Math 261: Practice Midterm 2 Solutions

1. (a)

$$\hat{v} = \mathrm{proj}_W v = \frac{v \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} u_2$$

$$= \frac{8 \cdot 4 + 7 \cdot (-1) + (-7) \cdot (-8)}{4^2 + (-1)^2 + (-8)^2} \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} + \frac{8 \cdot (-7) + 7 \cdot 4 + (-7) \cdot (-4)}{(-7)^2 + 4^2 + (-4)^2} \begin{pmatrix} -7 \\ 4 \\ -4 \end{pmatrix}$$

$$= \frac{81}{81} \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} + \frac{0}{81} \begin{pmatrix} -7 \\ 4 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix}.$$

(b) The distance is:

$$\| v - \hat{v} \| = \left\| \begin{pmatrix} 8 \\ 7 \\ -7 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \right\|$$

$$= \sqrt{4^2 + 8^2 + 1^2}$$

$$= 9.$$

(2) (a) Consider the matrix A whose columns are v_1, v_2, v_3:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

We row reduce A:

$$A \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix} \xrightarrow[R_3 \to \frac{R_3}{2}]{R_1 \to R_1 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to 2R_3 + R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ which has a pivot in every column.

Thus, the eq. $Ax = b$ has a sol. for every $b \in \mathbb{R}^3$.
In other words, $x_1 v_1 + x_2 v_2 + x_3 v_3 = b$ has a sol. for every $b \in \mathbb{R}^3$.
Hence, $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$.

(b) Note that:

$$2x + 3y + 5z = 0 \Rightarrow x + \frac{3}{2}y + \frac{5}{2}z = 0$$
$$\Rightarrow x = -\frac{3}{2}y - \frac{5}{2}z,$$

where y and z are free variables.

Thus $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3/2\,y - 5/2\,z \\ y \\ z \end{pmatrix}$$
$$= y \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix}.$$

So a basis is given by:

$$\left\{ \begin{pmatrix} -3/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5/2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(3) (a) The vectors v_1, v_2 and v_3 are lin. dep. when the row echelon form of the matrix A, whose columns are v_1, v_2, v_3, does NOT have a pivot in every column.

Now:

$$A \xrightarrow{R_3 \to R_3 - tR_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -t \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -t-1 \end{pmatrix}$$

The last column is not a pivot column when

$-t-1=0 \iff \boxed{t=-1.}$

In this case, the reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_3 = t \text{ is a free variable} \\ x_2 + x_3 = 0 \Rightarrow x_2 = -t \\ x_1 + x_3 = 0 \Rightarrow x_1 = -t. \end{array}$$

Thus: $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

In particular, a linear dependence relation is:

$$(-1) \cdot v_1 + (-1) \cdot v_2 + 1 \cdot v_3 = 0.$$

<u>Check</u>: $(-1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

(b) The eq. $Ax=0$ has a nontrivial sol

$\iff$ the columns of A are linearly dependent

$\iff t=-1$, by part (a).

4. (a) Note that

$$\begin{pmatrix} x \\ x-y \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ so}$$

$$W = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$$

The span of a collection of vectors is necessarily a subspace $\Rightarrow$ W is a subspace of $\mathbb{R}^3$.

(b) We row reduce A:

$$A \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix} \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 + 3R_1} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow{R_2 \to \frac{R_2}{7}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 14 & 1 & -4 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 14R_2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2 + R_3} \begin{pmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

There are pivots in columns 1, 2 and 3, thus:

$$\text{Basis of Col}(A) = \left\{\begin{pmatrix} 1 \\ -3 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 4 \end{pmatrix}\right\};$$

$$\text{Basis of Row}(A) = \left\{ (1,0,0,-\tfrac{10}{7}),\ (0,1,0,-\tfrac{2}{7}),\ (0,0,1,0) \right\}$$

Now, for Nul(A) solve the system ($Ax = 0$):

$$\left.\begin{aligned} &x_4 = t : \text{free variable} \\ &x_3 = 0 \\ &x_2 - 2/7\, x_4 = 0 \Rightarrow x_2 = \tfrac{2}{7} t \\ &x_1 - 10/7\, x_4 = 0 \Rightarrow x_1 = \tfrac{10}{7} t \end{aligned}\right\} \quad \text{Basis of Nul}(A) \text{ is: } \left\{\begin{pmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix}\right\}$$