}+d|}.$$

Writing $\delta = \frac{m}{\ell} \in [0, 1]$ we get

$$E \leq e^{\ell(2\pi\delta(\sin\theta - B) - (\ln|\Delta(iB)| - \ln|\Delta(e^{i\theta})|)) + h_B} + \sum_{\gamma \in \mathcal{R}} e^{\ell(2\pi\delta \sin\theta(1 - |ce^{i\theta}+d|^{-2}) - 12\ln|ce^{i\theta}+d|) - k' \ln|ce^{i\theta}+d|}.$$

When $k \to \infty$, the error term above approaches 0 as long as each of the following terms is negative:

$$\begin{aligned} &2\pi\delta(\sin\theta - B) - (\ln|\Delta(iB)| - \ln|\Delta(e^{i\theta})|) < 0 \qquad (2.2) \\ &2\pi\delta \sin\theta(1 - |ce^{i\theta} + d|^{-2}) - 12\ln|ce^{i\theta} + d| < 0 \quad \forall \gamma \in \mathcal{R}. \end{aligned}$$

The bounds (2.2) are guaranteed for all such $\theta \in [\alpha, \beta]$ if

$$\begin{aligned} &2\pi\delta(\sin\alpha - B) - (\ln|\Delta(iB)| - \ln|\Delta(e^{i\beta})|) < 0 \\ &2\pi\delta\sin\alpha(1 - |ce^{i\alpha} + d|^{-2}) - 12\ln|ce^{i\beta} + d| < 0 \quad \forall\gamma \in \mathcal{R} \ \ cd \geq 0 \\ &2\pi\delta\sin\alpha(1 - |ce^{i\beta} + d|^{-2}) - 12\ln|ce^{i\alpha} + d| < 0 \quad \forall\gamma \in \mathcal{R} \ \ cd < 0. \end{aligned} \tag{2.3}$$

Indeed, we use that $|\Delta(e^{i\theta})|$ is increasing on $[\alpha, \beta]$ ([Rav25, Prop 3.1]) and that $|ce^{i\theta} + d|^2 = c^2 + d^2 + 2cd\cos\theta$ is ≥ 1 on $[\frac{\pi}{2}, \frac{2\pi}{3}]$, and is increasing if and only if $cd < 0$. Finally, consider $I(\alpha, \beta, B)$ the minimum of the following:

- $\dfrac{\ln|\Delta(iB)| - \ln|\Delta(e^{i\beta})|}{2\pi(\sin\alpha - B)}$
- $\dfrac{12\ln|ce^{i\beta} + d|}{2\pi\sin\alpha(1 - |ce^{i\alpha} + d|^{-2})}$ for all $cd \geq 0$ and
- $\dfrac{12\ln|ce^{i\alpha} + d|}{2\pi\sin\alpha(1 - |ce^{i\beta} + d|^{-2})}$ for all $cd < 0$.

□

Thus, for each θ one can obtain a set of upper bounds for $\delta = \frac{m}{\ell}$ which guarantee the error $E \to 0$ as $k \to \infty$. One may reinterpret the approach of Duke-Jenkins and Raveh as bounding $I(\frac{\pi}{2}, 1.9, 0.75)$ and $I(1.9, \frac{2\pi}{3}, 0.65)$. Our method relies on the fact that different values of $B > 0$ are better suited for different ranges of θ, which we will execute using Sage Math [The25] in the following section.

3. Proof of Theorem A

The approximation from Lemma 3 provides the control needed to count zeros of $g_{k,m}$ on any portion of the arc $\mathcal{A}$.

Proposition 4. *Suppse $\frac{\pi}{2} \leq \alpha < \beta \leq \frac{2\pi}{3}$ and $B < \sin\alpha$. As $k \to \infty$ such that $\delta = \frac{m}{\ell} < \min(I(\alpha, \beta, B), \frac{3}{\pi})$, the Miller form $g_{k,m}$ at least*

$$\left\lfloor \frac{k\beta}{2\pi} + 2m\cos\beta \right\rfloor - \left\lceil \frac{k\alpha}{2\pi} + 2m\cos\alpha \right\rceil$$

roots on $\mathcal{A}_{\alpha,\beta} = \{e^{i\theta} \mid \alpha \leq \theta \leq \beta\}$.

Proof. Lemma 3 guarantees that, as $k \to \infty$:

$$|\overline{g}_{k,m}(e^{i\theta}) - 2\cos(k\theta/2 + 2\pi m\cos\theta)| < 2,$$

for all $\theta \in \mathcal{A}_{\alpha,\beta}$ as long as $\frac{m}{\ell} < I(\alpha, \beta, B)$. When this inequality is satisfied, as in [DJ08, Lemma 3] and [Rav25, §3.4] we see that $k\theta/2 + 2\pi m\cos\theta$ is increasing on $[\alpha, \beta]$ as $\delta \leq \frac{3}{\pi}$. This means that the expression takes at least

$$\left\lfloor \frac{k\beta}{2\pi} + 2m\cos\beta \right\rfloor - \left\lceil \frac{k\alpha}{2\pi} + 2m\cos\alpha \right\rceil + 1$$

consecutive integer multiple of π values in $[\alpha, \beta]$, and thus $2\cos(k\theta/2 + 2\pi m\cos\theta)$ changes sign in $[\alpha, \beta]$ at least $\left\lfloor \frac{k\beta}{2\pi} + 2m\cos\beta \right\rfloor - \left\lceil \frac{k\alpha}{2\pi} + 2m\cos\alpha \right\rceil$ times. □

The stage is now set for the proof of our first main result.

Theorem 5. *As $k \to \infty$ we have that*

$$\frac{1}{D}\left|\left\{\begin{array}{c}\textit{non-elliptic zeros}\\ \textit{of } g_{k,m} \textit{ on } \mathcal{A}\end{array}\right\}\right| \geq \begin{cases}1 & \delta < 0.6194\\ 1 - 2.9832(\delta - 0.6194) & 0.6194 \leq \delta < 0.9546\\ 0 & \delta \geq 0.9546.\end{cases}$$

Moreover, as $k \to \infty$ a subset of the roots of $g_{k,m}$ will become equidistributed on the arc from $\pi/2$ to

$$\Theta(\delta) = \begin{cases}2\pi/3 & \delta < 0.6194\\ 2\pi/3 - 2.9832(\delta - 0.6194)\pi/6 & 0.6194 \leq \delta < 0.9546\\ \pi/2 & \delta \geq 0.9546.\end{cases}$$

More precisely, there exists a decreasing piece-wise linear function $\mathcal{P} : [0,1] \to [0,1]$, explicitly computed in Sage and whose graph is given below, such that for k large enough

$$\frac{1}{D}\left|\left\{\begin{array}{c}\textit{non-elliptic zeros}\\ \textit{of } g_{k,m} \textit{ on } \mathcal{A}\end{array}\right\}\right| \geq \mathcal{P}\left(\frac{m}{\ell}\right).$$

Proof. We subdivide the interval $[\frac{\pi}{2}, \frac{2\pi}{3}]$ into $N = 1000$ equal intervals $I_r = [\frac{\pi}{2} + \frac{\pi r}{6N}, \frac{\pi}{2} + \frac{\pi(r+1)}{6N}]$, for $0 \leq r < N$, and approximate the Miller form $g_{k,m}$ on each arc $\mathcal{A}_r = e^{iI_r}$ separately.

On each interval $I_r = [\alpha_r, \beta_r]$ we choose $B_r \in (\frac{1}{2}\tan\frac{\beta_r}{2}, \sin\alpha_r)$ and compute the bound $\delta_r(B_r) = I(\alpha_r, \beta_r, B_r)$. Lemma 3 guarantees that, as $k \to \infty$,

$$\overline{g}_{k,m}(e^{i\theta}) = 2\cos(k\theta/2 + 2\pi m\cos\theta) + o(1)$$

for all $\theta \in I_r$ as long as $\frac{m}{\ell} < \delta_r(B_r)$ and $\frac{3}{\pi}$. We remark that the lower bound $B_r > \frac{1}{2}\tan\frac{\beta_r}{2} \geq \frac{1}{2} \geq \frac{1}{2}\cot\frac{\alpha_r}{2}$ implies that the set $\mathcal{R}$ (as defined in Lemma 3) is empty.

We want to choose $B_r \in (\frac{1}{2}\tan\frac{\beta_r}{2}, \sin\alpha_r)$ such that $\delta_r(B_r)$ is as large as possible. We perform an incremental search in Sage: as B_r varies from $\frac{1}{2}\tan\frac{\beta_r}{2}$ to $\sin\alpha_r$ in increments of 0.0005 we compute $\delta_r(B_r)$ and in the end choose B_r such that $\delta_r = \delta_r(B_r)$ is maximal. The following graph shows the values of B_r maximizing $\delta_r(B_r)$ under each corresponding arc $\mathcal{A}_r$, as well as the choice of $B_{\mathrm{DJ}} = 0.65$ and $B_{\mathrm{DJ}} = 0.75$ from [DJ08]. By inspection, B_r and δ_r are decreasing in r.

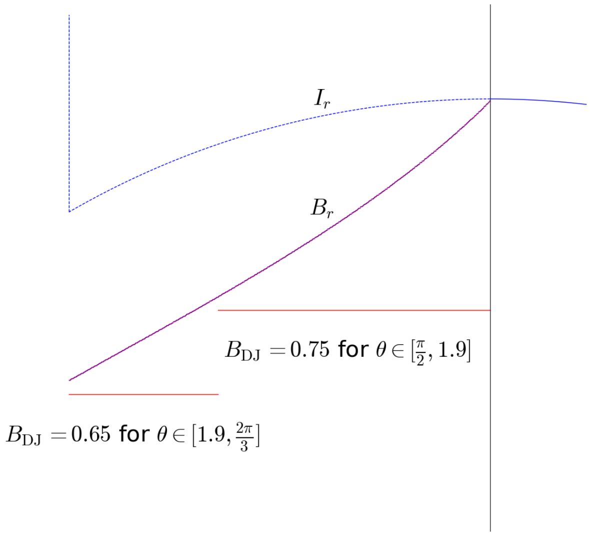


FIGURE 4. Choice of B_r for each arc $\mathcal{A}_r$

Suppose, now, that $\delta < \frac{3}{\pi} \approx 0.9549$. Since δ_r is decreasing in r, there exists a maximum r_0 such that $\delta < \delta_r$ for all $r \leq r_0$. We conclude that

$$\overline{g}_{k,m}(e^{i\theta}) = 2\cos(k\theta/2 + 2\pi m \cos\theta) + o(1)$$

for all $\theta \in [\frac{\pi}{2}, \beta_{r_0}]$. Then Proposition 4 applies for this interval and so the Miller form $g_{k,m}$ will have at least

$$\left\lfloor \frac{k\beta_{r_0}}{2\pi} + 2m\cos\beta_{r_0} \right\rfloor - \left\lceil \frac{k}{4} \right\rceil$$

roots on the arc $\mathcal{A}$, all of them lying on the arc corresponding to the angle interval $[\frac{\pi}{2}, \beta_{r_0}]$.

The computation in Sage gives the value

$$\delta_N = \delta_{1000} = 0.6194\ldots$$

which implies that if $\frac{m}{\ell} < \delta_N$ for $k \gg 0$, $g_{k,m}$ has at least

$$\left\lfloor \frac{k}{3} - m \right\rfloor - \left\lceil \frac{k}{4} \right\rceil = \ell - m$$

roots on $\mathcal{A}$. Since $g_{k,m}$ can have at most $\ell - m$ roots in the upper half plane, we conclude that every root of $g_{k,m}$ lies on $\mathcal{A}$ and define $\mathcal{P}(\delta) = 1$ for $\delta < 0.6194$.

Suppose that $\delta_{r_0+1} \leq \delta < \delta_{r_0}$. Then $g_{k,m}$ has at least

$$\left\lfloor \frac{k\beta_{r_0}}{2\pi} + 2m\cos\beta_{r_0} \right\rfloor - \left\lceil \frac{k}{4} \right\rceil > \frac{k\beta_{r_0}}{2\pi} + 2m\cos\beta_{r_0} - \frac{k}{4} - 2 > \frac{6\ell\beta_{r_0}}{\pi} + 2m\cos\beta_{r_0} - 3\ell - 2$$

roots on $\mathcal{A}$. This implies that

$$\frac{1}{D}\left|\{\text{zeros of } g_{k,m} \text{ on } \mathcal{A}\}\right| \geq \frac{\frac{6}{\pi}\beta_{r_0} + 2\delta\cos\beta_{r_0} - 3}{1-\delta} - \frac{2}{\ell}.$$

Fixing a small ε, say $\varepsilon = 10^{-10}$, if $k > 24/\varepsilon$ we may define

$$\mathcal{P}(\delta) = \min(0, \frac{\frac{6}{\pi}\beta_{r_0} + 2\delta\cos\beta_{r_0} - 3}{1-\delta} - \varepsilon)$$

for all $\delta \in (\delta_{r_0+1}, \delta_{r_0}]$.

Finally, set $\mathcal{P}(\delta) = 0$ for $\delta \in [\frac{3}{\pi}, 1]$. We notice that $\mathcal{P}(\delta) = 0$ for $\delta \geq 0.9546$.

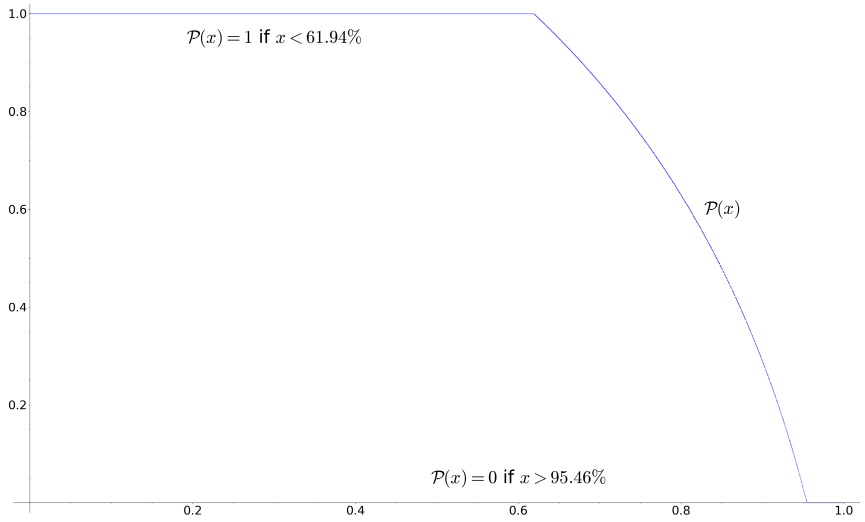


Figure 5. Lower bound $\mathcal{P}$ for the proportion of roots on $\mathcal{A}$

The first lower bound comes from the fact that the linear function connecting $(0.6194, 1)$ and $(0.9546, 0)$ lies below the graph of $\mathcal{P}$.

We may similarly define a function $\mathcal{T}(\delta)$ by setting $\mathcal{T}(\delta) = \frac{2\pi}{3}$ for $\delta < 0.6194$, $\mathcal{T}(\delta) = \beta_{r_0}$ if $\delta_{r_0+1} \leq \delta < \delta_{r_0}, \frac{3}{\pi}$, and $\mathcal{T}(\delta) = \frac{\pi}{2}$ otherwise.

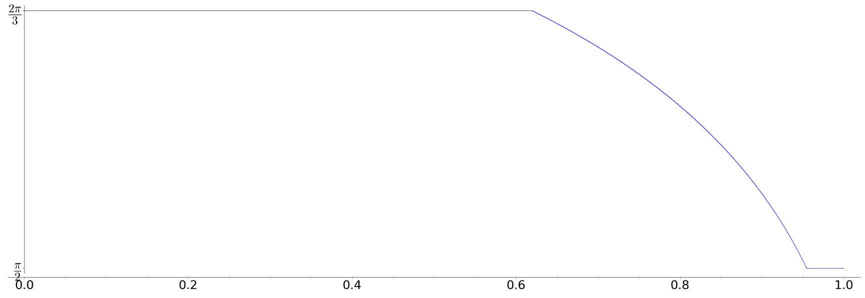


Figure 6. Angle $\mathcal{T}(\delta)$ with $\mathcal{A}_{\frac{\pi}{2}, \mathcal{T}(\delta)}$ containing Miller zeros

By the above, $\overline{g}_{k,m}(e^{i\theta}) = 2\cos(k\theta/2 + 2\pi m \cos\theta) + o(1)$ on the arc $[\frac{\pi}{2}, \mathcal{T}(\delta)]$ and the equidistribution was proved in [Rav25, §3.4] on any arc where this asymptotics holds.

The conclusion follows form the fact that Θ is the piece-wise linear lower hull of $\mathcal{T}$. $\square$

4. Zeros of weakly holomorphic forms

In this section, we extend our previous result to weakly holomorphic modular forms, i.e., modular forms that are holomorphic on $\mathfrak{H}$ with poles allowed at the cusp $i\infty$. Writing again $k = 12\ell + k'$ for some $\ell \in \mathbb{Z}$ and $k' \in \{0, 4, 6, 8, 10, 14\}$, the space of weight k weakly holomorphic modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is infinite-dimensional with a basis consisting of forms $g_{k,m} = q^m + O(q^{\ell+1})$ for $m \leq \ell$.

If $m \leq \ell - |\ell|$, Duke and Jenkins [DJ08] showed that all roots of $g_{k,m}$ are on the arc $\mathcal{A}$. We refine their result by showing that, for $k \ll 0$, all finite roots of $g_{k,m}$ are on $\mathcal{A}$ whenever $\dfrac{m}{\ell} > 115.98\%$, and moreover $g_{k,m}$ has some roots on $\mathcal{A}$ if $\dfrac{m}{\ell} > 110.26\%$. This is close to the experimentally obtained bounds, as seen in the following table.

k	ℓ	Not all roots on $\mathcal{A}$	No roots on $\mathcal{A}$
-12000	-1000	$m \geq -1154$	$m \geq -1090$
$-12000+4$	-1000	$m \geq -1155$	$m \geq -1096$
$-12000+6$	-1000	$m \geq -1151$	$m \geq -1089$
$-12000+8$	-1000	$m \geq -1156$	$m \geq -1105$
$-12000+10$	-1000	$m \geq -1152$	$m \geq -1096$
$-12000+14$	-1000	$m \geq -1153$	$m \geq -1104$

Given the similarity to the proof of Theorem 5, we outline the argument and concentrate on the adjustments needed in the present context.

Lemma 6. *If $B \geq 0.287$, the maximum of $|\Delta(x+iB)|$ is attained when $x = \frac{1}{2}$.*

Proof. Consider the function

$$f(x) = \log|\Delta(x+iB)|^2 = -2\pi B + \sum_{n=1}^{\infty}(1 - 2e^{-2\pi Bn}\cos(nx) + e^{-4\pi Bn})$$

on $[-\frac{1}{2}, \frac{1}{2}]$, with derivative $f'(x) = 2\sum_{n=1}^{\infty} ne^{-2\pi Bn}\sin(nx)$. It suffices to show that $f'(x) \geq 0$ on $[0, \frac{1}{2}]$ by symmetry. In fact, we'll show that

$$\frac{f'(x)}{2\sin x} = e^{-2\pi B} + \sum_{n=2}^{\infty} ne^{-2\pi nB}\frac{\sin(nx)}{\sin x} \geq 0$$

on this interval. Writing $A = e^{-2\pi B} < 0.1648$ we want to show that $A > \sum_{n=2}^{\infty} nA^n \dfrac{\sin(nx)}{\sin x}$. But Chebyshev polynomials imply that $|\frac{\sin(nx)}{\sin x}| \leq n$ and whenever $A < 0.1648$ we have

$$A > \frac{A^2+A}{(1-A)^3} - A = \sum_{n=2}^{\infty} n^2 A^n \geq \sum_{n=2}^{\infty} nA^n |\frac{\sin(nx)}{\sin x}|.$$

□

Remark 1. With a little more care, one can show that Lemma 6 holds for $B \geq 0.25$. However, it fails, for instance, when $B = 0.235$.

Lemma 7. *Let $\frac{\pi}{2} \leq \alpha \leq \beta \leq \frac{2\pi}{3}$. For each $B > 0.287$ such that $\operatorname{Im}\gamma e^{i\theta} \neq B$ for any γ as above and all $\theta \in [\alpha, \beta]$, there is a constant $h_{\alpha,\beta,B}$ such that*

$$\int_{-\frac{1}{2}+iB}^{\frac{1}{2}+iB} |G(\tau, e^{i\theta})| d\tau \leq e^{-2\pi mB + h_{\alpha,\beta,B}} \left(\frac{|\Delta(e^{i\theta})|}{|\Delta(\frac{1}{2}+iB)|}\right)^{\ell}.$$

Proof. The proof is identical to Lemma 2, except in using Lemma 6 to show that if $\ell < 0$

$$\max_{|x|\leq\frac{1}{2}}\left(\frac{|\Delta(e^{i\theta})|}{|\Delta(x+iB)|}\right)^{\ell} = \left(\frac{|\Delta(e^{i\theta})|}{|\Delta(\frac{1}{2}+iB)|}\right)^{\ell}.$$

□

Lemma 8. *Let* α, β *as above and* $0.287 < B < \sin\alpha$*, and let* $k < 0$*. There exists a constant* $I^{\mathrm{w}}(\alpha, \beta, B) \geq 1$ *such that for all* $\theta \in [\alpha, \beta]$ *we have*

$$\overline{g}_{k,m}(e^{i\theta}) = 2\cos\left(2\pi m\cos(\theta) + k\theta/2\right) + o(1),$$

as $k \to -\infty$ *and* $\frac{m}{\ell} > I^{w}(\alpha, \beta, B)$.

Proof. As in Lemma 8, write $\delta = \frac{m}{\ell} \in [1, \infty)$ and denote $\mathcal{R}$ the finite set of elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\operatorname{Im}\gamma e^{i\theta} \geq B$ for some $\theta \in [\alpha, \beta]$, excluding I_2 and S. From Lemma 7 we conclude that

$$\overline{g}_{k,m}(e^{i\theta}) - 2\cos\left(2\pi m\cos(\theta) + k\theta/2\right) = o(1)$$

as $k \to -\infty$ as long as

$$\begin{aligned} &2\pi\delta(\sin\theta - B) + \ln|\Delta(e^{i\theta})| - \ln|\Delta(\frac{1}{2}+iB)| > 0 \\ &2\pi\delta\sin\theta(1 - |ce^{i\theta}+d|^{-2}) - 12\ln|ce^{i\theta}+d| > 0 \quad \forall\gamma\in\mathcal{R}. \end{aligned} \tag{4.1}$$

The bounds (4.1) are guaranteed for all such $\theta \in [\alpha, \beta]$ if

$$\begin{aligned} &2\pi\delta(\sin\beta - B) - (\ln|\Delta(\frac{1}{2}+iB)| - \ln|\Delta(e^{i\alpha})|) > 0 \\ &2\pi\delta\sin\beta(1 - |ce^{i\beta}+d|^{-2}) - 12\ln|ce^{i\alpha}+d| > 0 \quad \forall\gamma\in\mathcal{R} \ \ cd \geq 0 \\ &2\pi\delta\sin\beta(1 - |ce^{i\alpha}+d|^{-2}) - 12\ln|ce^{i\beta}+d| > 0 \quad \forall\gamma\in\mathcal{R} \ \ cd < 0. \end{aligned} \tag{4.2}$$

Finally, consider $I^{\mathrm{w}}(\alpha, \beta, B)$ the maximum of the following:

- $\dfrac{\ln|\Delta(\frac{1}{2}+iB)| - \ln|\Delta(e^{i\alpha})|}{2\pi(\sin\beta - B)}$
- $\dfrac{12\ln|ce^{i\alpha}+d|}{2\pi\sin\beta(1-|ce^{i\beta}+d|^{-2})}$ for all $cd \geq 0$ and
- $\dfrac{12\ln|ce^{i\beta}+d|}{2\pi\sin\beta(1-|ce^{i\alpha}+d|^{-2})}$ for all $cd < 0$.

□

Theorem 9. *There exists a piece-wise constant function* $\mathcal{P}_-(x)$*, whose graph appears below, such that for* $k \ll 0$ *the proportion of roots of* $g_{k,m}$ *on the boundary arc is at least* $\mathcal{P}_-(\frac{m}{\ell})$.

Moreover, there exists a piece-wise linear function $\mathcal{T}_- : [0,1] \to [\frac{\pi}{2}, \frac{2\pi}{3}]$*, such that for* $\delta > 1$*, roots of* $g_{k,m}$ *equidistribute on the arc* $[\mathcal{T}_-(\delta), \frac{2\pi}{3}]$ *as* $\frac{m}{\ell} > \delta$.

Proof. The proof is identical to that of Theorem 5, using an explicit search in Sage. □

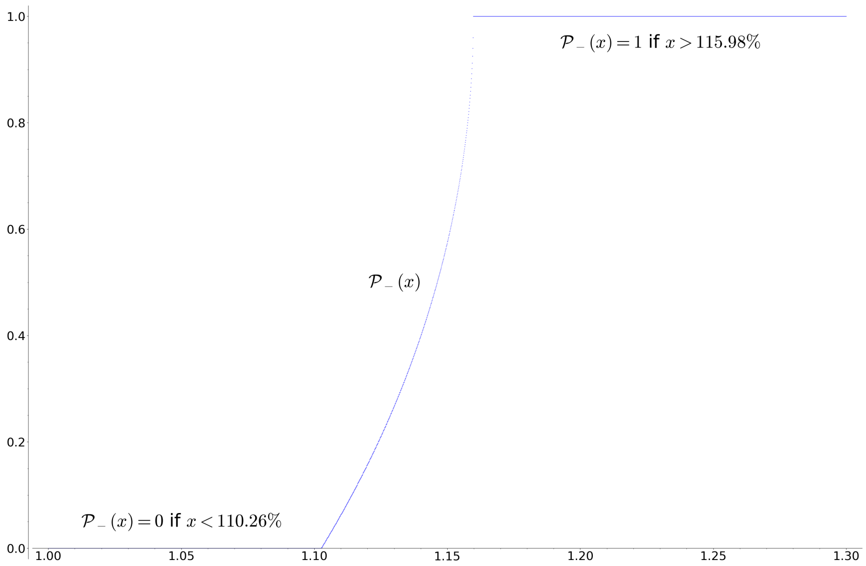


Figure 7. Lower bound $\mathcal{P}_-$ for the proportion of roots on $\mathcal{A}$

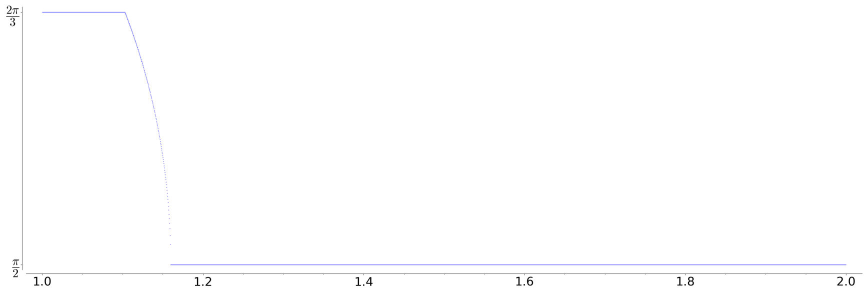


Figure 8. Angle $\mathcal{T}_-(\delta)$ with $\mathcal{A}_{\frac{\pi}{2},\mathcal{T}_-(\delta)}$ containing Miller zeros

5. Proof of Theorem B

Consider again the Miller basis $g_{k,m} = q^m + O(q^{\ell+1})$ of weight $k = 12\ell + k'$, with $k' \in \{0, 4, 6, 8, 10, 14\}$. For some monic polynomial P of degree $D = \ell - m$ we have

$$\frac{g_{k,m}}{\Delta^\ell E_{k'}} = P(j),$$

where $j = E_4^3/\Delta$ is the usual j-invariant. We refer to P as the Faber polynomial of $g_{k,m}$. Writing

$$P(X) = \sum_{d=0}^{D} y_d x^{D-d} \quad y_0 = 1,$$

we will be concerned with estimating the coefficients y_d when D grows with k, albeit more slowly. For all $r \geq 1$ we also set

$$j^r = \sum_{n=-r}^{\infty} c_{r,n} q^n,$$

so that

$$P(j) = \sum_{d=0}^{D} y_d j^{D-d} = \sum_{d=0}^{D} \sum_{n \geq -(D-d)} y_d c_{D-d,n} q^n.$$

Since

$$\frac{g_{k,m}}{\Delta^\ell E_{k'}} = q^{-D} \frac{1 + O(q^{D+1})}{(q^{-1}\Delta)^\ell E_{k'}},$$

it follows that

$$(q^{-1}\Delta)^{-\ell}(E_{k'})^{-1} = \sum_{d=0}^{D} \sum_{n=-(D-d)}^{0} y_d c_{D-d,n} q^{D+n} + O(q^{D+1}).$$

On the other hand, we may expand the power series formally as

$$(q^{-1}\Delta)^{-\ell}(E_{k'})^{-1} = \sum_{d=0}^{\infty} \mathcal{D}_{\ell,d} q^d.$$

Lemma 10. (1) *If* $d = O(|k|^{\frac{1}{2}-\varepsilon})$ *for some* $\varepsilon \in (0, \frac{1}{2})$ *then*

$$\mathcal{D}'_{\ell,d}, \mathcal{D}_{\ell,d} = \frac{(2|k|)^d}{d!} \left(1 + O\left(|k|^{-2\varepsilon}\right)\right).$$

(2) *If* $d \leq |\ell|$ *then*

$$\mathcal{D}'_{\ell,d}, \mathcal{D}_{\ell,d} = O\left(\left(\frac{2\alpha|k|}{d}\right)^d\right),$$

where $\alpha = \left(\frac{25}{24}\right)^{25}$.

Proof. Suppose $k > 0$, as the proof when $k < 0$ is almost identical. As in [Rud24, §4], we have

$$\mathcal{D}'_{\ell,d} = \sum_{\sum i r_i = d} \prod_i \binom{2k + r_i - 1}{r_i}.$$

By the Stirling approximation, we have

$$\begin{aligned}
\binom{2k+r-1}{r} &= \frac{2k}{2k+r}\binom{2k+r}{r} \\
&= \frac{2k}{2k+r} \frac{(2k+r)^{2k+r}}{r!(2k)^{2k}} \frac{1}{e^r} \sqrt{\frac{2k+r}{2k}} \left(1 + \frac{1}{24k} + O(k^{-2})\right) \\
&= \frac{(2k)^r}{r!} \left[\frac{1}{e}\left(1 + \frac{r}{2k}\right)^{1 + \frac{2k}{r} - \frac{1}{2r}}\right]^r \left(1 + \frac{1}{24k} + O(k^{-2})\right).
\end{aligned}$$

(1) If $r \le d = O(k^{\frac{1}{2}-\varepsilon})$, we see that

$$\begin{aligned}\log\left[\frac{1}{e}\left(1+\frac{r}{2k}\right)^{1+\frac{2k}{r}-\frac{1}{2r}}\right]^r &= (r+2k-\frac{1}{2})\log\left(1+\frac{r}{2k}\right)-r\\ &= (r+2k-\frac{1}{2})\left(\frac{r}{2k}-\frac{r^2}{8k^2}+O\left(\frac{r^3}{k^3}\right)\right)-r\\ &= \frac{r^2}{4k}-\frac{r}{4k}+\frac{r^2}{16k^2}+O\left(\frac{r^3}{k^2}\right)\\ &= \frac{r^2}{4k}+O\left(\frac{r}{k}\right)=O(k^{-2\varepsilon}).\end{aligned}$$

We conclude that

$$\sum_{\sum ir_i=d}\prod_i\binom{2k+r_i-1}{r_i}=\sum_{\sum ir_i=d}\prod_i\frac{(2k)^{r_i}}{r_i!}\left(1+\frac{r_i^2}{4k}+O\left(\frac{r}{k}\right)\right)\left(1+\frac{1}{24k}+O(k^{-2})\right)^{d'},$$

where $d' < \sqrt{2d}$ is the number of nonzero r_i. Thus $\left(1+\frac{1}{24k}+O(k^{-2})\right)^{d'}=1+O(k^{-\frac{3}{4}-\frac{\varepsilon}{2}})$, while $\prod\left(1+\frac{r_i^2}{4k}+O\left(\frac{r}{k}\right)\right)=1+O(k^{-2\varepsilon})$ for any choice of $r=(r_1,\dots,r_d)$. Therefore, it suffices to verify that the term with $r=(d,0,\dots,0)$ dominates the sum.

Note that the sum

$$\sum_{\sum ir_i=d}\prod_i\frac{(2k)^{r_i}}{r_i!}=\sum_{j=1}^{d}\binom{d-1}{j-1}\frac{(2k)^j}{j!}$$

can be computed using the Laguerre polynomial $L_d^{-1}(-2k)$. The terms in the latter sum are increasing in j as $d=O(k^{\frac{1}{2}-\varepsilon})$ so

$$\begin{aligned}\sum_{\sum ir_i=d}\prod_i\frac{(2k)^{r_i}}{r_i!} &= \frac{(2k)^d}{d!}+\frac{(2k)^{d-1}}{(d-1)!}+O\left(\frac{(d-2)(2k)^{d-2}}{(d-2)!}\right)\\ &= \frac{(2k)^d}{d!}\left(1+\frac{d}{2k}+O\left(\frac{d^3}{k^2}\right)\right)\end{aligned}$$

and the desired formula follows.

(2) If $r \le d \le \ell$, we have

$$\left[\frac{1}{e}\left(1+\frac{r}{2k}\right)^{1+\frac{2k}{r}-\frac{1}{2r}}\right]^r<\left[\frac{1}{e}\left(1+\frac{r}{2k}\right)^{1+\frac{2k}{r}}\right]^r<e^{-r}\alpha^r,$$

as $\frac{r}{2k}\le\frac{\ell}{2k}=\frac{1}{24}$. As above, the term that dominates corresponds to $r_1=d$, in which case Stirling's approximation gives

$$\mathcal{D}'_{\ell,d}=O\left(\frac{(2e^{-1}\alpha k)^d}{d!}\right)=O\left(\left(\frac{2\alpha k}{d}\right)^d\right).$$

Turning our attention to the coefficients $\mathcal{D}_{\ell,d}$, we recall that the coefficients in the expansion

$$\frac{1}{E_{k'}}=\sum_{n\ge 0}\beta_n q^n,$$

satisfy $\beta_n = O(e^{2\pi n})$. Indeed, when $k' = 6, 10, 14$ this follows from [HN22, Theorem 1]; for $k' = 4$ one has $\beta_n = O(e^{\pi\sqrt{3}n})$ by [HN22, Theorem 3]), and the statement for $k' = 8$ is clear since $E_8 = E_4^2$. Thus,

$$\mathcal{D}_{\ell,d} = \sum_{i=0}^{d} \mathcal{D}'_{\ell,i}\beta_{d-i} = \frac{(2k)^d}{d!}\left(1 + O\left(\frac{1}{k^{\frac{1}{2}+\varepsilon}}\right)\right),$$

if $d = O(k^{\frac{1}{2}-\varepsilon})$, while if $d \le \ell$, we have $\mathcal{D}_{\ell,d} = O\left(\left(\frac{2\alpha k}{d}\right)^d\right)$.

□

Finally, we recall some estimates for the coefficients of the powers of j. Namely, according to [BP05, Proposition 4.1], for all $r \ge 1$ and $n \in \mathbb{Z}$ such that $n \ge -r$ we have

$$c_{r,n} \le e^{2\pi n}1728^r.$$

Sharper estimates are available when $-r + 1 \le n \le -rc$, where $c = \dfrac{e^{2\pi}}{1728}$. In this range, [BP05, Proposition 4.2] says that

$$c_{r,n} \le (1728 - e^{2\pi})^{r+n}\frac{(-n)^n r^r}{(r+n)^{r+n}}.$$

Having assembled the necessary ingredients, we proceed to the central technical component of this section.

Proposition 11. *If $D = O(|k|^{1/2-\varepsilon})$ for some $\varepsilon > 0$, we have*

$$y_d = \frac{(2|k|)^d}{d!}\left(1 + O(|k|^{-2\varepsilon})\right),$$

for each $0 \le d \le D$.

Proof. Again, we assume $k > 0$, the other case being similar. We use induction on d; the base case is immediate because $y_0 = 1$.

Suppose $y_r = \frac{(2k)^r}{r!}(1 + O(k^{-2\varepsilon}))$ holds for all $0 \le r \le d - 1$. Since

$$\sum_{r=0}^{d} y_r c_{D-r,-(D-d)} = \mathcal{D}_{\ell,d} = \frac{(2k)^d}{d!}(1 + O(1/k^{\frac{1}{2}+\varepsilon})), \tag{5.1}$$

it suffices to check that

$$\frac{y_d - \mathcal{D}_{\ell,d}}{(2k)^d/d!} = \frac{d!}{(2k)^d}\sum_{r=0}^{d-1} y_r c_{D-r,-(D-d)} = O\left(\sum_{r=0}^{d-1}\frac{d!}{(2k)^{d-r}r!}c_{D-r,-(D-d)}\right)$$

is, in fact, $O(k^{-2\varepsilon})$.

When $D - d > c(D - r)$, and hence whenever $d < D(1 - c)$, we know that

$$c_{D-r,-(D-d)} \le (1728 - e^{2\pi})^{d-r}\frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}}.$$

Otherwise, the bound $c_{D-r,-(D-d)} \le 1728^{D-r}e^{-2\pi(D-d)}$ still applies.

Thus, for $d < D(1-c)$, we have

$$\sum_{r=0}^{d-1} \frac{d!}{(2k)^{d-r} r!} c_{D-r,-(D-d)} \leq \sum_{r=0}^{d-1} \frac{d!}{(2k)^{d-r} r!} (1728 - e^{2\pi})^{d-r} \frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}}$$

When $r = 0$, the term above is

$$\begin{aligned} \frac{d!}{(2k)^d} \frac{(1728 - e^{2\pi})^d D^D}{(D-d)^{D-d} d^d} &= O\left(\sqrt{d} \left(\frac{d(1728 - e^{2\pi})}{ke} \right)^d \left(1 + \frac{d}{D-d} \right)^{D-d} \right) \\ &= O\left(\left(\frac{dC}{k} \right)^d \right) = O(k^{-2\varepsilon}) \end{aligned}$$

for any $C > 1728 - e^{2\pi}$.

If $r \geq 1$, Stirling's bound gives

$$\begin{aligned} &\frac{d!}{(2k)^{d-r} r!} (1728 - e^{2\pi})^{d-r} \frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}} \\ &= O\left(\frac{d^d}{k^{d-r} r^r} (1728 - e^{2\pi})^{d-r} \frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}} \right), \end{aligned}$$

as $\sqrt{d/r} \leq 2^{d-r}$. Note that the derivative with respect to r of log of the big-O term is

$$\log(k) - \log(1728 - e^{2\pi}) - \log(D-r) + \log(d-r),$$

which is always positive if $D < k/(1728 - e^{2\pi})$.

Fix $t > 0$ such that $\left(\frac{1}{2} - \varepsilon\right)(2t+1) < t - 2\varepsilon$. Then $\frac{|y_d - \mathcal{D}_{\ell,d}|}{(2k)^d/d!}$ is

$$\begin{aligned} &O\left(\frac{(d-t)d^d}{k^t (d-t)^{d-t}} (1728 - e^{2\pi})^t \frac{(D-d+t)^{D-d+t}}{(D-d)^{D-d} t^t} + \frac{t d^d (1728 - e^{2\pi})}{k(d-1)^{d-1}} \frac{(D-(d-1))^{D-(d-1)}}{(D-d)^{D-d}} \right) \\ &= O\left(\frac{d(d(D-d+t))^t C_1^t}{k^t t^t} + \frac{td(D-d+t)C_2}{k} \right) = O\left(\frac{D^{2t+1} C_1^t}{k^t} + \frac{tD^2 C_2}{k} \right) = O(k^{-2\varepsilon}), \end{aligned}$$

for some constants C_1 and C_2.

If $d > D(1-c)$, we have to break up the sum into pieces based on $D - r$.

$$\begin{aligned} \frac{|y_d - \mathcal{D}_{\ell,r}|}{(2k)^d/d!} &= O\left(\sum_{r < D - c^{-1}(D-d)} \frac{d!}{(2k)^{d-r} r!} 1728^{D-r} e^{-2\pi(D-d)} \right) \\ &\quad + O\left(\sum_{r > D - c^{-1}(D-d)}^{d-1} \frac{d!}{(2k)^{d-r} r!} (1728 - e^{2\pi})^{d-r} \frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}} \right) \\ &= O\left(\sum_{r < D - c^{-1}(D-d)} \frac{d^d}{k^{d-r} r^r} 1728^{D-r} e^{-2\pi(D-d)} \right) + O(k^{-2\varepsilon}) \end{aligned}$$

As in the previous situation, we see that the term in the sum increases with r as long as $D < k/1728$.

When $d \leq D-1$, in the above sum we take $r = D - c^{-1}(D-d)$ and the sum is

$$= O\left(\frac{Dd^d}{(D-c^{-1}(D-d))^{D-c^{-1}(D-d)}}\left(\frac{1728^{1/c}}{e^{2\pi}k^{\frac{1-c}{c}}}\right)^{D-d}\right).$$

The derivative with respect to d of log of the above term is

$$\log d - c^{-1}\log(D-c^{-1}(D-d)) + \frac{1-c}{c}\log k - O(1),$$

which is positive whenever $D = o(k)$. Therefore, we may take $d = D-1$ and the sum is

$$= O\left(\frac{D(D-1)^{D-1}}{(D-c^{-1})^{D-c^{-1}}}\left(\frac{1728^{1/c}}{e^{2\pi}k^{\frac{1-c}{c}}}\right)\right) = O(k^{-1/2}),$$

as $D = O(k^{1/2-\varepsilon})$.

Finally, when $d = D$, $r \leq D-1$ and the terms in the sum are maximized when $r = D-1$. Therefore, the sum is

$$O\left(\frac{(D-1)D^D}{k(D-1)^{D-1}}1728e^{-2\pi}\right) = O\left(\frac{D^2}{k}\right) = O(k^{-2\varepsilon}).$$

□

We now apply the above approximation to Faber polynomials to the study of roots of $g_{k,m}$.

Corollary 12. *Suppose $D = O(k^{\frac{1}{2}-\varepsilon})$ for some $\varepsilon > 0$. Then $g_{k,m}$ has no non-elliptic roots on $\mathcal{A}$ for $k \gg 0$.*

Proof. By Proposition 11, the Faber polynomial of $g_{k,m}$ is

$$\sum_{r=0}^{D}\frac{(2k)^r}{r!}\left(1+O(k^{-2\varepsilon})\right)x^{D-r}.$$

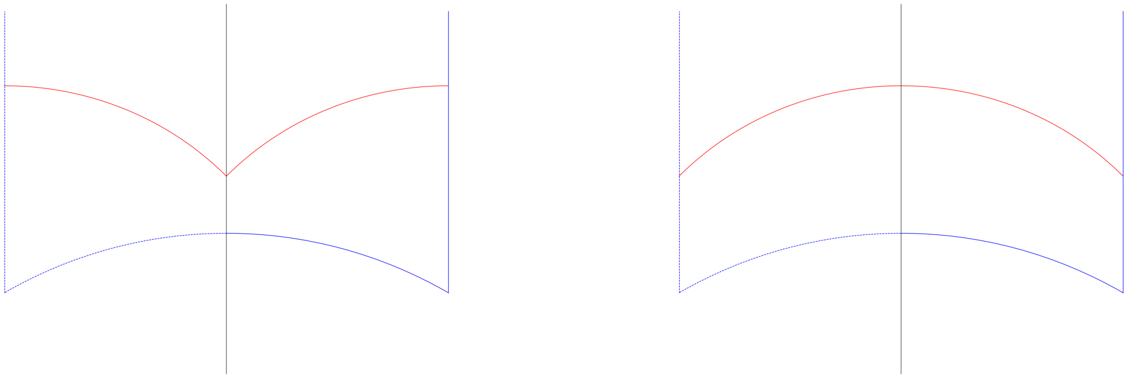
In particular, for $k \gg 0$, all the coefficients are positive and the polynomial cannot have nonnegative roots. Since $j(\mathcal{A}) = [0, 1728]$, the form $g_{k,m}$ cannot have non-elliptic roots on $\mathcal{A}$. □

Finally, when D is much smaller than k, we can prove Theorem B on the location of the zeros of $g_{k,m}$. To this end, we will use the following stability result for the preimage of the Szegő curve under the j-invariant.

Proposition 13. *For a real number $\delta > 0$, let $\mathcal{S}_\delta$ be the curve of points $\tau \in \mathcal{F}$ such that $\frac{24}{(1-\delta)j(\tau)} \in \mathcal{S}$. Then $\mathcal{S}_\delta - \frac{1}{2\pi}\log|1-\delta|$ approaches $\mathcal{L}_\pm$ as $\delta \to 1^\mp$.*

Proof. Note that $|j(z)| \gg 0$ then $\operatorname{Im} z = \frac{\log|j(z)|}{2\pi} + o(1)$ and the conclusion follows. □

Theorem 14. *Suppose $D \to \infty$ such that $D < \frac{\alpha\log|k|}{\log\log|k|}$ for some $\alpha \in (0,1)$. Then the zeros of $g_{k,m}$ asymptotically approach $\mathcal{S}_\delta = \mathcal{L}_\pm - \frac{1}{2\pi}\log|1-\delta|$, where $\delta = \frac{m}{\ell}$ and $\pm$ is the sign of k.*



(A) $\mathcal{S}_\delta$ when $\delta = 0.98$ (B) $\mathcal{S}_\delta$ when $\delta = 1.02$

Figure 9. Asymptotic location of zeros as $\delta \to 1$

Proof. Let $E_D(x) = \sum_{k=0}^{D} a_k x^{D-k}$, where $a_k = \frac{1}{k!}$, in which case, by Proposition 11 applied to $\varepsilon \in (\alpha/2, 1/2)$, the Faber polynomial satisfies

$$\frac{F(2kx)}{(2k)^D} = \sum_{k=0}^{D} b_k x^{D-k},$$

where $b_k = a_k(1+O(k^{-\alpha}))$. As in [Rud24, §5.1], we use Ostrowski's theorem which states that the roots of $E_D(x)$ and $F(2kx)$ are within $2D\Gamma(\sum |a_k - b_k|\Gamma^{-k})^{1/D}$, where $\Gamma = \max(a_k^{1/k}, b_k^{1/k})$.

In our case, we may take $\Gamma = 2$ for $k \gg 0$, in which case

$$\sum |a_k - b_k|\Gamma^{-k} = \sum_{k=0}^{D} \frac{O(k^{-\alpha})}{k!\Gamma^k} = O(e^{-\Gamma}k^{-\alpha}).$$

Thus, the roots of $F(2kx)$ and $E_D(x)$ are within $O(De^{-\Gamma/D}k^{-\alpha/D}) = O(Dk^{-\alpha/D})$ of each other. The hypothesis implies that $D = \frac{f(k)\log k}{\log\log k}$ for some $f(k) < \alpha$ so

$$\log D - \frac{\alpha \log k}{D} = (1 - \frac{\alpha}{f(k)})\log\log k + \log f(k) - \log\log\log k \to -\infty$$

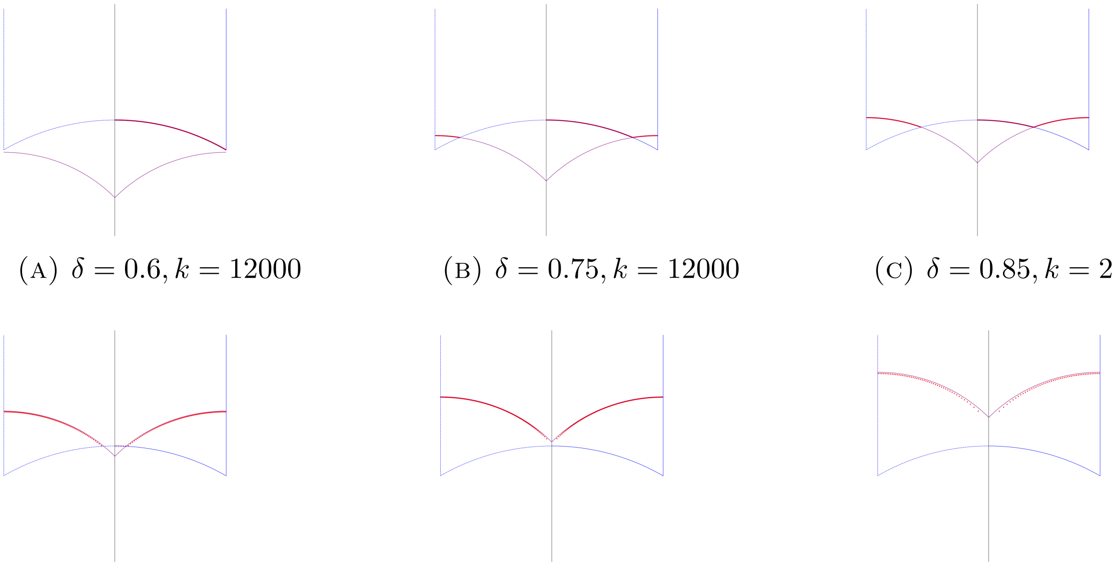
so the error above is $o(1)$.

Finally, by [Sze24] the roots of $E_D(\frac{1}{Dx})$ asymptotically approach the curve $\mathcal{S}$. However, by the above, the roots of $E_D(\frac{1}{Dx})$ asymptotically approach the roots of $F(\frac{2k}{Dx}) = F(\frac{24}{(1-\delta)x})$, and the result follows from Proposition 13. $\square$

6. CONJECTURAL BEHAVIOR OF ZEROS OF MILLER FORMS

The asymptotic between the zeros of the Miller forms $g_{k,m}$ and the logarithmic Szegő curve holds experimentally for all values of $\delta = \frac{m}{\ell}$.

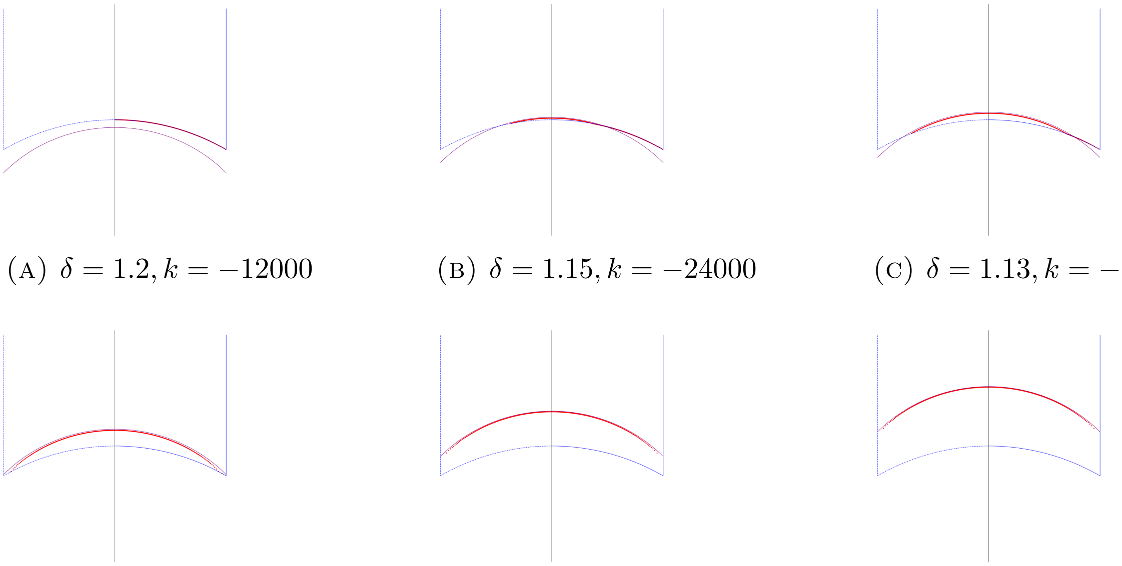
Conjecture 15. *For each $\delta > 0$, the zeros of $g_{k,m}$ asymptotically approach the curve $\mathcal{C}_\delta$, defined as the upper hull of the union of $\mathcal{A}$ and $\mathcal{S}_\delta$.*



(A) $\delta = 0.6, k = 12000$ (B) $\delta = 0.75, k = 12000$ (C) $\delta = 0.85, k = 24000$

(D) $\delta = 0.94, k = 36000$ (E) $\delta = 0.96, k = 120000$ (F) $\delta = 0.98, k = 240000$

Figure 10. Roots and $\mathcal{S}_\delta$ for several values of $\delta \in (0,1)$



(A) $\delta = 1.2, k = -12000$ (B) $\delta = 1.15, k = -24000$ (C) $\delta = 1.13, k = -24000$

(D) $\delta = 1.1, k = -36000$ (E) $\delta = 1.06, k = -60000$ (F) for $\delta = 1.03, k = -120000$

Figure 11. Roots and $\mathcal{S}_\delta$ for several values of $\delta \in (1,\infty)$

Remark 2. Let

$$\delta_{\mathcal{A}}^{+} = 1 - \frac{24}{W(e^{-1})e^{\sqrt{3}\pi}} = 0.6265\ldots \qquad \delta_{\mathcal{A}}^{-} = 1 + \frac{24}{W(e^{-1})e^{2\pi}} = 1.1609\ldots$$

$$\delta_{\mathcal{S}}^{+} = 1 - \frac{24}{e^{2\pi}} = 0.9551\ldots \qquad \delta_{\mathcal{S}}^{-} = 1 + \frac{24}{e^{\sqrt{3}\pi}} = 1.1040\ldots$$

The observation before Theorem B implies the following.

(1) If $\delta \in (0,1)$ then $\mathcal{C}_\delta = \mathcal{A}$ if $\delta < \delta^+_{\mathcal{A}}$ and $\mathcal{C}_\delta = \mathcal{S}_\delta$ if $\delta > \delta^+_{\mathcal{S}}$. If $\delta \in (\delta^+_{\mathcal{A}}, \delta^+_{\mathcal{S}})$, then $\mathcal{C}_\delta$ contains $\mathcal{A}_{\frac{\pi}{2},\mathcal{T}_\delta}$, where $-\cos\mathcal{T}_\delta$ is the solution to

$$g_+(x) = \sqrt{1-x^2} + \frac{1}{2\pi}\log(1-\delta).$$

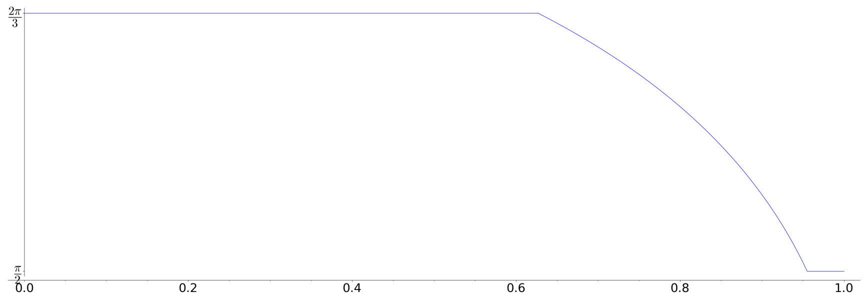


FIGURE 12. Conjectural value of $\mathcal{T}_\delta$ with $\mathcal{A}_{\frac{\pi}{2},\mathcal{T}_-(\delta)}$ containing Miller zeros

The conjectural value of $\mathcal{T}_\delta$ is very close to the computational values of $\mathcal{T}(\delta)$ in Figure 6.

(2) If $\delta \in (1,\infty)$ then $\mathcal{C}_\delta = \mathcal{A}$ if $\delta > \delta^-_{\mathcal{A}}$ and $\mathcal{C}_\delta = \mathcal{S}_\delta$ if $\delta < \delta^-_{\mathcal{S}}$.

The results of the previous sections prove parts of this Conjecture:

- By Theorem 5, Conjecture 15 holds as $k \to \infty$ and $\frac{m}{\ell} \to \delta < 0.6194$. Moreover, if $\delta < 0.9546$, the arc $\mathcal{A}_{\frac{\pi}{2},\mathcal{T}(\delta)} \subset \mathcal{C}_\delta$ contains some of the roots of $g_{k,m}$ for $\frac{m}{\ell} \to \delta$.
- By Theorem 9, Conjecture 15 holds as $k \to -\infty$ and $\frac{m}{\ell} \to \delta > 1.1598$. Moreover, if $\delta > 1.1026$, the arc $\mathcal{A}_{\mathcal{T}_-(\delta),\frac{2\pi}{3}} \subset \mathcal{C}_\delta$ contains some of the roots of $g_{k,m}$ for $\frac{m}{\ell} \to \delta$.
- Finally, by Theorem 14, Conjecture 15 holds if $|k| \to \infty$ and $D = \ell - m < \frac{\alpha \log|k|}{\log\log|k|}$ for some $\alpha \in (0,1)$.

Thus, we successfully proved Conjecture 15 at the two ends of the interval for δ. When δ is close, but not sufficiently close, to 1, we cannot account for the part of the curve $\mathcal{C}_\delta$ that is not on the arc $\mathcal{A}$, including showing that the curve $\mathcal{C}_\delta$ goes up as δ increases to 1. We have to contend with the following partial result.

Theorem 16. *If $k \to \infty$ then all the zeros of the Miller modular forms $g_{k,m}$ lie in the region $\{z \in \mathcal{F} \mid \operatorname{Im} z < \frac{2\log k}{1-c} + O(1)\}$, where $c = \frac{e^{2\pi}}{1728}$.*

We will prove Theorem 16 using bounds on the coefficients of Faber polynomials.

6.1. **Bounds on the coefficients of Faber polynomials.** When $D \le k$, Faber polynomials are no longer approximated by truncated exponential polynomials, but we can still bound the growth of their coefficients. Indeed, we may rewrite (5.1) as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ c_{D,-(D-1)} & 1 & 0 & 0 & 0 & 0 \\ c_{D,-(D-2)} & c_{D-1,-(D-2)} & 1 & 0 & 0 & 0 \\ c_{D,-(D-3)} & c_{D-1,-(D-3)} & c_{D-2,-(D-3)} & 1 & 0 & 0 \\ & & & \vdots & & \\ c_{D,0} & c_{D-1,0} & c_{D-2,0} & \dots & c_{1,0} & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_D \end{pmatrix} = \begin{pmatrix} \mathcal{D}_{\ell,0} \\ \mathcal{D}_{\ell,1} \\ \vdots \\ \mathcal{D}_{\ell,D} \end{pmatrix}.$$

Inverting, we get

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ u_{D,-(D-1)} & 1 & 0 & 0 & 0 & 0 \\ u_{D,-(D-2)} & u_{D-1,-(D-2)} & 1 & 0 & 0 & 0 \\ u_{D,-(D-3)} & u_{D-1,-(D-3)} & u_{D-2,-(D-3)} & 1 & 0 & 0 \\ & & & \vdots & & \\ u_{D,0} & u_{D-1,0} & u_{D-2,0} & \dots & u_{1,0} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{D}_{\ell,0} \\ \mathcal{D}_{\ell,1} \\ \vdots \\ \mathcal{D}_{\ell,D} \end{pmatrix},$$

giving $y_d = \sum_{i=0}^{d} u_{D-i,-(D-d)} \mathcal{D}_{\ell,i}$, where

$$u_{D-i,-(D-j)} = \sum_{D-j=i_s<i_{s-1}<\ldots<i_0=D-i} (-1)^s \prod_{r=0}^{s-1} c_{i_r,-i_{r+1}}.$$

Indeed, the inverse of lower unipotent matrices is given by $(I_m+N)^{-1} = \sum(-1)^s N^s$ where N is nilpotent. We conclude that

$$|u_{D-i,-(D-j)}| \leq \sum_{D-j=i_s<i_{s-1}<\ldots<i_0=D-i} \prod_{r=0}^{s-1} c_{i_r,-i_{r+1}}.$$

We denote

$$\begin{aligned} f(x,y) &= (x-y)\log(1728(1-c)) + x\log x - y\log y - (x-y)\log(x-y) \\ g(x,y) &= x\log 1728 - 2\pi y. \end{aligned}$$

Note that $f(x,cx) = g(x,cx)$ and $\lim_{y\to x} f(x,y) = 0$, which means that

$$\mathrm{BP}(x,y) = \begin{cases} f(x,y) & y \geq cx \\ g(x,y) & y < cx \end{cases}$$

is a continuous function on $\{(x,y) \mid x \geq y \geq 0\}$.

For convenience, we say a pair (x,y) is good if $y \geq cx$ and is bad if $y < cx$.

Lemma 17. *Given $z < x$, the function $h(y) = \mathrm{BP}(x,y) + \mathrm{BP}(y,z)$ on the interval (z,x) has the following monotonicity*

(x,y)	(y,z)	*Increasing*	*Descreasing*
good ($y \geq cx$)	*good* ($z \geq cy$)	$y < \frac{x+z}{2}$	$y > \frac{x+z}{2}$
good ($y \geq cx$)	*bad* ($z < cy$)	$y < \frac{x}{2-c}$	$y > \frac{x}{2-c}$
bad ($y < cx$)	*good* ($z \geq cy$)	*always*	
bad ($y < cx$)	*bad* ($z < cy$)	*always*	

Proof. Follows from

$$\begin{aligned}
\partial_y \left(f(x,y)+f(y,z)\right) &= \log(x-y)-\log(y-z)\\
\partial_y \left(f(x,y)+g(y,z)\right) &= \log(x-y)-\log y-\log(1-c)\\
\partial_y \left(g(x,y)+f(y,z)\right) &= \log y-\log(y-z)+\log\left(\frac{1-c}{c}\right)\\
\partial_y \left(g(x,y)+g(y,z)\right) &= -\log c.
\end{aligned}$$

□

Lemma 18. *If $D-j=i_s<i_{s-1}<\ldots<i_0=D-i$ then*

$$\prod_{r=0}^{s-1} c_{i_r,-i_{r+1}} \le \max(T_1,T_2,T_2',T_3,T_3',T_4),$$

where

$$\begin{aligned}
T_1 &= (1728-e^{2\pi})^{j-i}s^{j-i}\frac{(D-i)^{D-i}}{(D-j)^{D-j}(j-i)^{j-i}}\\
T_2 &= 2981^{D-i}1.25^{D-j}\\
T_2' &= (2393s)^{D-i}1.45^{j-i}\\
T_3 &= (2386s)^{D-i}2924^{-(D-j)}\\
T_3' &= 5849^{D-i}2924^{-(D-j)}\\
T_4 &= 1728^{D-i}e^{-2\pi(D-j)}.
\end{aligned}$$

When $D-j>c(D-i)$, it suffices to consider only T_1. In particular, if $\mathcal{A}=5849$ and $\mathcal{B}=1.5$ then

$$\prod_{r=0}^{s-1} c_{i_r,-i_{r+1}} \le \max(T_1,(\mathcal{A}s)^{D-i}\mathcal{B}^{D-j}).$$

Proof. We will bound

$$\sum_{r=0}^{s-1}\log c_{i_r,-i_{r+1}} \le \sum_{r=0}^{s-1}\mathrm{BP}(i_r,i_{r+1}).$$

To estimate an upper bound, we will determine the maximum of the continuous function

$$\mathrm{BP}(x_0,x_1,\ldots,x_n)=\sum_{r=0}^{s-1}\mathrm{BP}(x_r,x_{r+1}),$$

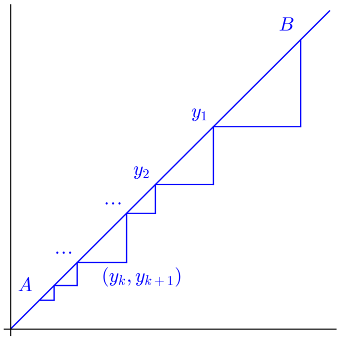
in the compact set

$$\{(x_0,\ldots,x_n)\mid A=x_n\le x_{n-1}\le\ldots\le x_0=B\}.$$

Let $(x_0,\ldots,x_n)$ be the point where the maximum is attained.

Since $\mathrm{BP}(x,x)=0$, we may instead assume that the maximum is attained at a point $(y_0,\ldots,y_m)$ such that $A=y_m<y_{m-1}<\ldots<y_0=B$, where $m\le n$, by eliminating duplicates.

We will represent the sequence $A=y_m<y_{m-1}<\ldots<y_0=B$ as a ladder



To describe $(y_0, \dots, y_m)$ at the maximum, we will repeatedly use Lemma 17. We denote $\mathcal{L}$ the line $y = cx$ and $\mathcal{L}'$ the line $y = \frac{1}{2-c}x$.

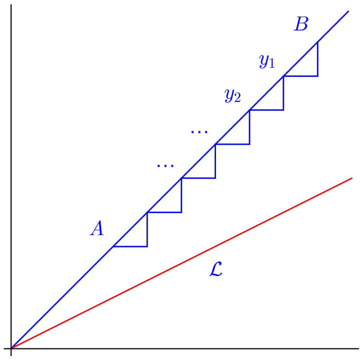
Suppose that $P_k = (y_k, y_{k+1})$ is bad. First, note that $P_{k-1} = (y_{k-1}, y_k)$ can't also be bad, since increasing y_k would increase the function. Thus P_{k-1} is good and located on the line $\mathcal{L}'$, or else increasing or decreasing y_k would increase the function.

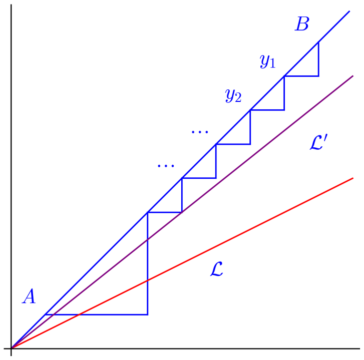
Suppose $P_k = (y_k, y_{k+1})$ is good, but not on the line $\mathcal{L}$. If P_{k-1} is bad, we could increase y_k to increase the value of the function, which means P_{k-1} is also good. If P_{k-1} is on the line $\mathcal{L}$, then $y_{k-1} + y_{k+1} > \frac{1}{c}y_k > 2y_k$, so we may increase y_k keeping both P_k and P_{k-1} good to increase the value of the function. If P_{k-1} is not on the line $\mathcal{L}$, it must be that $2y_k = y_{k-1} + y_{k+1}$. We conclude that every good point not on the line $\mathcal{L}$ must be preceeded by evenly spaced out good points, also not on the line $\mathcal{L}$.

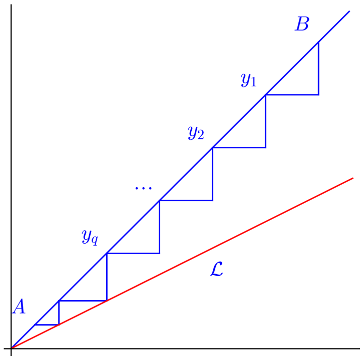
Finally, if P_k is on the line $\mathcal{L}$, by the same argument, the previous point P_{k-1} must be good with $y_{k-1} - y_k \geq y_k - y_{k+1}$.

Therefore, there are four possible configurations:

(1) All points $P_0, \dots, P_{m-1}$ are good, not on $\mathcal{L}$, with $y_0, \dots, y_m$ evenly spaced out.
(2) The last point P_{m-1} is bad, P_{m-2} is on $\mathcal{L}'$, and $y_0, \dots, y_{m-1}$ are evenly spaced out.
(3) The points $P_0, \dots, P_{q-1}$ are good, not on $\mathcal{L}$, with $y_0, \dots, y_q$ evenly spaced out, and $P_q, \dots, P_{m-1}$ are all on $\mathcal{L}$.
(4) The last, degenerate, case is $m = 1$, $y_0 = B$, $y_1 = A$.







In each of these cases, we can determine $y_0, \dots, y_m$ exactly. For convenience, we denote $[x, y] = x \log x - y \log y - (x - y) \log(x - y) \leq x \log 2$.

(1) For $1 \leq u \leq m$, $y_i = B - (B-A)i/m$. In this case, the value of the function is

$$\begin{aligned}
\mathrm{BP}(y_0,\ldots,y_m) &= \sum_{i=0}^{m-1} \left((y_i - y_{i+1}) \log(1728(1-c)) + y_i \log y_i - y_{i+1} \log y_{i+1} - \frac{B-A}{m} \log \frac{B-A}{m} \right) \\
&= (B-A)(\log m + \log(1728(1-c))) + B \log B - A \log A - (B-A) \log(B-A) \\
&= (B-A)(\log m + \log(1728(1-c))) + [B, A],
\end{aligned}$$

which is maximized when $m = n$.

(2) The remaining three cases can only occur if $A < \mathcal{B}$. We have $y_m = A$, $y_{m-1} = A + a$, $y_{m-2} = A + a + b$, $\ldots$, $y_0 = B = A + a + (m-1)b$ with $y_{m-1} = \frac{1}{2-c} y_{m-2}$. This gives $b = (1-c)(A+a)$ and

$$a = \frac{B + A}{1 + (m-1)(1-c)} - A.$$

In this case, denoting $N = 1 + (m-1)(1-c)$ the value of the function is

$$\begin{aligned}
&g(A+a, A) + \mathrm{BP}(y_0,\ldots,y_{m-1}) \\
&\quad = (A+a)\log 1728 - 2\pi A + (B - A - a)(\log(m-1) + \log(1728(1-c))) \\
&\qquad + B \log B - (A+a)\log(A+a) - (B-A-a)\log(B-A-a) \\
&\quad = B(\log(1728) + \log(N-1)) - (A+a)\log(N-1) + [B, A+a] \\
&\quad \leq B(\log 1728 + \log 2) + (B - \frac{A+B}{N}) \log(N-1).
\end{aligned}$$

When $m = 2$, $N = 2 - c$ and the quantity above is

$$\leq B \left(\log 1728 + \log 2 + \left(1 - \frac{1}{2-c} \right) \log(1-c) \right) - \frac{\log(1-c)}{2-c} A \leq 8B + 0.22A.$$

When $m \geq 2$, $N = 1 + (1-c)(m-1) > 2$ and the quantity above is maximized when $m = n$ giving

$$\leq B \left(\log 1728 + \log 2 + \left(1 - \frac{2}{N} \right) \log(N-1) \right) + (B-A) \frac{\log(N-1)}{N-1}$$

$$< B\left(\log 1728 + \log 2 + \log(1-c) + \log(n)\right) + e^{-1}(B-A) < (7.78 + \log n)B + e^{-1}(B-A).$$

(3) For $0 \leq i \leq m - q$ we have $y_{m-i} = c^{-i}A$, and $y_j = B - (B - c^{q-m}A)j/(m-q)$ for $0 \leq j \leq q$. In this case, the value of the function is

$$A \left(\frac{c^{q-m} - 1}{c^{-1} - 1} \right) \left(c^{-1} \log 1728 - 2\pi \right) + (B - c^{q-m}A)(\log(q) + \log(1728(1-c))) + \left[B, c^{q-m}A \right]$$

$$\leq B \log 2 + A \left(\frac{c^{q-m} - 1}{c^{-1} - 1} \right) \left(c^{-1} \log 1728 - 2\pi \right) + (B - c^{q-m}A)(\log(q) + \log(1728(1-c)))$$

$$= B \log 2 - A \left(\frac{\log 1728 - 2\pi c}{1-c} \right) + B(\log(q) + \log(1728(1-c))) +$$

$$+ c^{q-m} A \left(\frac{(\log 1728 - 2\pi)c}{1-c} - (\log(q) + \log(1-c)) \right)$$

Note that the last term is negative if $q \geq 3$. Keeping in mind that $Ac^{q-m} \leq B$ we get that this is

$$\leq B \log 2 - A \left(\frac{\log 1728 - 2\pi c}{1-c} \right) + B(\log(q) + \log(1728(1-c)))$$

$$\leq B \left(\log 2 + \log(1728(1-c)) + \log n \right) - A \left(\frac{\log 1728 - 2\pi c}{1-c} \right) \leq (7.78 + \log n)B - 7.98A$$

if $q \geq 3$ and

$$\leq B \left(\log 2 + \frac{\log 1728 - 2\pi c}{1-c} \right) - A \left(\frac{\log 1728 - 2\pi c}{1-c} \right) \leq 8.68B - 7.98A$$

if $q \leq 2$.

We remark that this case can only occur if $A \leq cB$.

(4) In the degenerate case when $y_0 = B$ and $y_1 = A$ the value of the function is

$$B \log 1728 - 2\pi A.$$

□

Proposition 19. *We have* $y_d = O\left(\max(U_1, U_2, U_3)\right)$, *where*

$$\begin{aligned} U_1 &= \left(\frac{2e\alpha k}{d} \right)^d \\ U_2 &= \frac{(\alpha k)^R (2e)^d (1728 - e^{2\pi})^{d-R} (D-R)^{D-R}}{R^R (D-d)^{D-d}} \\ U_3 &= \left(\alpha \mathcal{B}^c k d) \right)^{\frac{d}{1-c}}. \end{aligned}$$

Here $\alpha = \left(\frac{25}{24}\right)^{25}$, U_3 *is considered only when* $d > (1-c)D$, *and* U_2 *only when* $R = \frac{1}{2}(D - \sqrt{D^2 - 4k/C})$ *is real with* $R < d$, *where* $C = \alpha^{-1} e^2 (1728 - e^{2\pi})$.

Proof. We know that

$$y_d = \sum_{r=0}^{d} u_{D-r,-(D-d)} \mathcal{D}_{\ell,r},$$

with $\mathcal{D}_{\ell,d} = O\left(\left(\frac{2\alpha |k|}{d} \right)^d \right)$ by Lemma 10, where $\alpha = \left(\frac{25}{24}\right)^{25}$. Here,

$$|u_{D-r,-(D-d)}| \leq \sum_{D-d = i_s < i_{s-1} < \ldots < i_0 = D-r} \prod c_{i_u, i_{u+1}} \leq \sum_{s=1}^{d-r} \frac{(d-r)^s}{s!} \max \prod c_{i_u, i_{u+1}}.$$

Whenever $D - d > c(D-r)$ (which is always guaranteed when $d < D(1-c)$), we know from Lemma 18 that

$$\max \prod c_{i_u, i_{u+1}} \leq (1728 - e^{2\pi})^{d-r} s^{d-r} \frac{(D-r)^{D-r}}{(D-d)^{D-d} (d-r)^{d-r}}.$$

Thus, for $d < D(1-c)$, we have

$$
\begin{aligned}
|y_d| &= O\left(\sum_{r=0}^{d}\left(\frac{2\alpha k}{r}\right)^r (1728-e^{2\pi})^{d-r}\frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}}\sum_{s=1}^{d-r}\frac{(d-r)^s}{s!}s^{d-r}\right)\\
&= O\left(\sum_{r=0}^{d}\left(\frac{2\alpha k}{r}\right)^r (1728-e^{2\pi})^{d-r}\frac{(D-r)^{D-r}}{(D-d)^{D-d}}(d-r)e^{d-r}\right)\\
&= O\left(\sum_{r=0}^{d}\frac{(\alpha k)^r(2e)^d}{r^r}(1728-e^{2\pi})^{d-r}\frac{(D-r)^{D-r}}{(D-d)^{D-d}}\right),
\end{aligned}
$$

since $d-r \le 2^{d-r}$. The derivative with respect to r of log of the term above is

$$\log(\alpha k) - \log r - 2 - \log(1728-e^{2\pi}) - \log(D-r).$$

This is positive whenever $r(D-r) \le k/C$, where $C = \alpha^{-1}e^2(1728-e^{2\pi})$. This is always true if

(1) $D^2 - 4k/C < 0$ or
(2) $d < R = \frac{1}{2}(D - \sqrt{D^2-4k/C})$.

In this case, the term in the sum is maximized when $r = d$ and the sum is

$$|y_d| = O\left(\left(\frac{2e\alpha k}{d}\right)^d\right).$$

If $D^2 > 4k/C$ and $d > R$, the terms in the sum attains its maximum either at $r = R$ or at $r = d$ and we obtain

$$|y_d| = O\left(\max\left(\left(\frac{2e\alpha k}{d}\right)^d, \frac{(\alpha k)^R(2e)^d(1728-e^{2\pi})^{d-R}(D-R)^{D-R}}{R^R(D-d)^{D-d}}\right)\right).$$

If $d > D(1-c)$, we have to break up the sum into pieces based on $D-r$. We use the bounds from Lemma 18.

$$
\begin{aligned}
|y_d| &= O\left(\sum_{r<D-c^{-1}(D-d)}\left(\frac{2\alpha k}{r}\right)^r \mathcal{A}^{D-r}\mathcal{B}^{D-d}\sum_{s=0}^{d-r}\frac{(d-r)^s s^{D-r}}{s!}\right)\\
&\quad + O\left(\sum_{r>D-c^{-1}(D-d)}^{d}\left(\frac{2\alpha k}{r}\right)^r (1728-e^{2\pi})^{d-r}\frac{(D-r)^{D-r}}{(D-d)^{D-d}(d-r)^{d-r}}\sum_{s=1}^{d-r}\frac{(d-r)^s}{s!}s^{d-r}\right)\\
&= O\left(\sum_{r<D-c^{-1}(D-d)}\frac{(\alpha k)^r(2e)^d}{r^r}\mathcal{A}^{D-r}\mathcal{B}^{D-d}(d-r)^{D-r}\right)\\
&\quad + O\left(\sum_{r>D-c^{-1}(D-d)}^{d}\frac{(\alpha k)^r(2e)^d}{r^r}(1728-e^{2\pi})^{d-r}\frac{(D-r)^{D-r}}{(D-d)^{D-d}}\right).
\end{aligned}
$$

The second sum has the same upper bounds as above. For the first sum, note that $(\alpha k\mathcal{A}^{-1})^r \le (\alpha k\mathcal{A}^{-1})^{D-c^{-1}(D-d)} < (\alpha k\mathcal{A}^{-1})^{d/(1-c)}$ (if $\alpha k > \mathcal{A}$) and $r^{-r}(d-r)^{D-r} \le d^D$, yielding the bound $O((\alpha\mathcal{B}^c kd)^{d/(1-c)})$. □

6.2. **Proof of Theorem 16.** Suppose $g_{k,m}(z) = 0$ for some $z \in \mathcal{F}$. Then $j(z)$ is a root of the Faber polynomial $\sum_{d=0}^{D} y_{D-d} x^d$ and thus, by Lagrange's inequality

$$|j(z)| \leq 2 \max(|y_d|^{1/d}).$$

By Proposition 19, we have

$$|y_d|^{1/d} = O\left(\max\left(\frac{2e\alpha k}{d}, (\alpha \mathcal{B}^c k d)^{\frac{1}{1-c}}, U_2^{1/d}\right)\right),$$

where

$$U_2 = \frac{(\alpha k)^R (2e)^d (1728 - e^{2\pi})^{d-R} (D-R)^{D-R}}{R^R (D-d)^{D-d}}$$

is counted only when $D^2 \geq 4k/C$ and $R = \frac{1}{2}(D - \sqrt{D^2 - 4k/C}) < d$. We see that

$$\frac{\log U_2}{d} \leq \log(2e\alpha(1728 - e^{2\pi})) + \frac{(D-R)\log(D-R) - (D-d)\log(D-d)}{d}.$$

By the Mean Value Theorem, the latter equals $1 + \log T$ for some $T \in (D-d, D)$ and so $\frac{\log U_2}{d} \leq \log(2e^2\alpha(1728 - e^{2\pi})) + \log(kD)$. Therefore,

$$|j(z)| = \max(O(k), O((kD)^{\frac{1}{1-c}}), O(kD))) = O(k^{\frac{2}{1-c}}).$$

Finally, if $|j(z)| \gg 0$ then $\operatorname{Im} z = \log |j(z)| + O(1)$ and the conclusion follows.

7. ALGEBRAIC ZEROS

As noted in the introduction, Kohnen [Koh03] proved that the only possible algebraic zeros of E_k in $\mathcal{F}$ are ρ or i, a conclusion that ultimately rests on the fact that all zeros of E_k lie on $\mathcal{A}$. Since the Miller forms $g_{k,m}$ do not, in general, share this feature, the possibility of additional algebraic zeros arises. While a specific instance of this phenomenon for weakly holomorphic Miller forms has been previously observed in [JSS14], a systematic investigation has not been undertaken. In this section, we obtain a complete classification of all such forms within the range $\ell - m \leq 25$.

Proposition 20. *Suppose $\ell - m \leq 25$. Then the Miller form $g_{k,m}$ has a non-elliptic algebraic root if and only if $m = \ell - 1$ and the form appears in the following table:*

Zero	*Values of k*
$\frac{1+\sqrt{-19}}{2}$	442740, 442864, 442494, 442988, 442618, 442742
$\frac{1+\sqrt{-27}}{2}$	6144372, 6144496, 6144126, 6144620, 6144250, 6144374
$\frac{1+\sqrt{-43}}{2}$	442368372, 442368496, 442368126, 442368620, 442368250, 442368374
$\frac{1+\sqrt{-67}}{2}$	73598976372, 73598976496, 73598976126, 73598976620, 73598976250, 73598976374
$\frac{1+\sqrt{-163}}{2}$	131268706320384372, 131268706320384496, 131268706320384126, 131268706320384620, 131268706320384250, 131268706320384374.

Proof. If z is algebraic zero of $g_{k,\ell-D}$, then [Koh03, Thm 1] implies that there exists $d > 0$ such that $j(z)$ is a root of its Faber polynomial, where $z = \sqrt{-d}$ or $\frac{1+\sqrt{-d}}{2}$ depending on $-d \equiv 0, 1 \pmod 4$. Thus the minimal polynomial of $j(z)$, the Hilbert Class Polynomial of

$-d$ whose degree is the class number of the corresponding order in $\mathbb{Q}(\sqrt{-d})$, is an irreducible factor of the Faber polynomial of $g_{k,\ell-D}$.

Our proof relies on two computations: (1) a computation of the Faber polynomials $F_{k',D} \in \mathbb{Z}[\ell][x]$ of $g_{k,\ell-D}$ symbolically for small values of D and (2) a database of all Hilbert Class Polynomials of degree at most D. We computed the former ourselves, and relied on [Wat04] and [Kla12] for a complete classification of orders of class number up to 100.

When $D = 1$, the Faber polynomial of $g_{k,\ell-1}$ is $x + 24\ell - 744 + \dfrac{2k'}{B_{k'}}$, where $B_{k'}$ is the k'-th Bernoulli number. The following table makes these values explicit.

k'	Faber polynomial of $g_{k,\ell-1}$
0	$x + 24\ell - 744$
4	$x + 24\ell - 984$
6	$x + 24\ell - 240$

k'	Faber polynomial of $g_{k,\ell-1}$
8	$x + 24\ell - 1224$
10	$x + 24\ell - 480$
14	$x + 24\ell - 720$

We conclude that we must seek the possible algebraic zeros among the orders with class number 1 in quadratic imaginary fields, for which d is in the set

$$\{-3, -4, -7, -8, -11, -12, -16, -19, -27, -28, -43, -67, -163\}.$$

The table of algebraic roots follows from a computation of ℓ corresponding to these j-values.

For every other discriminant $-d$ with Hilbert Class Polynomial $H(x)$ of degree $\leq D$ we checked that the $F_{k',D}(x, \ell) \not\equiv 0 \pmod{H(x)}$, by verifying that the coefficients of the residue, which are polynomials in ℓ, have no integral roots. □

We remark that the computation above runs exponentially slow, for $D = 10, 20, 30, 40, 50$ requiring 2 seconds, $1, 5, 23, 90$ minutes.

Finally, we conclude with a weaker computational result for $25 < D \leq 100$.

Proposition 21. *As $T \to \infty$, for all values of $k = 12\ell + k' < T$ except, possibly, for $O(\sqrt{T})$ values, the Miller form $g_{k,m}$ has no non-elliptic algebraic roots whenever $25 < \ell - m \leq 100$.*

Proof. A non-elliptic algebraic root z of $g_{k,m}$ yields a root $j(z)$ of the Faber polynomial $F(x)$ of $g_{k,m}$. The Galois closure of $\mathbb{Q}(j(z))/\mathbb{Q}$ has dihedral Galois group so it suffices to verify that $F(x)$ is irreducible and has non-dihedral Galois group. Fixing $D \leq 100$ and k', the Faber polynomial $F_{k',D}(x, \ell) \in \mathbb{Q}[\ell][x]$ is a family of polynomials over $\mathbb{Q}[\ell]$, and we denote $G_{k',D}$ the Galois group of $F_{k',D}(x, y)$ over $\mathbb{Q}(y)$. By the Hilbert Irreducibility Theorem [Ser97, §9], for all but $O(\sqrt{T})$ values of $\ell < T$, the Galois group of $F_{k',D}(x, \ell)$ is also $G_{k',D}$.

For each k' and each D, we found in Sage (at least) two primes p and q among the first 1000 primes such that (1) $F_{k',D}(x, 0)$ is irreducible mod p and (2) $F_{k',D}(x, 0)$ is a product of a linear term and an irreducible polynomial mod q. This implies that the Galois group of $F_{k',D}(x, \ell)$ is neither cyclic nor dihedral for a positive density of ℓ (whenever ℓ is a multiple of $p^a q^b$ for some a, b larger than the power of p and q in the denominators of $F_{k',D}(x, \ell)$), which implies that $G_{k',\ell}$ is neither cyclic nor dihedral. We conclude that the Galois group of $F_{k',D}(x, \ell)$ is $G_{k',\ell}$ for all but $O(\sqrt{T})$ values of $\ell < T$, and the result follows.

□

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