CONGRUENCES FOR HECKE EIGENVALUES VIA PERIOD POLYNOMIALS

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Abstract. We study a conjecture motivated by Coleman and Stein's work on approximating eigenforms of infinite slope by those of finite slopes, which was recast by Rustom as a congruence for the second Fourier coefficient. We establish the weight condition required for this coefficient to be divisible by nine, and obtain the full conjecture under a natural ramification hypothesis. The proof uses period polynomials, especially Zagier's description of the Hecke action on them.

1. Introduction

A conjecture of Kiming, Rustom and Wiese [KRW16, Conjecture 1] asserts that, for any prime power p^m and $N \geq 1$ coprime to p, the set of reductions modulo p^m of eigenforms of level N is finite. The case $m = 1$ is classical, originating in ideas of Serre and Tate and carried out in full generality by Jochnowitz [Joc82]. For $m > 1$, explicit evidence is much more limited; prominent results include the level-one congruences modulo 128 obtained by Rustom [Rus19].

Earlier, and independently of the above conjecture, Coleman and Stein [CS04, Conjecture 3.4] developed for $N \in \{1, 3, 9\}$, a detailed prediction for the possible systems of eigenvalues modulo 9, stemming from their study of the approximation of infinite-slope eigenforms by finite-slope eigenforms. One consequence of their prediction is that the system of eigenvalues modulo 9 attached to the weight 2 cuspform on $X_0(27)$ should not arise from any eigenform of level dividing 9. This complements their finding that the weight 2 cusp form on $X_0(32)$ is not 2-adically approximable by eigenforms of tame level one and finite slope, which employs an extension of a theorem of Hatada [Hat79] on systems of eigenvalues modulo 8.

For $N = 1$, Rustom [Rus19, Theorem 9.2] showed that the finiteness assertion modulo 9 would follow from the following condition involving only the second Fourier coefficient a_2. Another congruence in [Hat79] already implies that a_2 is divisible by 3, but that analysis does not address the more delicate behavior of a_2 modulo 9.

Throughout this paper, an eigenform shall mean a cuspidal Hecke eigenform for $\mathrm{SL}_2(\mathbb{Z})$, normalized so that its first Fourier coefficient is 1.

Conjecture 1.1. *Let $f = \sum_{n\geq 1} a_n q^n$ be an eigenform of weight k. Then*

$$a_2 \equiv \begin{cases} 0 \pmod{9\overline{\mathbb{Z}}_3} \text{ if } k \equiv 4 \pmod 6, \\ 3 \text{ or } 6 \pmod{9\overline{\mathbb{Z}}_3} \text{ if } k \not\equiv 4 \pmod 6. \end{cases}$$

The principal result proved here is the following necessary condition for the vanishing congruence formulated in the conjecture.

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Theorem 1.2. *Let $f = \sum_{n\geq 1} a_n q^n$ be an eigenform of weight k. Assume that*

$$a_2 \equiv 0 \pmod{9\overline{\mathbb{Z}}_3}.$$

Then $k \equiv 4 \pmod 6$.

Hatada's argument is formulated in terms of the action of Hecke operators on lattices generated by periods of modular forms, while Rustom uses Merel's theory of modular symbols. We work instead in the closely related language of period polynomials. In particular, we make essential use of a theorem of Zagier [Zag90, Theorem 2] that gives an explicit description of the action of Hecke operators on period polynomials. This approach is inspired by an article of Kohnen [Hoh99], where it is shown that $a_2 \not\equiv 0 \pmod{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of the coefficient field K_f of f lying above 5. Kohnen also indicated that the method should be applicable to more general congruence problems, an observation validated by this note.

Building on our Theorem, we establish a conditional proof of the full conjecture.

Corollary 1.3. *Conjecture 1.1 holds assuming that* 3 *is unramified in the field of coefficients of the eigenform.*

The hypothesis in the Corollary is precisely what allows the residue-field information obtained from period polynomials to be lifted to a congruence modulo 9. Moreover, it appears to impose no restriction in practice. Indeed, Rustom [Rus19, p. 3] notes that the hypothesis is satisfied for all weights $k \leq 1000$. Therefore, the conjecture holds unconditionally in that range.

2. Period polynomials

Let S_k denote the space of level one cusp forms of even weight $k \geq 12$. For $0 \leq m \leq k-2$, the m-th period of an element $f = \sum_{n\geq 1} a_n q^n \in S_k$ is defined as

$$r_m(f) = \int_0^{i\infty} f(z) z^m dz = \frac{m!}{(-2\pi i)^{m+1}} L(f, m+1),$$

where the integral is taken along the imaginary axis, and $L(f,s) = \sum_{n\geq 1} a_n n^{-s}$ is the L-series of f. The period polynomial of f is the polynomial

$$r_f(x) = \sum_{m=0}^{k-2} (-1)^m \binom{k-2}{m} r_m(f) x^{k-2-m}.$$

Write $r_f^{\pm}(x) = \frac{1}{2}\left(r_f(x) \pm r_f(-x)\right)$ for the even and odd parts of r_f, respectively.

Let V_{k-2} be the set of polynomials of degree $\leq k-2$. For any integral matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with nonzero determinant and $P \in V_{k-2}$ define

$$(P \mid \gamma)(x) = (cx+d)^{k-2} P\left(\frac{ax+b}{cx+d}\right).$$

This gives a right action on V_{k-2}, which can be extended linearly to formal sums of matrices. Consider the matrices:

$$\varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$U = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Obviously, $(P \mid \varepsilon)(x) = P(-x)$. Moreover, it is known (see, for example, [KZ84, Section 1.1]) that both $r_f^{\pm}$ belong to the space

$$W_{k-2} := \{P \in V_{k-2} : P \mid (1+S) = P \mid (1+U+U^2) = 0\}.$$

Having recalled the standard properties of period polynomials, we now introduce our principal technical input: a formula of Zagier [Zag90, Section 6] expressing the period polynomial under the action of the Hecke operator T_ℓ in terms of a finite, explicitly described set of matrices. Specializing to $\ell = 2$, Zagier's formula states that if $f \in S_k$ then

$$r^{\pm}_{T_2(f)} = \sum_{M \in \mathrm{Man}_2} r_f^{\pm} \mid M,$$

where

$$\mathrm{Man}_2 = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right\}.$$

We first illustrate the utility of this formula by recovering the divisibility-by-3 result mentioned in the introduction, for it emerges quite naturally from the framework developed above. Related proofs can be found in [Hat79] and [Rus19], where the argument is expressed in terms of modular symbols.

Lemma 2.1. *Let $f = \sum_{n \geq 1} a_n q^n \in S_k$ be an eigenform. Then $a_2/3$ is an algebraic integer.*

Proof. By Manin's rationality theorem for periods (see [KZ84, p. 202]), there exists a period $\omega^- \in \mathbb{R}$ such that the polynomial $P_K = r_f^-/\omega^-$ has coefficients in K_f, the number field generated by the Fourier coefficients of f. Let $\mathcal{O}_f$ be the ring of integers of K_f.

Fix a 3-adic embedding $K_f \hookrightarrow \overline{\mathbb{Q}}_3$, and let $\mathfrak{p} \mid 3$ be the corresponding prime of K_f. Let R be the ring of integers of the $\mathfrak{p}$-adic completion of K_f. This is a complete discrete valuation ring. Let π be a uniformizer, and normalize $v_\mathfrak{p}$ by $v_\mathfrak{p}(\pi) = 1$. Write $P_K(x) = \sum_i c_i x^i$, and let $m = \min_{c_i \neq 0} v_\mathfrak{p}(c_i)$. Define $P(x) = \pi^{-m} P_K(x)$. Then $P \in R[x]$, and at least one coefficient of P is a unit of R. By construction, P is nonzero and satisfies the same relations as r_f^-, namely:

$$a_2 P = \sum_{M \in \mathrm{Man}_2} P \mid M, \qquad P \mid (1+S) = 0, \qquad P \mid (1+U+U^2) = 0.$$

We now compute the T_2-action modulo the ideal $3R$. Viewing the integer matrices modulo 3, we readily establish the following congruences:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \equiv \varepsilon \pmod 3 \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \equiv -\varepsilon \pmod 3.$$

Furthermore, we observe that

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \equiv -SU\varepsilon \pmod 3 \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \equiv SU^2\varepsilon \pmod 3.$$

Combining these congruences yields

$$\begin{aligned} a_2 P &= P \mid \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right) \\ &\equiv P \mid \big(\varepsilon + (-\varepsilon) + (-SU)\varepsilon + SU^2 \varepsilon \big) \pmod{3R}, \\ &\equiv P \mid (I_2 + (-I_2) + (-SU) + SU^2) \mid \varepsilon \pmod{3R}. \end{aligned}$$

Note that $-I_2$ acts trivially because k is even, so $P \mid A = P \mid -A$ for every matrix A. Thus

$$P \mid (I_2 + (-I_2) + (-SU) + SU^2) = 2P + P \mid (SU + SU^2),$$

and since $2P \equiv -P \pmod{3R}$ and $-P = P \mid S$, it follows that

$$a_2 P \equiv \Big[P \mid S \mid (1 + U + U^2) \Big] \mid \varepsilon \equiv 0 \mid \varepsilon \equiv 0 \pmod{3R}.$$

Because some coefficient u of P is a unit in R, the corresponding coefficient of $a_2 P$ gives $a_2 u \in 3R$, and so $a_2 \in 3R$. Equivalently, $v_{\mathfrak{p}}(a_2) \geq v_{\mathfrak{p}}(3)$. As $\mathfrak{p} \mid 3$ was arbitrary, this inequality holds at every prime of K_f above 3. At the other primes, the element 3 is a unit, and a_2 is already an algebraic integer. Therefore $a_2/3$ is integral at every finite prime of K_f, concluding that $a_2/3 \in \mathcal{O}_f$. □

3. Proof of the main results

We now refine the period-polynomial argument from the preceding section. Modulo 3, the matrices appearing in Zagier's formula admit particularly simple reductions, so that their effect is governed essentially by the conditions defining the space W_{k-2}. Modulo 9, this coarse reduction no longer suffices, for one has to keep the first 3-adic corrections to the powers of 2 appearing in the slash action. These corrections introduce derivatives of the period polynomial and lead to a differential-difference operator. Isolating this operator is the key step; the computations are explicit, but the obstruction on the weight becomes visible only in this form.

With this preparatory discussion in place, we restate the main theorem and proceed to its proof.

Theorem 1.2. *Let $f = \sum_{n \geq 1} a_n q^n \in S_k$ be an eigenform. Assume that*

$$a_2 \equiv 0 \pmod{9\overline{\mathbb{Z}}_3}.$$

Then $k \equiv 4 \pmod 6$.

Proof. Fix a 3-adic embedding $K_f \hookrightarrow \overline{\mathbb{Q}}_3$ for which the hypothesis $a_2 \in 9\overline{\mathbb{Z}}_3$ holds, and let R be the ring of integers of the corresponding completion. By the same normalization as in Lemma 2.1, choose $P \in R[x]$ proportional to the odd period polynomial, with at least one coefficient a unit.

Let $w = k - 2$ and write

$$P = \sum_{\substack{n=1 \\ n \text{ odd}}}^{k-3} c_n x^n.$$

Using Zagier's formula, we obtain

$$\begin{aligned} a_2 P &= P \mid \left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \right) \\ &= P(2x) + 2^w P\left(\frac{x}{2}\right) + (x+1)^w P\left(\frac{2x}{x+1}\right) + 2^w P\left(\frac{x+1}{2}\right). \end{aligned}$$

For every odd $n \geq 1$, we have $2^n = (-1+3)^n \equiv -1 + 3n \pmod 9$, and since w is even we also get $2^{w-n} \equiv -1 + 3(w-n) \pmod 9$. Thus the contribution of the first two terms is

$$P(2x) + 2^w P\left(\frac{x}{2}\right) \equiv \sum_{\substack{n=1 \\ n \text{ odd}}}^{k-3} c_n(-2+3w)x^n \equiv (-2+3w)P(x) \pmod{9R[x]}.$$

For the third term, set

$$A(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{k-3} c_n x^n (x+1)^{w-n} = (x+1)^w P\left(\frac{x}{x+1}\right).$$

Since

$$P \mid U = P \mid \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = x^w P\left(\frac{x-1}{x}\right),$$

we see that $A(x)$ is just $P \mid U$ evaluated at $x+1$. However, $P \mid U = -P - P \mid U^2$, and

$$P \mid U^2 = P \mid ST^{-1} = (P \mid S) \mid T^{-1} = (-P) \mid T^{-1} = -P(x-1).$$

showing that

$$A(x) = P(x) - P(x+1).$$

Moreover, since

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{k-3} n c_n x^n (x+1)^{w-n} = x(x+1)A'(x) - wxA(x).$$

we obtain that

$$(x+1)^w P\left(\frac{2x}{x+1}\right) \equiv -(1+3wx)(P(x)-P(x+1)) + 3x(x+1)(P'(x)-P'(x+1)) \pmod{9R[x]}.$$

Similarly,

$$2^w P\left(\frac{x+1}{2}\right) \equiv (3w-1)P(x+1) - 3(x+1)P'(x+1) \pmod{9R[x]}.$$

Combining these four terms gives

$$3bP = a_2 P \equiv 3D_w(P) \pmod{9R[x]},$$

where

$$\begin{aligned} D_w(P) = (w-1)P &+ x(x+1)\big(P'(x) - P'(x+1)\big) \\ &- wx\big(P(x) - P(x+1)\big) + wP(x+1) - (x+1)P'(x+1). \end{aligned}$$

Hence

$$bP \equiv D_w(P) \pmod{3R[x]}.$$

Let $\mathfrak{m}$ be the maximal ideal of R, and let $\kappa = R/\mathfrak{m}$ be the residue field. Since $b \in 3R \subseteq \mathfrak{m}$, we have $\overline{b} = 0 \in \kappa$. Also, since P has a unit coefficient, its reduction $\overline{P} \in \kappa[x]$ is nonzero, which gives $D_w(\overline{P}) = 0$ in $\kappa[x]$.

Let $\delta \in \{0, 1, 2\}$ be the residue class of w modulo 3. Over the field κ, define

$$H_\delta(Q) = xQ' - \delta Q$$

and

$$L_\delta(Q) = (x+1)\big(H_\delta(Q)(x) - H_\delta(Q)(x+1)\big).$$

Then for every polynomial $Q \in \kappa[x]$ we have

$$D_w(Q) = (2\delta - 1)Q + L_\delta(Q)$$

Therefore $0 = D_w(\overline{P}) = (2\delta - 1)\overline{P} + L_\delta(\overline{P})$, or equivalently

$$L_\delta(\overline{P}) = -(2\delta - 1)\overline{P}.$$

If $k \not\equiv 4 \pmod 6$, then $\delta \in \{0, 1\}$, which leads to either $L_0(\overline{P}) = \overline{P}$ or $L_1(\overline{P}) = 2\overline{P}$. The subsequent Lemma rules out both of these possibilities, and completes the proof.

□

Lemma 3.1. *Let K be a field of characteristic 3, let $P \in K[x]$ be a nonzero odd polynomial. For $\delta \in K$ define*

$$H_\delta = xP' - \delta P,$$

and

$$L_\delta(P) = (x+1)(H_\delta(x) - H_\delta(x+1)).$$

Then $L_0(P) \neq P$ and $L_1(P) \neq 2P$. Moreover, if $\lambda \in K^\times$ then $L_2(P) \neq \lambda P$.

Proof. Suppose that P has degree d, and that $P(x) = a_d x^d + a_{d-2}x^{d-2} + O(x^{d-4})$ with $a_d \neq 0$. Then

$$H_\delta(x) = (d-\delta)a_d x^d + (d-2-\delta)a_{d-2}x^{d-2} + O(x^{d-4})$$

and

$$H_\delta(x) - H_\delta(x+1) = -(d-\delta)a_d\left[dx^{d-1} + \binom{d}{2}x^{d-2}\right] + O(x^{d-3}),$$

which shows that

$$L_\delta(P)(x) = -d(d-\delta)a_d x^d - \binom{d+1}{2}(d-\delta)a_d x^{d-1} + O(x^{d-2}).$$

If $d \equiv 0 \pmod 3$ then clearly $\deg(L_\delta) < d$, so $L_\delta(P)$ cannot equal any non-zero scalar multiple of $P(x)$.

If $d \equiv 1 \pmod 3$ then $L_\delta(P)(x) = (\delta - 1)a_d x^d + (\delta - 1)a_d x^{d-1} + O(x^{d-2})$. For $L_\delta(P)$ to be a scalar multiple of $P(x)$, it must mirror P's structure. Because $P(x)$ is odd, it has no x^{d-1} term, meaning $L_\delta(P)$ must also have a zero coefficient for x^{d-1}, i.e.,

$$(\delta - 1)a_d = 0$$

Since $a_d \neq 0$, we must have $\delta = 1$. However, if $\delta = 1$, the x^d coefficient $(\delta-1)a_d$ also becomes 0. This means $\deg(L_1(P)) < d$, so $L_1(P) \neq 2P$. For $\delta \in \{0, 2\}$, the required x^{d-1} term fails to vanish.

If $d \equiv -1 \pmod 3$ then the leading term of $L_\delta(P)$ is $-(\delta + 1)a_d x^d$, and the x^{d-1} term vanishes. We consider three cases.

(1) Let $\delta = 0$. The leading coefficient of $L_0(P)$ evaluates to $-a_d$, whereas that of P is a_d. Thus $L_0(P) \neq P$.
(2) Let $\delta = 1$. This time the leading coefficient of $L_1(P)$ is $-2a_d \equiv a_d \pmod 3$, whereas that of $2P$ is $2a_d \equiv -a_d \pmod 3$. Hence $L_1(P) \neq 2P$.
(3) Let $\delta = 2$. Then the leading coefficient of $L_2(P)$ is $0 \pmod 3$. Since $\deg(L_2(P)) < d$, we have $L_2(P) \neq \lambda P$ for all $\lambda \neq 0$.

□

We now explain how the ramification of 3 enters the argument. Our computations above determines the reduction of $a_2/3$ in the residue field of the local coefficient ring. If 3 is unramified, then the maximal ideal is precisely $3R$, so this information is equivalent to knowing a_2 modulo $9R$. In the ramified case, the same reduction gives only a congruence modulo the maximal ideal, which is weaker than a congruence modulo $3R$.

Proof of Corollary 1.3. We use the notation from the proof of Theorem 1.2. The condition that 3 is unramified in K_f implies that $3R = \mathfrak{m}$, the maximal ideal of R.

First suppose $k \equiv 4 \pmod 6$. Then $\delta = 2$, so $\overline{b}\,\overline{P} = L_2(\overline{P})$. By Lemma 3.1, we must have $\overline{b} = 0$, giving $a_2 \equiv 0 \pmod{9\mathbb{Z}_3}$.

Now suppose $k \not\equiv 4 \pmod 6$, so $\delta \in \{0, 1\}$. The computation in the proof of Theorem 1.2 shows that

$$bP \equiv D_w(P) \pmod{3R[x]},$$

where $b = a_2/3 \in R$. We emphasize that this part does not require that $a_2 \in 9R$; indeed, this assumption was only used in the theorem to conclude that $b \in 3R$. Then

$$\overline{b}\,\overline{P} = (2\delta - 1)\overline{P} + L_\delta(\overline{P}).$$

If $\overline{P}$ has degree N and leading term $u \neq 0$, then the coefficient of x^N in $L_\delta(\overline{P})$ is $-N(N-\delta)u$. This yields

$$\overline{b} = (2\delta - 1) - N(N - \delta).$$

It is easy to see that for $\delta \in \{0, 1\}$ we have $\overline{b} \in \{1, 2\}$. Thus in either case $b \equiv 1$ or $2 \pmod{3R}$. Equivalently, $a_2 \equiv 3$ or $6 \pmod{9\mathbb{Z}_3}$. □

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