Math 261: Practice Midterm 1 Solutions

1. (a) We row reduce the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 5 & -1 & 3 & 2 \\ -2 & -10 & 3 & -8 & -1 \\ 3 & 15 & -3 & 9 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{cccc|c} 1 & 5 & -1 & 3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 5 & 0 & 1 & 5 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

which is the reduced row echelon form.

(b) (i) Columns 1 and 3 are pivot columns.

(ii) By (i) we get that x_1 and x_3 are <u>basic</u> variables. Thus, x_2 and x_4 are <u>free</u> variables.

(iii) Since there are free variables, the system has infinitely many solutions. We write the solution set as follows:

$$\begin{aligned} x_1 + 5x_2 + x_4 &= 5 \\ x_3 - 2x_4 &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 5 - 5z - t \\ x_3 &= 3 + 2t \end{aligned}$$

$$\text{free} \begin{cases} x_2 = z \\ x_4 = t \end{cases}$$

Thus, the general solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 - 5z - t \\ z \\ 3 + 2t \\ t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

where z and t are any real numbers.

(2) The zero vector is a linear combination of V_1, V_2 and V_3 if there exist scalars c_1, c_2 and c_3 such that: $c_1V_1 + c_2V_2 + c_3V_3 = 0$.

Equivalently: $c_1\begin{pmatrix}3\\-3\\6\end{pmatrix} + c_2\begin{pmatrix}5\\-2\\1\end{pmatrix} + c_3\begin{pmatrix}-4\\4\\-8\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$,

which is the same as solving the system:

$$\begin{aligned} 3c_1 + 5c_2 - 4c_3 &= 0 \\ -3c_1 - 2c_2 + 4c_3 &= 0 \\ 6c_1 + c_2 - 8c_3 &= 0 \end{aligned}$$

We reduce the augmented matrix:

$$\left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_1 + R_2} \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left(\begin{array}{ccc|c} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{R_2}{3}]{R_1 \to \frac{R_1}{3}} \left(\begin{array}{ccc|c} 1 & 5/3 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - \frac{5}{3}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced Row Echelon Form

pivot columns

We get: $x_1 - \frac{4}{3}x_3 = 0$
$x_2 = 0$ $\Rightarrow x_1 = \frac{4}{3}x_3 = \frac{4}{3}t$.
$x_3 = t$

The solution is $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}\frac{4}{3}t\\0\\t\end{pmatrix} = t\begin{pmatrix}4/3\\0\\1\end{pmatrix}$.

This gives, for example:

$\frac{4}{3} \cdot V_1 + 0 \cdot V_2 + 1 \cdot V_3 = 0$.

3. (a) We use the augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & 7 & 0 & 1 & 0 \\ -1 & 1 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 2 & 7 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -9 & 4 & -2 \\ 0 & 1 & 0 & -17 & 7 & -3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -3 & 1 \\ 0 & 1 & 0 & -17 & 7 & -3 \\ 0 & 0 & 1 & 5 & -2 & 1 \end{array}\right)$$

Thus $A^{-1} = \begin{pmatrix} 8 & -3 & 1 \\ -17 & 7 & -3 \\ 5 & -2 & 1 \end{pmatrix}$

(b) We write the system as a matrix equation:

$$Ax = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

This way, the solution is:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 & -3 & 1 \\ -17 & 7 & -3 \\ 5 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 12 \\ -28 \\ 9 \end{pmatrix}$$

4. (a) We compute

$$AB = \begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 5\cdot 2 + 1\cdot 4 & 5\cdot 0 + 1\cdot 3 \\ 3\cdot 2 + (-2)\cdot 4 & 3\cdot 0 + (-2)\cdot 3 \end{pmatrix} = \begin{pmatrix} 14 & 3 \\ -2 & -6 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 2\cdot 5 + 0\cdot 3 & 2\cdot 1 + 0\cdot(-2) \\ 4\cdot 5 + 3\cdot 3 & 4\cdot 1 + 3\cdot(-2) \end{pmatrix} = \begin{pmatrix} 10 & 2 \\ 29 & -2 \end{pmatrix}$$

Thus, $AB \neq BA$, so the two matrices do not commute.

(b) Recall that:

$$(AB)^t = B^t A^t$$

Thus: $B^t A^t = \begin{pmatrix} 14 & 3 \\ -2 & -6 \end{pmatrix}^t = \begin{pmatrix} 14 & -2 \\ 3 & -6 \end{pmatrix}$.

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5. (a) We use the cofactor expansion along row 1:

$$\det A = (-1)^{1+1}\cdot 1\cdot \det\begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix} + (-1)^{1+2}\cdot 3\cdot \det\begin{pmatrix} 2 & 5 \\ 4 & 1 \end{pmatrix} + (-1)^{1+3}\cdot 2\cdot \det\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= 1\cdot(3\cdot 1 - 5\cdot 1) - 3\cdot(2\cdot 1 - 5\cdot 4) + 2(2\cdot 1 - 3\cdot 4)$$

$$= -2 - 3\cdot(-18) + 2\cdot(-10)$$

$$= \boxed{32}.$$

(b) Since $\det A \neq 0$, it follows that A is invertible.