A GENERALIZED LEHMER CONJECTURE FOR THE TRACE OF T_3

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Abstract. The expectation that Ramanujan's tau function does not vanish, commonly known as Lehmer's Conjecture, has inspired several extensions to broader settings. In this paper, we focus on one such direction, proposed by Rouse, concerning the non-vanishing of traces of Hecke operators T_n. We refine an algorithm originally introduced by Rouse to resolve the case $n = 2$, and, together with tools from our earlier work on the case $n = 3$ in level one, we settle the conjecture for T_3 in full generality. We also discuss an implication of our result for the non-vanishing of all coefficients of the characteristic polynomial of T_3.

1. Introduction

The modular discriminant Δ is the normalized Hecke eigenform in the space $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ of cusp forms of full level and weight 12, and is given by:

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n, \text{ with } q = e^{2\pi i z}.$$

In 1947, Lehmer initiated the study of vanishing of its Hecke eigenvalues $\tau(n)$, leading to what is now commonly known as Lehmer's Conjecture: the assertion that $\tau(n)$ is never zero. Given that $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is one-dimensional, $\tau(n)$ is precisely the trace of the Hecke operator T_n acting on this space. This naturally raises the question of whether the trace of T_n can ever be zero.

More generally, we can consider the trace of T_n acting on the space $S_{2k}(\Gamma_0(N))$ of cusp forms of level N and weight $2k$, denoted by $\mathrm{Tr}_{2k}(\Gamma_0(N), n)$. In this setting, the "Generalized Lehmer Conjecture" was originally posed in 2006 by Rouse [Rou06, Conjecture 1.5] as follows:

Conjecture 1. *If $n \geq 1$ is not a square, $\gcd(N, n) = 1$ and $2k = 12$ or $2k \geq 16$, then*

$$\mathrm{Tr}_{2k}(\Gamma_0(N), n) \neq 0.$$

As supporting evidence, Rouse proved the conjecture for $n = 2$. It is worth noting that in the level one case, the conjecture for $n = 2$ also follows from the work of Chiriac and Jorza [CJ22] who have shown that the trace $\mathrm{Tr}_{2k}(\mathrm{SL}_2(\mathbb{Z}), 2)$ takes on no repeated values as k varies, with the sole exception of zero, which occurs only when the space $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is trivial. Further progress on the $n = 3$ case was achieved by Chiriac, Kurzenhauser, and Williams [CKW24], who verified the Conjecture in level one and several other levels. In fact, their work furnishes a systematic way of demonstrating that the trace for a fixed level is non-vanishing, though they were limited in only being able to treat one level at a time. Our primary objective is to overcome this limitation, providing a comprehensive resolution of the $n = 3$ case in all levels.

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Theorem 1. *Conjecture 1 is true for $n = 3$.*

Our proof employs a structured computational strategy. Specifically, we implemented an algorithm described by Rouse and grounded in the Eichler-Selberg trace formula. This reduces the problem to the vanishing of coefficients in a finite list of rational functions, which can be thought of as "modified traces". For $n = 2$, the algorithm yields a list of size 8, enabling Rouse to verify the non-vanishing of the trace of T_2 by checking each case directly. In contrast, for $n = 3$, it returns 240 cases—a number too large for manual verification. A central component of our work was optimizing this implementation to further reduce the computational burden. After refining Rouse's algorithm, the number of cases requiring explicit verification decreased to a manageable 21. Second, we addressed these remaining cases individually: many were resolved via ad hoc modular observations, while for the few that resisted such analysis, we applied the p-adic method from [CKW24].

A related line of inquiry, inspired by the "Generalized Lehmer Conjecture", investigates whether coefficients beyond the first in the characteristic polynomial of T_n can vanish. In this direction, it was shown in [Cla+24] that the second coefficient—when defined—of the characteristic polynomial of T_2 acting on $S_{2k}(N)$ with N odd, vanishes only when $2k = 2$ and $N \in \{33, 37, 57\}$. Recent work by Ross and Xue [RX25] determines the sign of the even-indexed coefficients of the characteristic polynomial of T_n when N or k is sufficiently large. A similar statement is provided for the odd-indexed coefficients, under the additional assumption that the corresponding trace does not vanish; see [RX25, Corollary 4.3]. Our main result complements their work for $n = 3$, allowing us to eliminate the trace condition in the following application:

Corollary 1. *If $2k = 12$ or $2k \geq 16$ is fixed, then any given coefficient of the characteristic polynomial of T_3 acting on $S_{2k}(\Gamma_0(N))$ is nonzero for all but finitely many N coprime to 3.*

We now give an overview of the paper's structure. In Section 2, we introduce the necessary notation and provide background on the Eichler-Selberg Trace Formula. Section 3 outlines the reduction step and details the optimizations used to make the problem computationally feasible. In Section 4, we present the non-vanishing arguments for each "modified trace". Finally, Section 5 concludes with a discussion of the prospects for extending our methods for larger values of n.

2. The Trace Formula

Throughout, we fix $n \geq 1$ to be a non-square integer that is coprime to the level N. Our starting point is a version of the Eichler-Selberg Trace Formula due to Hijikata [Hij74], which makes explicit how the contributing terms depend on the weight and level. It says that for $2k \geq 4$ we have

$$\mathrm{Tr}_{2k}(\Gamma_0(N), n) = -\sum_{s \in \mathbb{Z}} a(s, k, n) \sum_{f \mid t(s,n)} b(s, f, n) c(s, f, N, n).$$

These quantities are defined as follows.

- Call an integer s *admissible* if $s^2 - 4n$ is negative or is a positive square.

- For an admissible integer s, let t_0 be the largest integer for which $t_0^2 \mid (s^2-4n)$. Then define

$$t(s,n) = \begin{cases} t_0 & \text{if } (s^2-4n)/t_0^2 \equiv 1 \pmod 4 \\ t_0/2 & \text{if } (s^2-4n)/t_0^2 \equiv 2,3 \pmod 4. \end{cases}$$

- For an admissable integer s, let y and $\overline{y}$ be the roots of $x^2 - sx + n = 0$. Define

$$a(s,k,n) = \begin{cases} \frac{1}{2} \cdot \frac{y^{2k-1} - \overline{y}^{2k-1}}{y - \overline{y}} & \text{if } s^2 - 4n < 0 \\ \frac{\min\{|y|, |\overline{y}|\}^{2k-1}}{|y - \overline{y}|} & \text{if } s^2 - 4n \text{ is a positive square.} \end{cases}$$

- Fix $f \mid t(s,n)$ and let

$$b(s,f,n) = \begin{cases} \frac{h((s^2-4n)/f^2)}{\omega((s^2-4n)/f^2)} & \text{if } s^2 - 4n < 0 \\ \frac{1}{2}\varphi\left(\frac{\sqrt{s^2-4n}}{f}\right) & \text{if } s^2 - 4n \text{ is a positive square.} \end{cases}$$

  Here φ is the totient function, $h(-d)$ is the class number for the imaginary order R of discriminant $-d$, and $\omega(-d)$ is half the number of units in R.
- For $f \mid t(s,n)$ and a fixed prime ℓ, let $v = v_\ell(N)$ and $b = v_\ell(f)$. Let A be the number of distinct solutions modulo ℓ^{v+b} to the system

$$x^2 - sx + n \equiv 0 \pmod{\ell^{v+2b}} \text{ and } 2x \equiv s \pmod{\ell^b}.$$

  Similarly, let B be the number of distinct solutions modulo ℓ^{v+b} to the system

$$x^2 - sx + n \equiv 0 \pmod{\ell^{v+2b+1}} \text{ and } 2x \equiv s \pmod{\ell^b}.$$

  Define

$$c(s,f,N,n,\ell) = \begin{cases} A & \text{if } (s^2-4n)/f^2 \not\equiv 0 \pmod{\ell} \\ A+B & \text{if } (s^2-4n)/f^2 \equiv 0 \pmod{\ell} \end{cases}$$

  and $c(s,f,N,n) = \prod_{\ell \mid N} c(s,f,N,n,\ell)$.

For convenience, we take the above constants to be zero when s is not an admissible integer. We also note that the constants are symmetric in the s component.

We also record how the trace formula manifests in the generating function

$$R(\Gamma_0(N), n; x) = \sum_{k \geq 1} \mathrm{Tr}_{2k}(\Gamma_0(N), n) x^{k-1}.$$

More precisely, as observed in [FOP04, Theorem 3.3], we have:

$$R(\Gamma_0(N), n; x) = \sigma_1(n) + \sum_{\substack{d \mid n \\ d < \sqrt{n}}} \sum_{f \mid \left(\frac{n}{d} - d\right)} \frac{b(\frac{n}{d}+d, f, n)c(\frac{n}{d}+d, f, N, n)}{d^2x - 1}$$
$$- \frac{1}{2} \sum_{\substack{s \in \mathbb{Z} \\ s^2 - 4n < 0}} \sum_{f \mid t(s,n)} b(s,f,n)c(s,f,N,n) \frac{nx+1}{n^2x^2 + (2n - s^2)x + 1}.$$

3. The Reduction Step

A key step in the proof of Theorem 1 is reducing the problem to finitely many cases. To this end, a general algorithm was developed in [Rou06, Lemma 3.2], though the details of its implementation are not straightforward. While Rouse worked with the notion of "projective equivalence," we found it more convenient to introduce the concept of a "modified trace."

Definition 1. Let ε be a vector of length $\lfloor 2\sqrt{n}\rfloor + 1$ consisting of only 0's and 1's, i.e.

$$\varepsilon = \begin{pmatrix}\varepsilon_0 & \varepsilon_1 & \dots & \varepsilon_{\lfloor 2\sqrt{n}\rfloor}\end{pmatrix}$$

where $\varepsilon_s \in \{0,1\}$. We will use ε to delete terms from $\mathrm{Tr}_{2k}(\Gamma_0(N), n)$ and from $R(\Gamma_0(N), n; x)$. We define the **modified trace** by

$$\begin{aligned}\mathrm{Tr}_{2k}(\Gamma_0(N), n, \varepsilon) = &-\sum_{s^2-4n<0} \varepsilon_s a(s,k,n) \sum_{f\mid t(s,n)} b(s,f,n)c(s,f,N,n)\\ &-\sum_{s^2-4n>0} a(s,k,n) \sum_{f\mid t(s,n)} b(s,f,n)c(s,f,N,n)\end{aligned}$$

where we take the convention that $\varepsilon_{-s} = \varepsilon_s$. If we further take ε_s to be 1 when $|s| > 2\sqrt{n}$, we can further simplify this formula to

$$\mathrm{Tr}_{2k}(\Gamma_0(N), n, \varepsilon) = -\sum_{s\in\mathbb{Z}} \varepsilon_s a(s,k,n) \sum_{f\mid t(s,n)} b(s,f,n)c(s,f,N,n).$$

Up to a constant, the generating function for the modified traces is

$$\begin{aligned}R(\Gamma_0(N), n, \varepsilon; x) = \sigma_1(n) &+ \sum_{\substack{d\mid n\\ d<\sqrt{n}}} \sum_{f\mid\left(\frac{n}{d}-d\right)} \frac{b(\frac{n}{d}+d, f, n)c(\frac{n}{d}+d, f, N, n)}{d^2x-1}\\ &- \frac{1}{2}\sum_{\substack{s\in\mathbb{Z}\\ s^2-4n<0}} \sum_{f\mid t(s,n)} \varepsilon_s b(s,f,n)c(s,f,N,n)\frac{nx+1}{n^2x^2+(2n-s^2)x+1}.\end{aligned}$$

The constant term is not relevant in our discussion, since we only aim to prove that the modified traces are non-vanishing for $2k \geq 16$ (or $2k = 12$).

We will also make use of a constant $M(n)$ defined below.

Definition 2. Let $\ell \nmid n$ be a prime for which there exists either s with $\ell \mid (s^2-4n)$ or $d < \sqrt{n}$ with $\ell \mid (n/d - d)$. For such ℓ, define

$$M(n,\ell) = \max\left(\left\{v_\ell(s^2-4n) : s^2 < 4n\right\} \cup \left\{v_\ell(n/d-d) : d \mid n, d < \sqrt{n}\right\}\right) + 1,$$

and for all other primes ℓ set $M(n,\ell) = 0$. Finally, define

$$M(n) = \prod_\ell \ell^{M(n,\ell)}.$$

n	2	3	5	6	7	8
$M(n)$	49	1936	1397792	13225	467856	24910081

Table 1. First few values of $M(n)$ for $n \geq 1$ non-square.

With this notation, Rouse's algorithm [Rou06, Lemma 3.2] can be interpreted as follows:

Proposition 1. *For a fixed $n \geq 1$ that is not a square and any level N coprime to n, there exist $N_0 \mid M(n)$, $\lambda \in \mathbb{Q}^\times$, and a vector ε such that for all $2k \geq 4$*

$$\mathrm{Tr}_{2k}(\Gamma_0(N), n) = \lambda \, \mathrm{Tr}_{2k}(\Gamma_0(N_0), n, \varepsilon).$$

3.1. **Explicit Examples.** Before continuing with further discussion of the reduction step, we provide explicit examples of Proposition 1 in action.

Example 1. For $n = 3$ and $N = 5$, the trace formula says that

$$\mathrm{Tr}_{2k}(\Gamma_0(5), 3) = \sum_{s \in \mathbb{Z}} a(s, k, 3) \sum_{f \mid t(s,3)} b(s, f, 3) c(s, f, 5, 3).$$

The only admissible integers in this case are $s \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$. We claim that this will be equal to a scalar times $\mathrm{Tr}_{2k}(\mathrm{SL}_2(\mathbb{Z}), 3, \varepsilon)$. To that end, we can work out the constants involved explicitly:

s	0	± 1	± 2	± 3	± 4
$t(s,3)$	2	1	1	1	2

Table 2. $t(s,3)$ for the admissible choices of s

(s,f)	$(0,1)$	$(0,2)$	$(\pm 1,1)$	$(\pm 2,1)$	$(\pm 3,1)$	$(\pm 4,1)$	$(\pm 4,2)$
$b(s,f,3)$	1	1/3	1	1	1/3	1/2	1/2
$c(s,f,5,3)$	0	0	2	0	0	2	2
$c(s,f,1,3)$	1	1	1	1	1	1	1

Table 3. The other relevant constants to compute.

Hence, we have

$$\begin{aligned}\mathrm{Tr}_{2k}(\Gamma_0(5), 3) &= \sum_{s \in \mathbb{Z}} a(s, k, 3) \sum_{f \mid t(s,3)} b(s, f, 3) c(s, f, 5, 3) \\ &= 4a(1, k, 3) + 4a(4, k, 3).\end{aligned}$$

On the other hand, choosing $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 0, 0)$ gives

$$\mathrm{Tr}_{2k}(\mathrm{SL}_2(\mathbb{Z}), 3, \varepsilon) = 2a(1, k, 3) + 2a(4, k, 3).$$

Therefore

$$\mathrm{Tr}_{2k}(\Gamma_0(5), 3) = 2 \cdot \mathrm{Tr}_{2k}(\mathrm{SL}_2(\mathbb{Z}), 3, \varepsilon),$$

as claimed.

Example 2. If $N \mid M(n)$, then $\mathrm{Tr}_{2k}(\Gamma_0(N), n) = \mathrm{Tr}_{2k}(\Gamma_0(N), n, \varepsilon)$ where ε is the vector of all 1's.

3.2. **Outline of the Reduction Argument.** The following is a corollary to Proposition 1 and is what we will use to prove the $n = 3$ case.

Corollary 2. *To verify the "Generalized Lehmer Conjecture" for fixed $n \geq 1$ non-square, it suffices to check that*

$$\mathrm{Tr}_{2k}(\Gamma_0(N_0), n, \varepsilon) \neq 0$$

for $2k = 12$ or $2k \geq 16$ for all choices of $N_0 \mid M(n)$ and ε.

We note that $\mathrm{Tr}_{2k}(\Gamma(N_0), n, \varepsilon)$ is given by an explicit linear recurrence which we can read off using the formula for $R(\Gamma_0(N_0), n, \varepsilon; x)$. There are a total of $d(M(n)) \cdot 2^{\lfloor 2\sqrt{n} \rfloor + 1}$ recurrences that need to be checked. Table 4 gives a rough idea of how quickly this quantity grows. It will turn out that this bound can be significantly improved; see Section 3.3 for the $n = 3$ case and Proposition 3 for an improved bound for general n.

n	2	3	5	6	7	8
$d(M(n)) \cdot 2^{\lfloor 2\sqrt{n} \rfloor + 1}$	24	240	1728	288	4800	1728

TABLE 4. Number of choices (N_0, ε) for the first few n

The following definition will allow us to quickly compare modified traces to see if they produce the same linear recurrence.

Definition 3. For s such that $s^2 - 4n < 0$, let $\mathbf{c}^-(s, N, n)$ be the vector consisting of $c(s, f, N, n)$ for all $f \mid t(s, n)$. Similarly, for $d \mid n$ and $d < \sqrt{n}$, let $\mathbf{c}^+(d, N, n)$ be the vector consisting of $c(n/d + d, f, N, n)$ for all $f \mid (n/d - d)$. Define

$$V(n, N) = \big(\mathbf{c}^-(0, N, n) \quad \dots \quad \mathbf{c}^-(\lfloor 2\sqrt{n} \rfloor, N, n) \quad \mathbf{c}^+(1, N, n) \quad \dots \quad \mathbf{c}^+(\cdot, N, n)\big)$$

and

$$V(n, N, \varepsilon) = \big(\varepsilon_0 \mathbf{c}^-(0, N, n) \quad \dots \quad \varepsilon_{\lfloor 2\sqrt{n} \rfloor} \mathbf{c}^-(\lfloor 2\sqrt{n} \rfloor, N, n) \quad \mathbf{c}^+(1, N, n) \quad \dots \quad \mathbf{c}^+(\cdot, N, n)\big)$$

where $\cdot$ represents the largest divisor of n smaller than $\sqrt{n}$.

We note that the only parts of $\mathrm{Tr}_{2k}(\Gamma_0(N), n, \varepsilon)$ that depend on N are the constants $c(s, f, N, n)$. In particular, if $V(n, N_1, \varepsilon^{(1)}) = \lambda V(n, N_2, \varepsilon^{(2)})$ then $\mathrm{Tr}_{2k}(\Gamma_0(N_1), n, \varepsilon^{(1)}) = \lambda \, \mathrm{Tr}_{2k}(\Gamma_0(N_2), n, \varepsilon^{(2)})$. This reduces the question of when the traces are scalar multiples of each other into a question about vectors that can be done by inspection.

3.3. **Optimizing the Reduction.** In the case of $n = 2$, there are only 8 cases that need to be checked. For $n = 3$, there are 240 choices of $N_0 \mid M(3)$ and ε, which makes an ad hoc solution infeasible. Therefore, some improvements are required to reduce the problem to a significantly smaller number of cases. We make two optimizations here:

(1) If two choices of (N_0, ε) would give the linear recurrence, then we only need to check non-vanishing for one of them. More specifically, if $V(N_0, n, \varepsilon^{(1)}) = \lambda V(N_0, n, \varepsilon^{(2)})$ for some $\lambda \in \mathbb{Q}^\times$ then we only need to check non-vanishing for one of the pairs $(N_0, \varepsilon^{(i)})$. As a consequence, we note that if $\mathbf{c}^-(s, N, n) = \mathbf{0}$ then the choice of ε_s does not matter.

(2) There are certain choices of (N_0, ε_s) which make the non-vanishing of the modified trace trivial. Indeed, suppose that ε is chosen such that $\varepsilon_s c(s, f, N_0, n) = 0$ for all $|s| < 2\sqrt{n}$ and f. Then $\mathrm{Tr}_{2k}(\Gamma_0(N_0), \varepsilon) < 0$ for all k, and in particular is non-vanishing [Rou06, Proof of Theorem 1.7]. This allows us to ignore several N_0 where we may otherwise need to check that the modified trace doesn't vanish.

In Section 5 we discuss the feasibility of applying these optimizations to $n = 5$ and beyond. Proposition 3 applies this reasoning to obtain an upper bound on the number of cases for arbitrary n.

We now categorize $V(3, N_0, \varepsilon)$ subject to the above optimizations. We start by recording $V(3, N_0)$ for the divisors of $M(3) = 1936 = 2^4 \cdot 11^2$ in Table 5. From this table, we immediately see that we won't need to check any case where the level N_0 is in the set $\{8, 16, 44, 88, 176, 484, 968, 1936\}$. This leaves only 7 levels to verifying non-vanishing in. Further imposing that $\varepsilon_s c(s, f, N_0, 3) \neq 0$ for some s further reduces this to 71 cases. When we group the choices of $\varepsilon^{(i)}$ with $V(3, N_0, \varepsilon^{(1)}) = V(3, N_0, \varepsilon^{(2)})$, we obtain only 25 cases. Of these, 4 are actually a scalar multiple of a lower level, bringing the final number of cases to 21, which we list in Table 6.

N_0	$V(n, N_0)$
1	(1 1 1 1 1 1 1)
2	(2 0 0 1 0 2 2)
4	(2 0 0 0 0 4 2)
8	(0 0 0 0 0 6 2)
11	(0 0 1 2 0 2 2)
16	(0 0 0 0 0 6 2)
22	(0 0 0 2 0 4 4)
44	(0 0 0 0 0 8 4)
88	(0 0 0 0 0 12 4)
121	(0 0 0 2 0 2 2)
176	(0 0 0 0 0 12 4)
242	(0 0 0 2 0 4 4)
484	(0 0 0 0 0 8 4)
968	(0 0 0 0 0 12 4)
1936	(0 0 0 0 0 12 4)

TABLE 5. $V(3, N_0)$ for $N_0 \mid M(n)$. N_0 is bolded if $V(3, N_0)$ contains a non-zero $\mathbf{c}^+(s, N_0, n)$.

i	N_0	ε	$V(3, N_0, \varepsilon)$
1	1	$(1,0,0,0)$	(1 1 0 0 0 1 1)
2	1	$(0,1,0,0)$	(0 0 1 0 0 1 1)
3	1	$(1,1,0,0)$	(1 1 1 0 0 1 1)
4	1	$(0,0,1,0)$	(0 0 0 1 0 1 1)
5	1	$(1,0,1,0)$	(1 1 0 1 0 1 1)
6	1	$(0,1,1,0)$	(0 0 1 1 0 1 1)
7	1	$(1,1,1,0)$	(1 1 1 1 0 1 1)
8	1	$(0,0,0,1)$	(0 0 0 0 1 1 1)
9	1	$(1,0,0,1)$	(1 1 0 0 1 1 1)
10	1	$(0,1,0,1)$	(0 0 1 0 1 1 1)
11	1	$(1,1,0,1)$	(1 1 1 0 1 1 1)
12	1	$(0,0,1,1)$	(0 0 0 1 1 1 1)
13	1	$(1,0,1,1)$	(1 1 0 1 1 1 1)
14	1	$(0,1,1,1)$	(0 0 1 1 1 1 1)
15	1	$(1,1,1,1)$	(1 1 1 1 1 1 1)
16	2	$(1,*,0,*)$	(2 0 0 0 0 2 2)
17	2	$(0,*,1,*)$	(0 0 0 1 0 2 2)
18	2	$(1,*,1,*)$	(2 0 0 1 0 2 2)
19	4	$(1,*,*,*)$	(2 0 0 0 0 4 2)
20	11	$(*,1,0,*)$	(0 0 1 0 0 2 2)
21	11	$(*,1,1,*)$	(0 0 1 2 0 2 2)

TABLE 6. $V(3, N_0, \varepsilon)$ for non-trivial choices of (N_0, ε), duplicates removed.

4. Verifying Non-vanishing

We have reduced the problem to verifying that a list of 21 linear recurrences are non-vanishing. To achieve this, we combine two approaches: reduction modulo suitable integers and p-adic methods developed by Mignotte and Tzanakis. For each recurrence, we work with its generating function and the initial terms of its power series expansion. These data are compiled in Tables 10 and 11 of Appendix A.

4.1. **Modular Arithmetic.** We first note that if the generating function of a linear recurrence $\{a_n\}$ is $P(x)/Q(x)$, then the characteristic polynomial of $\{a_n\}$ is $x^k Q(1/x)$ (where $k = \deg Q(x)$). We can therefore reduce this characteristic polynomial modulo a convenient integer m and show that the sequence modulo m is non-vanishing. The reduction is necessarily periodic, so it will suffice to find a threshold after which the sequence is periodic with no zero terms. In each of the cases below, the sequence becomes periodic after $n = 3$. We will only show all the details for a handful of the series, but we list the appropriate values of m in Table 7.

i	m	$a_n \pmod{m}$
1	2	a_{n-2}
2	4	a_{n-3}
3	5	$3a_{n-1} + 4a_{n-2} + 2a_{n-3} + 2a_{n-4}$
4	4	$3a_{n-1} + a_{n-2} + a_{n-3}$
5	3	$2a_{n-1} + 2a_{n-2}$
7	4	$3a_{n-1} + a_{n-2} + 2a_{n-3} + a_{n-4} + 3a_{n-5} + 3a_{n-6}$
8	7	$4a_{n-1} + 2a_{n-2} + 2a_{n-3}$
9	3	a_{n-1}
10	2	$a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5}$
11	2	a_{n-6}
12	3	$2a_{n-1} + 2a_{n-2}$
13	4	$3a_{n-1} + a_{n-2} + 2a_{n-3} + a_{n-4} + 3a_{n-5} + 3a_{n-6}$
14	4	$a_{n-1} + 2a_{n-3} + 2a_{n-4} + 3a_{n-6} + a_{n-7}$
16	2	a_{n-2}
17	2	$a_{n-1} + a_{n-2} + a_{n-3}$
21	3	$2a_{n-3}$

Table 7. Recurrence relations for each sequence reduced modulo m. We omit the moduli for sequences $i = 6, 15, 18, 19, 20$, which require different methods.

For $i = 2$, we have a linear recurrence $\{a_n\}$ with generating function $(36x^3 - 10x^2 - 9x - 2)/(9x^3 - 4x^2 - 4x - 1)$. This lets us read off that $9a_{n-3} - 4a_{n-2} - 4a_{n-1} = a_n$ for $n \geq 2$. Reducing modulo 4 gives that $a_n \equiv a_{n-3} \pmod{4}$. This reduces the non-vanishing of $i = 2$ to checking that $a_0 \equiv 2a_1 \equiv a_2 \equiv 2 \pmod{4}$.

For $i = 8$, the sequence $\{a_n\}$ satisfies $27a_{n-3} - 36a_{n-2} + 12a_{n-1} = 3a_n$. Reducing modulo 7 and rearranging gives that for $n \geq 3$ we have

$$a_n \equiv 4a_{n-1} + 2a_{n-2} + 2a_{n-3} \pmod{7}.$$

It follows that a_n modulo 7 is periodic and cycles between $4, 3,$ and 1.

In the cases of $i = 5$ and $i = 12$, we need to reduce modulo 3 even though the first term does not make sense modulo 3. If we ignore the first term (say by letting the sequence begin at $n = 1$ instead of $n = 0$), we will obtain sequences which are periodic for $n \geq 2$.

4.2. **p-adic Methods.** The remaining five power series appear to resist the methods used above. While it may still be possible to find a modulus with the desired properties, we instead transition to p-adic methods. These were previously employed in [CKW24] to establish non-vanishing of the trace in levels 1, 2, and 4, which correspond to the sequences with indices $i = 15, 18$ and 19, respectively. We present a slightly simplified setup; specifically, we fix the parameters $A = 1$ and $c = 0$.

Consider a k-th degree linear recurrence $\{u_n\}$ with characteristic polynomial $g(x)$. Assume that g has integer coefficients and k distinct complex roots $\omega_1, \ldots, \omega_k$. For a finite set of solutions $\mathcal{M}$ to the equation $u_n = 0$, [MT91] provides a method of certifying that $\mathcal{M}$ is in fact the set of *all* solutions. Suppose that we can produce a prime $p \geq 3$ such that:

- the prime p does not divide the discriminant of g or any of its coefficients;
- each root ω_i is a p-adic unit;
- there is a positive integer S such that each ω_i satisfies $\omega_i^S \equiv 1 \pmod{p}$.

Then the following result from [MT91] holds.

Proposition 2. *Let (p, S) be as above and let $\mathcal{M}$ be a finite set of solutions $m \in \mathbb{Z}$ to the equation $u_m = 0$. Let $\mathcal{P}$ be a complete set of residues modulo S such that $\mathcal{M} \subseteq \mathcal{P}$. Assume the following conditions hold:*

(1) *$u_m = 0$ for all $m \in \mathcal{M}$;*
(2) *for $n \in \mathcal{P}$, if $u_n \equiv 0 \pmod{p}$, then $n \in \mathcal{M}$;*
(3) *for each $m \in \mathcal{M}$, $u_{m+S} \not\equiv u_m \pmod{p^2}$.*

Then $u_n = 0$ implies $n \in \mathcal{M}$.

As mentioned previously, in [CKW24] it was shown that the trace is non-vanishing for three of the five remaining sequences, namely $i = 15, 18, 19$. It thus remains to show that the technique works for $i = 6$ and $i = 20$.

Lemma 1. (a) *For $i = 6$, the pair $(p, S) = (59, 29)$ works.*
(b) *For $i = 20$, the pair $(p, S) = (23, 11)$ works.*

Proof. (a) From the generating function, we find that $g(x) = (x^2+2x+9)(x^2+5x+9)(x-1)$ is the characteristic polynomial of

$$u_n = \mathrm{Tr}_{2n}(\mathrm{SL}_2(\mathbb{Z}), 3, (0, 1, 1, 0)).$$

The polynomial $x^2 + 2x + 9$ has roots

$$\omega_1 = 12 + 43 \cdot 59 + 28 \cdot 59^2 + O(59^3) \text{ and } \omega_2 = 45 + 15 \cdot 59 + 30 \cdot 59^2 + O(59^3).$$

Similarly, the polynomial $x^2 + 5x + 9$ has roots

$$\omega_3 = 5 + 55 \cdot 59 + 57 \cdot 59^2 + O(59^3) \text{ and } \omega_4 = 49 + 3 \cdot 59 + 1 \cdot 59^2 + O(59^3).$$

The fifth root of $g(x)$ is $\omega_5 = 1$. It's easy to compute that

$$\left(\frac{12}{59}\right) = \left(\frac{45}{59}\right) = \left(\frac{5}{59}\right) = \left(\frac{49}{59}\right) = \left(\frac{1}{59}\right) = 1.$$

Thus $\omega_i^{29} \equiv 1 \pmod{59}$ for each i. Here, $\mathcal{M} = \{2, 5\}$ so we choose $\mathcal{P} = \{0, 1, \ldots, 28\}$. Conditions (1) and (2) can be checked by direct calculation, and for (3) we have that

$$u_{2+29} \equiv 2773 \not\equiv u_2 \pmod{59^2} \text{ and } u_{5+29} \equiv 2714 \not\equiv u_5 \pmod{59^2}.$$

(b) Here we have the sequence

$$\tilde{u}_n = \mathrm{Tr}_{2n}(\Gamma_0(11), 3, (*, 1, 0, *))$$

with characteristic polynomial $\tilde{g}(x) = (x^2 + 5x + 9)(x - 1)$. Besides 1, the roots of $\tilde{g}$ are:

$$\omega_1 = 16 + 17 \cdot 23 + 12 \cdot 23^2 + O(23^3) \text{ and } \omega_2 = 2 + 5 \cdot 23 + 20 \cdot 23^2 + O(23^3).$$

As in (a), we can easily check that

$$\left(\frac{16}{23}\right) = \left(\frac{2}{23}\right) = \left(\frac{1}{23}\right) = 1,$$

so $\omega_i^{11} \equiv 1 \pmod{23}$. Choosing $\mathcal{M} = \{2\}$ and $\mathcal{P} = \{0, 1, \ldots, 10\}$, we see that (1) and (2) are satisfied. For (3), we simply note that $\tilde{u}_{2+11} \equiv 483 \not\equiv \tilde{u}_2 \pmod{23^2}$.

□

5. Concluding Remarks

Based on the preliminary estimates in Table 4, the number of recurrences to verify grows quickly with n. The following observation reflects a significant refinement achieved via our method.

Proposition 3. *Fix a non-square integer $n \geq 2$. The number of recurrences that need to be checked to establish Conjecture 1 for T_n does not exceed*

$$\sum_{N_0 \mid M(n)} 2^{(\#\text{ of nonzero } \mathbf{c}^-(s, N_0, n)\text{'s})} - 1.$$

This refined bound reduces the number of recurrences to check by incroporating the two optimizations we made in Section 3. Namely, each vanishing $\mathbf{c}^-(s, N_0, n)$ eliminates a corresponding choice of ε_s, and cases in which all ε_s are zero can be omitted entirely.

n	2	3	5	6	7
Refined Bound	8	25	47	55	144

Table 8. Refined bounds on the number of recurrences to check.

For $n = 5$, the number of recurrences can be further reduced to 43 by discarding those $V(5, N_0, \varepsilon)$'s that are scalar multiples of one other. Consequently, a proof of the non-vanishing of the trace of T_5 appears to be within reach.

On the other hand, $V(n, 1)$ consists entirely of 1's for every n, so each distinct choice of ε yields a distinct modified trace. It follows that

$$2^{\lfloor 2\sqrt{n} \rfloor + 1} - 1$$

is a sharp lower bound on the number of cases that must be checked. While this still represents an improvement over the formula from Table 4, it is clear that computationally challenges may arise quite early.

We conclude with a comment on the weight restriction in Conjecture 1. In level one, the condition $2k = 12$ or $2k \geq 16$ is equivalent to requiring that the space of cusp forms $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be non-trivial. However, this equivalence does not hold at higher levels. In other words, for each weight $2k \in \{2, 4, 6, 8, 10, 14\}$ there are levels N for which $\dim S_{2k}(\Gamma_0(N)) \geq 1$, yet the trace vanishes. The following table provides such examples for T_2:

$2k$	2	4	6	8	10	14
N	19	9	23	113	113	23
$\dim S_{2k}(\Gamma_0(N))$	1	1	9	66	84	25

Table 9. Pairs $(2k, N)$ for which $\dim S_{2k}(\Gamma_0(N)) \geq 1$ but $\mathrm{Tr}_{2k}(\Gamma_0(N), 2) = 0$.

It would be interesting to gain a more conceptual understanding of this phenomenon.

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Appendix A. Generating functions and power series

i	N_0	ε	$R(\Gamma_0(N_0), 3, \varepsilon; x)$
1	1	$(1,0,0,0)$	$\frac{36x^2-17x-7}{9x^2-6x-3}$
2	1	$(0,1,0,0)$	$\frac{36x^3-10x^2-9x-2}{9x^3-4x^2-4x-1}$
3	1	$(1,1,0,0)$	$\frac{324x^4-103x^2-37x-4}{81x^4-9x^3-48x^2-21x-3}$
4	1	$(0,0,1,0)$	$\frac{36x^3-22x^2-2}{9x^3-7x^2-x-1}$
5	1	$(1,0,1,0)$	$\frac{324x^4-108x^3-52x^2-16x-4}{81x^4-36x^3-30x^2-12x-3}$
6	1	$(0,1,1,0)$	$\frac{324x^5-45x^4-62x^3-30x^2-6x-1}{81x^5-18x^4-35x^3-21x^2-6x-1}$
7	1	$(1,1,1,0)$	$\frac{2916x^6+405x^5-657x^4-386x^3-102x^2-15x-1}{729x^6+81x^5-369x^4-294x^3-117x^2-27x-3}$
8	1	$(0,0,0,1)$	$\frac{108x^3-120x^2+41x-8}{27x^3-36x^2+12x-3}$
9	1	$(1,0,0,1)$	$\frac{108x^4-90x^3+9x^2+3x-2}{27x^4-27x^3+x-1}$
10	1	$(0,1,0,1)$	$\frac{972x^5-621x^4-42x^3+13x^2-2x-5}{243x^5-189x^4-45x^3-3x^2-3x-3}$
11	1	$(1,1,0,1)$	$\frac{972x^6-351x^5-207x^4+9x^3+3x^2-5x-1}{243x^6-108x^5-108x^4-18x^3-4x^2-4x-1}$
12	1	$(0,0,1,1)$	$\frac{972x^5-945x^4+318x^3-110x^2+22x-5}{243x^5-270x^4+63x^3-39x^2+6x-3}$
13	1	$(1,0,1,1)$	$\frac{972x^6-675x^5+63x^4-18x^3-6x^2+x-1}{243x^6-189x^5-27x^4-18x^3-7x^2-x-1}$
14	1	$(0,1,1,1)$	$\frac{8748x^7-4374x^6-324x^5-264x^4+20x^3-24x^2-2}{2187x^7-1215x^6-540x^5-306x^4-78x^3-36x^2-9x-3}$
15	1	$(1,1,1,1)$	$\frac{8748x^8-1944x^7-1512x^6-252x^5}{2187x^8-486x^7-945x^6-486x^5-180x^4-62x^3-21x^2-6x-1}$
16	2	$(1,*,0,*)$	$\frac{12x^2-3x-1}{3x^2-2x-1}$
17	2	$(0,*,1,*)$	$\frac{36x^3-13x^2+2x-1}{9x^3-7x^2-x-1}$
18	2	$(1,*,1,*)$	$\frac{108x^4-12x^3}{27x^4-12x^3-10x^2-4x-1}$
19	4	$(1,*,*,*)$	$\frac{12x^2}{3x^2-2x-1}$
20	11	$(*,1,0,*)$	$\frac{36x^3-x^2-4x-1}{9x^3-4x^2-4x-1}$
21	11	$(*,1,1,*)$	$\frac{324x^5+9x^4+4x^3+14x^2+8x+1}{81x^5-18x^4-35x^3-21x^2-6x-1}$

TABLE 10. Generating functions for the 21 choices of N_0 and ε we need to check.

i	N_0	ε	$R(\Gamma_0(N_0), 3, \varepsilon; x)$
1	1	(1, 0, 0, 0)	$\frac{7}{3} + x - 7x^2 + 17x^3 - 55x^4 + 161x^5 - 487x^6 + 1457x^7 - 4375x^8$ $+13121x^9 - 39367x^{10} + 118097x^{11} + O(x^{12})$
2	1	(0, 1, 0, 0)	$2 + x - 2x^2 - 14x^3 + 73x^4 - 254x^5 + 598x^6 - 719x^7 - 1802x^8$ $+15466x^9 - 61127x^{10} + 166426x^{11} + O(x^{12})$
3	1	(1, 1, 0, 0)	$\frac{4}{3} + 3x - 8x^2 + 4x^3 + 19x^4 - 92x^5 + 112x^6 + 739x^7 - 6176x^8$ $+28588x^9 - 100493x^{10} + 284524x^{11} + O(x^{12})$
4	1	(0, 0, 1, 0)	$2 - 2x + 10x^2 - 14x^3 - 74x^4 + 262x^5 + 130x^6 - 2630x^7 + 4078x^8$ $+15502x^9 - 67718x^{10} - 4094x^{11} + O(x^{12})$
5	1	(1, 0, 1, 0)	$\frac{4}{3} + 4x^2 + 4x^3 - 128x^4 + 424x^5 - 356x^6 - 1172x^7 - 296x^8$ $+28624x^9 - 107084x^{10} + 114004x^{11} + O(x^{12})$
6	1	(0, 1, 1, 0)	$1 + 9x^2 - 27x^3 + 9x^5 + 729x^6 - 3348x^7 + 2277x^8$ $+30969x^9 - 128844x^{10} + 162333x^{11} + O(x^{12})$
7	1	(1, 1, 1, 0)	$\frac{1}{3} + 2x + 3x^2 - 9x^3 - 54x^4 + 171x^5 + 243x^6 - 1890x^7 - 2097x^8$ $+44091x^9 - 168210x^{10} + 280431x^{11} + O(x^{12})$
8	1	(0, 0, 0, 1)	$\frac{8}{3} - 3x - 4x^2 + 8x^3 + 53x^4 + 80x^5 - 244x^6 - 1459x^7 - 2188x^8$ $+6560x^9 + 39365x^{10} + 59048x^{11} + O(x^{12})$
9	1	(1, 0, 0, 1)	$2 - x - 10x^2 + 26x^3 - x^4 + 242x^5 - 730x^6 - x^7 - 6562x^8$ $+19682x^9 - x^{10} + 177146x^{11} + O(x^{12})$
10	1	(0, 1, 0, 1)	$\frac{5}{3} - x - 5x^2 - 5x^3 + 127x^4 - 173x^5 + 355x^6 - 2177x^7 - 3989x^8$ $+22027x^9 - 21761x^{10} + 225475x^{11} + O(x^{12})$
11	1	(1, 1, 0, 1)	$1 + x - 11x^2 + 13x^3 + 73x^4 - 11x^5 - 131x^6 - 719x^7 - 8363x^8$ $+35149x^9 - 61127x^{10} + 343573x^{11} + O(x^{12})$
12	1	(0, 0, 1, 1)	$\frac{5}{3} - 4x + 7x^2 - 5x^3 - 20x^4 + 343x^5 - 113x^6 - 4088x^7 + 1891x^8$ $+22063x^9 - 28352x^{10} + 54955x^{11} + O(x^{12})$
13	1	(1, 0, 1, 1)	$1 - 2x + x^2 + 13x^3 - 74x^4 + 505x^5 - 599x^6 - 2630x^7 - 2483x^8$ $+35185x^9 - 67718x^{10} + 173053x^{11} + O(x^{12})$
14	1	(0, 1, 1, 1)	$\frac{2}{3} - 2x + 6x^2 - 18x^3 + 54x^4 + 90x^5 + 486x^6 - 4806x^7 + 90x^8$ $+37530x^9 - 89478x^{10} + 221382x^{11} + O(x^{12})$
15	1	(1, 1, 1, 1)	$252x^5 - 3348x^7 - 4284x^8$ $+50652x^9 - 128844x^{10} + 339480x^{11} + O(x^{12})$
16	2	(1, ∗, 0, ∗)	$1 + x - 11x^2 + 25x^3 - 83x^4 + 241x^5 - 731x^6 + 2185x^7 - 6563x^8$ $+19681x^9 - 59051x^{10} + 177145x^{11} + O(x^{12})$
17	2	(0, ∗, 1, ∗)	$1 - 3x + 9x^2 - 15x^3 - 75x^4 + 261x^5 + 129x^6 - 2631x^7 + 4077x^8$ $+15501x^9 - 67719x^{10} - 4095x^{11} + O(x^{12})$
18	2	(1, ∗, 1, ∗)	$12x^3 - 156x^4 + 504x^5 - 600x^6 - 444x^7 - 2484x^8$ $+35184x^9 - 126768x^{10} + 173052x^{11} + O(x^{12})$
19	4	(1, ∗, ∗, ∗)	$-12x^2 + 24x^3 - 84x^4 + 240x^5 - 732x^6 + 2184x^7 - 6564x^8$ $+19680x^9 - 59052x^{10} + 177144x^{11} + O(x^{12})$
20	11	(∗, 1, 0, ∗)	$1 - 3x^2 - 15x^3 + 72x^4 - 255x^5 + 597x^6 - 720x^7 - 1803x^8$ $+15465x^9 - 61128x^{10} + 166425x^{11} + O(x^{12})$
21	11	(∗, 1, 1, ∗)	$-1 - 2x + 19x^2 - 41x^3 - 74x^4 + 271x^5 + 859x^6 - 5978x^7$ $+6355x^8 + 46471x^9 - 196562x^{10} + 158239x^{11} + O(x^{12})$

Table 11. Generating functions written as power series.