Math 261: Practice Final Solutions

1. (a) Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Then

- $T(x+y) = T\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ -(x_3+y_3) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ -y_3 \end{pmatrix} = T(x) + T(y)$;

- If c - scalar, then:

$$T(c \cdot x) = T\begin{pmatrix} cx_1 \\ cx_2 \\ cx_3 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ -cx_3 \end{pmatrix} = c\begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix} = cT(x).$$

Thus, T is a linear transf.

(b) Note that $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, so the standard matrix of T is $\begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}$.

This means that $T(x) = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} x$ for all $x \in \mathbb{R}^2$.

We want to find x s.t. $T(x) = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$.

Row reduce: $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 4 & 5 & 8 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & -4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -4 \end{array}\right)$.

Thus $x = \begin{pmatrix} 7 \\ -4 \end{pmatrix}$.

Check: $T\begin{pmatrix} 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$.

(2.) The standard matrix of T is:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & -2 & 4 \end{pmatrix}.$$

The kernel of T is the Null space of A, and the image of T is the Column space of A.

Row reducing we get:

$$A \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

x_1 - free var; $x_2 = x_3 = 0$.

Basis of Nul(A) is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Basis of Col(A) is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 1 \\ 4 \end{pmatrix} \right\}$.

(3.) Since $\lambda_1 \neq \lambda_2$, at least one of them is nonzero, say $\lambda_1 \neq 0$.

Now, $\{v_1, v_2\}$ is linearly independent if and only if the vectors are not scalar multiples of one another.

Assume this is false, then $v_1 = c v_2$ for some scalar $c \neq 0$.

Then $Av_1 = A(cv_2)$. But $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$, so we get: $\lambda_1 v_1 = c \lambda_2 v_2$ (1)

Moreover, $v_1 = cv_2 \Rightarrow \lambda_1 v_1 = c\lambda_1 v_2$ (2)

From (1) and (2) we obtain:

$$c\lambda_2 v_2 = c\lambda_1 v_2 \Rightarrow c(\lambda_1 - \lambda_2) v_2 = 0.$$

But $c \neq 0$ and $\lambda_1 - \lambda_2 \neq 0$ (since $\lambda_1 \neq \lambda_2$).

This forces $v_2 = 0$, which also cannot happen since eigenvectors by definition are nonzero.

In conclusion, our assumption was false, so $\{v_1, v_2\}$ is linearly indep.

4. Let $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$.

· <u>Eigenvalues</u>: Consider the char. poly

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2).$$

Thus A has two eigenvalues: $\lambda = 1$ and $\lambda = 2$.

In particular, we know A is diagonalizable.

· <u>Eigenvectors</u>

<u>$\lambda = 1$</u>; Reduce $A - \lambda I_2 = \begin{pmatrix} 3 & -3 \\ 2 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 - \text{free} \\ x_1 = x_2 . \end{matrix}$

Basis for this eigenspace: $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

<u>$\lambda = 2$</u>: Reduce $A - \lambda I_2 = \begin{pmatrix} 2 & -3 \\ 2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 - \text{free} \\ x_1 = \frac{3}{2} x_2 . \end{matrix}$

Basis for this eigenspace: $\left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}$.

· <u>Diagonalize A</u>:

Form $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $P = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$.

This gives: $A = PDP^{-1}$.

Now: $A^{100} = PD^{100}P^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1^{100} & 0 \\ 0 & 2^{100} \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 3 \cdot 2^{100} \\ 1 & 2^{101} \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 2^{100} - 2 & 3 - 3 \cdot 2^{100} \\ 2^{101} - 2 & 3 - 2^{101} \end{pmatrix}.$$