Homework 3/ Practice Midterm 1

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1. Consider the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Use induction to prove that F_{3n} is even for each natural number n.

   (Hint: Find a relation between $F_{3(k+1)}$ and F_{3k}. Then use the inductive hypothesis.)

2. (a) Show that every integer can be written in the form $44x + 17y$ where x and y are integers.
   (b) Find integers x and y such that $44x + 17y = 100$.
   (c) Describe all integer solutions of the equation $44x + 17y = 100$.

3. Prove that every perfect square is either of the form $4k$ or $4k + 1$ for some $k \in \mathbb{Z}$.

4. Let $a, b, c \in \mathbb{N}$. Prove that $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$.

5. (a) How many positive integers less than 101 have an *odd* number of positive divisors?
   (b) How many positive integers k have the property that

$$\operatorname{lcm}(6^6, 8^8, k) = 12^{12}?$$

Remark. For Problems 4 and 5 you may use (without proof) the following fact: if

$$m = p_1^{a_1} \dots p_r^{a_r}, n = p_1^{b_1} \dots p_r^{b_r}, \text{ and } k = p_1^{c_1} \dots p_r^{c_r},$$

then

$$\gcd(m, n, k) = p_1^{d_1} \dots p_r^{d_r} \text{ and } \operatorname{lcm}(m, n, k) = p_1^{e_1} \dots p_r^{e_r},$$

where $d_i = \min(a_i, b_i, c_i)$ and $e_i = \max(a_i, b_i, c_i)$.