Some Hypergeometric Identities and Related Leonard Pairs

by

Taiyo Summers Terada

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Dissertation Committee:
John Caughman, Chair
Derek Garton
Paul Terwilliger
J.J.P. Veerman

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Abstract

The notion of a Leonard pair was introduced by Terwilliger in 2001 [48] to simplify Leonard's theorem, which classifies the orthogonal polynomials in the terminating branch of the Askey-Wilson scheme. In the same year, Kresch and Tamvakis [32] made a conjecture about a certain ${}_4F_3$ hypergeometric series while studying the arithmetic analogues of the standard conjectures for the Grassmanian $G(2, n)$. The ${}_4F_3$ series appearing in their conjecture is closely related to a family of orthogonal polynomials in the Askey-Wilson scheme. Consequently, the theory of Leonard pairs provides a useful framework for understanding their conjecture.

In this dissertation, we present our proof of the Kresch-Tamvakis conjecture (a result we first published in [6]). To do so, we construct a specific Leonard pair A, A^* and a related sequence of matrices B_i. We identify the hypergeometric series in question with the eigenvalues of these matrices. We then use a result from mathematical physics known as the Biedenharn-Elliot identity to prove that the entries of the B_i are nonnegative, and, from this, we obtain the conjectured bound from the Perron-Frobenius theorem.

The Leonard pair studied here has many special properties related to spin models and strongly regular graphs. We formulate a number of results exploring these connections, and we prove a generalization that holds for a larger family of Leonard pairs.

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1 Introduction

The notion of a Leonard pair in its modern form was introduced by Terwilliger in 2001 [48] (although Leonard pairs over $\mathbb{R}$ appeared implicitly in his work dating back to 1987 [47]). These algebraic objects were used to simplify the framework and results of Leonard [33], and Bannai and Ito [1] on the classification of the terminating branch of the Askey scheme of orthogonal polynomials. The theory of Leonard pairs is an active area of research in algebraic combinatorics, with applications to the study of association schemes, distance regular graphs, combinatorial designs, orthogonal polynomials, special function theory, hypergeometric functions, representation theory, knot theory, and quantum mechanics [1, 14, 43, 38, 37, 21, 52, 41].

In this dissertation, we use the theory of Leonard pairs to prove some results about hypergeometric series. To describe our results, we need to introduce some terminology. Formal definitions will be given in the next section (see Definition 2.1.1), but loosely, a Leonard pair is a pair of diagonalizable linear transformations A, A^* over a finite vector space, with the property that each transformation acts on an eigenbasis of the other one in an irreducible tridiagonal fashion.

Let z denote an indeterminate and let p, q be positive integers. Let us briefly recall the notion of the hypergeometric series ${}_pF_q(z)$. For any real number a and nonnegative integer n, define

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1). \tag{1.1}$$

Given real numbers $\{a_i\}_{i=1}^p$ and $\{b_i\}_{i=1}^q$, the corresponding ${}_pF_q$ hypergeometric series is defined by

$$ {}_pF_q\left[\begin{matrix} a_1,\ a_2,\ \dots\ a_p \\ b_1,\ b_2,\ \dots\ b_q \end{matrix} ;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}. \tag{1.2} $$

One of our two main results (Theorem 4.57) resolves a 22-year old open conjecture that was well-known in the special functions community [32, 26, 36, 46], [50, Problem 11.5]. That conjecture, first put forth in 2001 by Kresch and Tamvakis [32] in their study of intersection theory in algebraic geometry, can be stated as follows.

Conjecture 1.0.1. *[32, Conjecture 2] For any positive integer D and any integers i, j $(0 \leq i, j \leq D)$, the absolute value of the following hypergeometric series is at most 1:*

$$ {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ;1\right]. \tag{1.3} $$

Note 1.0.2. Conjecture 1.0.1 appears in [32, Conjecture 2] with

$$ n = i, \qquad s = j, \qquad T = D+1. $$

The proof of this result was found in 2023 (by Caughman and the present author), and was recently published in [6]. A complete presentation of the proof also appears in Chapter 4 of this dissertation.

The other main result of this dissertation regards a formula related to so-called Boltzmann pairs for certain Leonard pairs. These Boltzmann pairs can be useful in finding spin models [40, 41] which, in turn, can be used to construct link invariants, including the well-known Jones polynomial and Kauffman polynomial [29, 30, 27, 28, 13]. We will show that, under certain circumstances, these Boltzmann pairs satisfy formulas that give an expression for a specific sum of products of hypergeometric series. Indeed, for the same Leonard pair we explored in our work on the Kresch-Tamvakis conjecture, this formula implies a connection to a family of feasible parameters of strongly regular graphs. The corresponding graphs have been conjectured to always exist [22]. We hope this algebraic connection may lead to further results in this direction.

The dissertation is organized as follows. In Chapter 2, we recall the definition of a Leonard pair and discuss some of the basic related terminology and properties. We introduce an example that we will use throughout the thesis. We further discuss some basics of Leonard pairs, related orthogonal polynomials and discuss the intersection matrices. In Chapter 3, we discuss association schemes and distance regular graphs and how their so called intersection and Krein parameters relate to Leonard pairs. In Chapter 4, we present the proof of the Kresch-Tamvakis conjecture. In Chapter 5, we recall the definition of a modular Leonard triple and a spin-Leonard pair, and discuss how they correspond. We prove a formula that certain Racah type spin-Leonard pairs satisfy. This result implies an algebraic connection between a family of strongly regular graphs and the Leonard pair in our running example. Using the work of Curtin [10, 11] we show a similar formula for the Racah case holds for some other families of spin-Leonard pairs. We state some hypergeometric formulas implied by these formulas. Finally, in Chapter 6, we state some further directions of research.

Throughout this dissertation, the square root of a nonnegative real number is understood to be nonnegative.

2 Leonard Pairs

In this chapter, we define Leonard pairs and review some of their basic properties, which will be used throughout the dissertation.

2.1 Definition

Let $\mathbb{K}$ be an algebraically closed field with characteristic 0. A matrix $B \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is called *tridiagonal* whenever each nonzero entry lies on the diagonal, the subdiagonal, or the superdiagonal. Assume that B is tridiagonal. Then B is called *irreducible* whenever each entry on the subdiagonal is nonzero, and each entry on the superdiagonal is nonzero. (See Lemma A.1.3 in Appendix A.1 for more details.)

We now recall the definition of a Leonard pair.

Definition 2.1.1. Let V be a finite dimensional vector space over $\mathbb{K}$. Two transformations $A : V \to V$ and $A^* : V \to V$ are called a *Leonard pair on* V if both of the following (i),(ii) hold:

(i) there is a basis for V such that the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal, and

(ii) there is a basis for V such that the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

The above Leonard pair A, A^* is said to be *over* $\mathbb{K}$.

Note 2.1.2. According to a common notational convention, the symbol A^* denotes the conjugate-transpose of A. We are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

Let d denote a non-negative integer. Let V be a vector space of dimension $d+1$ over $\mathbb{K}$ and let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the algebra of all $(d+1) \times (d+1)$ matrices over $\mathbb{K}$. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra of all linear operators on V. For this dissertation, we will work exclusively in finite dimension, where these two spaces are known to be isomorphic [15].

Let A, A^* be a Leonard pair on V. We will use $\mathcal{B}$ to refer to a fixed basis of V, with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal. We will sometimes refer to this as the *standard basis* of the Leonard pair A, A^*. Expressing A, A^* in this basis, we use the parameter conventions:

$$A = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta_0^* & & & & \mathbf{0} \\ & \theta_1^* & & & \\ & & \ddots & & \\ & & & \theta_{d-1}^* & \\ \mathbf{0} & & & & \theta_d^* \end{pmatrix}. \qquad (2.4)$$

Because of the isomorphism between $\mathrm{Mat}_{d+1}(\mathbb{K})$ and $\mathrm{End}(V)$, we often equate A and A^* with their matrices represented in the basis $\mathcal{B}$ (unless specified otherwise).

We will use $\mathcal{B}^*$ to refer to a fixed basis of V, with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Expressing A, A^* in this basis we use the parameter conventions:

$$A = \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & \theta_1 & & & \\ & & \ddots & & \\ & & & \theta_{d-1} & \\ \mathbf{0} & & & & \theta_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} a_0^* & b_0^* & & & \mathbf{0} \\ c_1^* & a_1^* & b_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ \mathbf{0} & & & c_d^* & a_d^* \end{pmatrix}. \tag{2.5}$$

By assumption $b_i, c_{i+1} \neq 0$, for $0 \leq i < d$. Typically c_0, c_0^*, b_d, and b_d^* are not defined, but we will use the convention that $c_0 = c_0^* = b_d = b_d^* = 0$. We will also refer to d as the *diameter* of A, A^*.

We will need to discuss the bases $\mathcal{B}$ and $\mathcal{B}^*$ of V often enough that it will be convenient to define the maps from $\mathrm{End}(V) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ that take linear transformations of V to their corresponding matrix in a desired basis.

Definition 2.1.3. [50, Def. 3.2] Let (A, A^*) be a Leonard pair over V and let $\mathcal{B}$ be the basis of V where A is irreducible tridiagonal and A^* diagonal. The map $\flat : \mathrm{End}(V) \to \mathrm{Mat}_{d+1}(\mathbb{K})$ takes each linear transformation X of V to its associated matrix, as represented in basis $\mathcal{B}$. We note that $\flat$ is a $\mathbb{K}$-algebra isomorphism.

Note that, to get the map that represents a transformation as a matrix in basis $\mathcal{B}^*$, we can apply Definition 2.1.3 to the Leonard pair (A^*, A).

For convenience, we define a few more parameters. Define $k_i = \prod_{j=1}^{i} \frac{b_{j-1}}{c_j}$, and let K be the diagonal matrix

$$K = \mathrm{diag}(k_0, k_1, \ldots, k_d). \tag{2.6}$$

The parameter ν is defined by

$$\nu = \sum_{i=0}^{d} k_i. \tag{2.7}$$

The values k_i^* are defined similarly in terms of the b_i^* and c_i^*. By [50, Def. 2.6, Lem. 3.8] we have $\nu^* = \nu$.

2.2 An example of a Leonard pair

A *graph* G is a pair of sets $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a nonempty set, and $\mathcal{E}$ is a (possibly empty) set of two element subsets of $\mathcal{V}$. An element of $\mathcal{V}$ is called a *vertex* and an element of $\mathcal{E}$ is called an *edge*. This is sometimes referred to as a *simple* graph, which means that $\mathcal{E}$ is not a multi-set (there are no multi-edges), its members are not ordered pairs (edges are undirected), and no edge is allowed from a vertex to itself (no loops). We will assume all graphs are simple unless specified otherwise. A graph can be depicted visually as a set of points representing the vertices connected by curves or lines representing the edges. An example is shown in Figure 2.1.

As we will see in Chapter 3, Leonard pairs can sometimes be associated with certain graphs. Here is an example that happens to be associated with the 4-cube (or tesseract) graph, depicted in Figure 2.1. With $d = 4$, $\mathbb{K} = \mathbb{R}$, $V = \mathbb{R}^5$, let

$$A = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}. \tag{2.8}$$

We can see in Figure 2.1, that if we take the partition of the vertex set $\{X_0(u), X_1(u), X_2(u), X_3(u), X_4(u)\}$ as shown by the dashed ellipses, then the $(i, i-1)$, (i, i), and $(i, i+1)$ entries of A, correspond to the number of neighbors any vertex in the set $X_i(u)$ has in $X_{i-1}(u)$, $X_i(u)$, and $X_{i+1}(u)$, respectively.

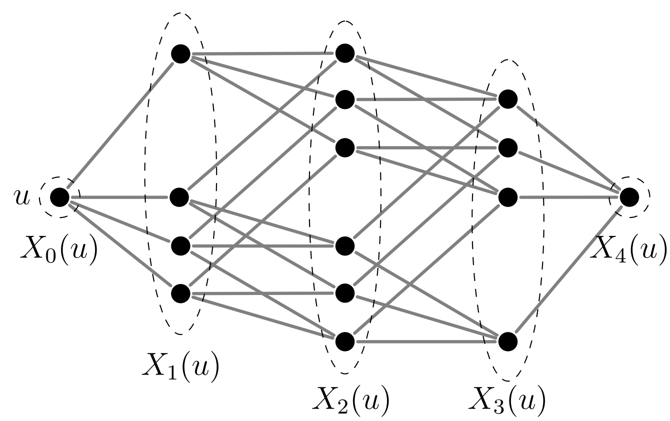

Figure 2.1: Tesseract. The dashed ellipses show the partite sets $X_i(u)$. These contain the vertices at distance i from the vertex u shown on the far left.

If we consider the following matrix,

$$P = \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}, \tag{2.9}$$

then, by matrix multiplication, we find that $P^2 = (2^4)I$, $AP = PA^*$, and $PA = A^*P$. These facts easily imply that A, A^* is a Leonard pair.

The combinatorial connection this Leonard pair enjoys with the graph above offers many advantages to aid in its study. For one, many parameters of interest must be nonnegative integers, because they count some combinatorial property of the structure. However, not all Leonard pairs arise from graphs in this way. Indeed, the next example, which will be very important in our subsequent work, does not have such an immediate connection to graphs. Nevertheless we will eventually show that some parameters of interest can still be shown to be nonnegative. As we will see, this nonnegativity can have important implications, including the Kresch-Tamvakis Conjecture [6].

2.3 Another example of a Leonard pair

In order to present our next example of a Leonard pair we need a few definitions.

Definition 2.3.1. Fix any integer $d \geq 0$. For each integer i $(0 \leq i \leq d)$, define

$$c_i = \frac{3(d-i+1)i(d+i+1)}{d(d+2)(2i+1)}, \quad a_i = \frac{3i(i+1)}{d(d+2)}, \quad b_i = \frac{3(d-i)(i+1)(d+i+2)}{d(d+2)(2i+1)}, \tag{2.10}$$

and let $\theta^*_i = 3 - 2a_i$. Define A and A^* as the following matrices in $\mathrm{Mat}_{d+1}(\mathbb{R})$:

$$A = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta^*_0 & & & & \mathbf{0} \\ & \theta^*_1 & & & \\ & & \ddots & & \\ & & & \theta^*_{d-1} & \\ \mathbf{0} & & & & \theta^*_d \end{pmatrix}. \tag{2.11}$$

Definition 2.3.2. We define a matrix $P \in \mathrm{Mat}_{d+1}(\mathbb{R})$ with the following entries:

$$P_{i,j} = (2j+1)\, {}_4F_3 \left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ d+2,\ -d \end{matrix} ; 1 \right] \qquad (0 \leq i, j \leq d). \tag{2.12}$$

As in our previous example, the matrix P is useful to verify that the given matrices A, A^* form a Leonard pair.

Lemma 2.3.3. ([49, Ex. 5.10] and [50, Thm. 4.9]) *The following hold:*

(i) $P^2 = (d+1)^2 I$*;*

(ii) $PA = A^* P$*;*

(iii) $PA^* = AP$*;*

(iv) the pair A, A^* *is a Leonard pair over* $\mathbb{R}$*.*

Proof. The calculations establishing *(i)–(iii)* are the following special case of [49, Ex. 5.10] and [50, Thm. 4.9]:

$$d = d, \qquad \theta_0 = \theta_0^* = 3, \qquad s = s^* = r_1 = 0, \qquad r_2 = d+1, \qquad h = h^* = \frac{-6}{d(d+2)}.$$

Item *(iv)* follows from items *(i)–(iii)*. □

The Leonard pairs from [49, Ex. 5.10] are said to have *Racah type*. So the Leonard pair A, A^* in Lemma 2.3.3 has Racah type. This Leonard pair is *self-dual* in the sense of [40, p. 5]. In particular, we see from Lemma 2.3.3 that for all i, $c_i = c_i^*$, $a_i = a_i^*$, $b_i = b_i^*$, and $\theta_i = \theta_i^*$. Hence when discussing the Leonard pair in Def. 2.3.1 (and self-dual Leonard pairs in general) we will omit the $*$ notation on these parameters.

We note that, for the Leonard pair in Def. 2.3.1, it is straightforward to show that

$$k_i = 2i + 1 \qquad (0 \leq i \leq d), \tag{2.13}$$

which means that

$$\nu = \sum_{i=0}^{d} (2i + 1) = (d + 1)^2. \tag{2.14}$$

2.4 Leonard systems

A useful structure related to a Leonard pair is an object known as a Leonard system. We will define them briefly, but we refer the interested reader to the work of Terwilliger ([48, 49, 50]) for more detailed information.

Before we give the definition of a Leonard system we note a few observations.

Definition 2.4.1. A matrix $A \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is called *multiplicity-free* if it has $d+1$ distinct eigenvalues.

Lemma 2.4.2. *Let A, A^* be a Leonard pair on vector space V of dimension $d+1$ over $\mathbb{K}$. The eigenvalues of A are mutually distinct elements of $\mathbb{K}$, and the eigenvalues of A^* are mutually distinct elements of $\mathbb{K}$.*

Proof. We prove the result for A, and the proof for A^* is essentially the same. By the definition of a Leonard pair (Def. 2.1.1), there is basis of V composed of eigenvectors of A. First, this means the eigenvalues of A are in $\mathbb{K}$. And, in this basis, the matrix representing A is diagonal. Hence, for any repeated eigenvalue, λ, on the diagonal, only one factor of $x - \lambda I$ is needed in the minimal polynomial $\mu(x)$ to zero all of them. Hence the minimal polynomial has no repeated roots.

We now show that the degree of $\mu($ is the dimension of V. Again by Definition 2.1.1, there is a basis of V where A is irreducible tridiagonal. Let B be the matrix representing A in this basis. Since B is irreducible tridiagonal, we will show that B^i has a nonzero ith upper diagonal and all jth upper diagonals for $j > i$ are zero. Hence the set $\{I = B^0, B, B^1, \ldots, B^d\}$ is linearly independent, which will mean the degree of $\mu(x)$ is $d+1$, which will complete our proof.

To prove our claim about the form of B^i, recall that

$$B = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}.$$

We proceed by induction. Clearly the condition on the diagonals holds for $i = 0$ and $i = 1$. Now fix any $i \geq 1$ and suppose B^i satisfies the condition on the diagonals. Let the ith upper diagonal be $(x_0, x_1, \ldots, x_{d-i})$. By assumption $x_k \neq 0$ for $(0 \leq k \leq d-i)$. Let $(y_0, y_1, \ldots, y_{d-i-1})$ be the $(i+1)$th upper diagonal of B^{i+1}. Since $B^{i+1} = B^i B$, we have that $y_k = b_{i+k} x_k \neq 0$. And similarly, the jth upper diagonals of B^{i+1}, for $j > i+1$, are zero, since they are given all in terms of entries in the lth upper diagonals of B^i for $l > i$. □

Given any matrix $A \in \mathcal{A} = \mathrm{Mat}_{d+1}(\mathbb{K})$ with distinct eigenvalues $\{\theta_i\}_{i=0}^{d}$, let I be the identity element of $\mathcal{A}$, and let

$$E_i = \prod_{j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j}. \tag{2.15}$$

Then, as shown in [49], the following properties hold:

(i) $AE_i = \theta_i E_i$ $(0 \leq i \leq d)$.

This can be seen by the fact that $AE_i - \theta_i E_i$ is a multiple of the minimum polynomial $\mu(A) = 0$.

(ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$.

If $i \neq j$, then $E_i E_j$ is again a multiple of the minimum polynomial. If $i = j$, express each vector $v \in V$ in the basis of eigenvectors of A as $v = \sum_{i=0}^{d} s_i v_i$. We can see from (2.15) that for $k \neq i$, $E_i v_k$ has a scalar factor of $\theta_k - \theta_k = 0$, and so we get $E_i v = \left(\prod_{j \neq i} \frac{\theta_i - \theta_j}{\theta_i - \theta_j}\right) s_i v_i = s_i v_i$, hence $E_i E_i v = s_i v_i = E_i v$.

(iii) $\sum_{i=0}^{d} E_i = I$.

Given $v \in V$ we have $\sum_{i=0}^{d} E_i v = \sum_{i=0}^{d} E_i \left(\sum_{j=0}^{d} s_j v_j\right) = \sum_{i=0}^{d} E_i s_i v_i = \sum_{i=0}^{d} s_i v_i = v = Iv$.

(iv) $A = \sum_{i=0}^{d} \theta_i E_i$.

As above, we have $Av = A \sum_{i=0}^{d} s_i v_i = \sum_{i=0}^{d} \theta_i s_i v_i = \sum_{i=0}^{d} E_i s_i v_i = \sum_{i=0}^{d} E_i v = \left(\sum_{i=0}^{d} E_i\right) v$.

The matrix E_i is called the *primitive idempotent* of A *associated with* θ_i. From now on, we will use $\mathcal{A}$ to refer to a $\mathbb{K}$-algebra that is isomorphic to $\mathrm{Mat}_{d+1}(\mathbb{K})$. Now suppose $A, A^* \in \mathcal{A}$ form a Leonard pair. Then $\mathcal{A}$ is usually referred to as the *ambient* algebra of the pair (A, A^*). Let $\langle A \rangle$ and $\langle A^* \rangle$ be the subalgebras of $\mathcal{A}$ generated by A and A^* respectively. Then (i)-(iv) implies that the E_i form a basis of $\langle A \rangle$ as a

$\mathbb{K}$-vector space [49].

We now give the definition of a Leonard system.

Definition 2.4.3. [[48], Definition 1.4] By a *Leonard system* in $\mathcal{A}$ we mean a sequence $\Phi := (A; A^*; E_{i\,i=0}^{\,d}; E_{i\ i=0}^{*d})$ that satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element in $\mathcal{A}$.

(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of A.

(iii) $E_0^*, E_1^*, \ldots, E_d^*$ is an ordering of the primitive idempotents of A^*.

(iv) $E_i A^* E_j = \begin{cases} 0, & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1, \end{cases}$ for $(0 \leq i, j \leq d)$.

(v) $E_i^* A E_j^* = \begin{cases} 0, & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1, \end{cases}$ for $(0 \leq i, j \leq d)$.

We refer to d as the diameter of Φ and say Φ is over $\mathbb{K}$. We call $\mathcal{A}$ the *ambient algebra* of Φ.

The next lemma tells us the connection between Leonard pairs and Leonard systems.

Lemma 2.4.4. *[50, Lem. 1.2] Let $A, A^* \in \mathcal{A}$. Then the pair A, A^* is a Leonard pair in $\mathcal{A}$ if and only if the following hold.*

(i) Both A and A^* are multiplicity-free.

(ii) There exists an ordering of the primitive idempotents of A, $E_0, E_1, \ldots, E_d$, and the primitive idempotents of A^*, $E_0^*, E_1^*, \ldots, E_d^*$, such that

$(A; A^*; \{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is a Leonard system in $\mathcal{A}$.

Proof. The necessity of (i) was proven in Lemma 2.4.2. For (ii), let $v_0, v_1, \ldots, v_d$ be a basis satisfying Definition 2.1.1(ii). Hence for each i, the vector v_i is an eigenvector of A for eigenvalue θ_i. Let E_i be the corresponding primitive idempotent of A. Similarly, let $v_0^*, v_1^*, \ldots, v_d^*$ be a basis satisfying Definition 2.1.1(i), and let $E_0^*, E_1^*, \ldots, E_d^*$ be the corresponding primitive idempotents of A^*. Conditions (i)-(iii) of Definition 2.4.3 are satisfied.

The v_i^* are eigenvectors of A^*. Working in this basis $\{v_0^*, v_1^*, \ldots, v_d^*\}$ gives us that A is irreducible tridiagonal. We know from (2.15) that $E_j^* v_i^* = \delta_{i,j} v_i^*$, and hence

$$(E_i^*)_{j,k} = \begin{cases} 1, & \text{if } j = k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Hence the tridiagonal shape of A gives us (v). Working in the basis of $\{v_0, v_1, \ldots, v_d\}$, the same argument gives us (iv).

For the other direction, let $(A; A^*; E_{i\,i=0}^{\,d}; E_{i\;i=0}^{*d})$ denote a Leonard system in $\mathcal{A}$. For $0 \leq i \leq d$ let v_i denote a nonzero vector in $E_i V$. Then the $\{v_0, v_1, \ldots, v_d\}$ is a basis for V that satisfies Definition 2.1.1(ii). Similarly, for $0 \leq i \leq d$, let v_i^* denote a nonzero vector in $E_i^* V$. Then the sequence $\{v_0^*, v_1^*, \ldots, v_d^*\}$ is a basis for V that satisfies Definition 2.1.1(i). Hence A, A^* is a Leonard pair in $\mathcal{A}$. □

The following lemma is implied by the proof above.

Lemma 2.4.5. *Let A, A^* be a Leonard pair on vector space V. Take a basis of V such that A is irreducible tridiagonal and A^* is diagonal. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ be the eigenvalues of A and A^* respectively. Then the idempotents of A_i^* are given by* *the matrices E_i^*, where the entries satisfy*

$$(E_i^*)_{j,k} = \begin{cases} 1, & \text{if } j = k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.4.6. Let A, A^* and B, B^* denote Leonard pairs on V and V' respectively. We say the pair A, A^* is *isomorphic* (as a Leonard pair) to the pair B, B^* if there exists a $\mathbb{K}$-algebra isomorphism $\sigma : \text{End}(V) \to \text{End}(V')$ such that $\sigma(A) = B$, $\sigma(A^*) = B^*$.

Proofs of the following can be found in [50].

Theorem 2.4.7. *[50, Thm. 2.1] Let Φ be a Leonard system as in Definition 2.4.3. Then the elements*

$$A^r E_0^* A^s \quad (0 \leq r, s \leq d) \tag{2.16}$$

form a basis for the $\mathbb{K}$-vector space $\mathcal{A}$

Corollary 2.4.8. *[50, cor. 2.1] Let A, A^* denote a Leonard pair on V. Then the set $\{A, A^*\}$ generates End(V).*

This last corollary states that the ambient algebra of a Leonard pair on V is all of $\text{End}(V)$.

2.5 Automorphisms and anti-automorphisms

Let V be a vector space over $\mathbb{K}$. We say $\sigma : \text{End}(V) \to \text{End}(V)$ is an *automorphism* on the transformations of V if it is $\mathbb{K}$-linear and, for any two linear transformations X, Y on V, we have $\sigma(XY) = \sigma(X)\sigma(Y)$.

Lemma 2.5.1. *[42, Noether-Skolem, Cor. 9.122], A map $\sigma : End(V) \to End(V)$ is a $\mathbb{K}$-algebra automorphism if and only if there exists an invertible $S \in End(V)$ such that $\sigma(X) = SXS^{-1}$ for all $X \in End(V)$. We say that S represents σ.*

Also note that for any $a \in \mathbb{K}$, aS represents the same automorphism σ as S. Furthermore, given any P that represents the same σ as S, we must have that for all $X \in \mathrm{End}(V)$, $S^{-1}PX = XS^{-1}P$, which implies that $S^{-1}P$ is in the center of $\mathrm{End}(V)$. But this center is $\{\alpha I : \alpha \in \mathbb{K}\}$, so, by uniqueness of inverses in $\mathrm{End}(V)$, $P = aS$ for some $a \in \mathbb{K}$. This proves the following lemma.

Lemma 2.5.2. *Let σ be an automorphism on $End(V)$. Assume that $S \in End(V)$ represents σ. Then $S' \in End(V)$ represents σ if and only if there is a nonzero $a \in \mathbb{K}$, such that $S' = aS$.*

We say $\tau : \mathrm{End}(V) \to \mathrm{End}(V)$ is an *antiautomorphism* on the transformations of V if it is $\mathbb{K}$-linear, and, for any two linear transformations X, Y on V, we have $\tau(XY) = \tau(Y)\tau(X)$.

Theorem 2.5.3. *[50, Thm. 2.2] Let A, A^* denote a Leonard pair in $\mathcal{A}$. Then there exists a unique antiautomorphism $\dagger$ of $\mathcal{A}$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. Moreover, $X^{\dagger\dagger} = X$ for all $X \in \mathcal{A}$.*

In the proof of [50, Thm. 2.2] it is shown that the matrix $K = \mathrm{diag}(k_0, k_1, \ldots, k_d)$ defined in (2.6), represents the antiautomorphism $\dagger$ given in Theorem 2.5.3. In other words, $X^\dagger = K^{-1}X^\top K$.

The following lemma also follows from the Noether-Skolem theorem.

Lemma 2.5.4. *[Noether-Skolem, [42], Cor. 9.122] A map $\tau : End(V) \to End(V)$ is* *a $\mathbb{K}$-algebra antiautomorphism if and only if there exists an invertible $R \in End(V)$ such that $\tau(X) = RX^{\top}R^{-1}$ for all $X \in End(V)$. We say that R represents τ.*

2.6 Orthogonal polynomials

Using a Leonard pair (A, A^*) of diameter d, with A in tridiagonal form and A^* diagonal, we can define a sequence of polynomials from a three-term recurrence.

Definition 2.6.1. With reference to Definition 2.1.1 and (2.4,2.5), let

$$u_{-1}(\lambda), u_0(\lambda), u_1(\lambda), \ldots, u_d(\lambda)$$

be the unique sequence of polynomials in $\mathbb{K}[\lambda]$ satisfying:

$$u_{-1}(\lambda) = 0, \qquad u_0(\lambda) = 1,$$

$$\lambda u_i(\lambda) = b_i u_{i+1}(\lambda) + a_i u_i(\lambda) + c_i u_{i-1}(\lambda) \qquad (0 \leq i \leq d-1). \tag{2.17}$$

This recurrence can be associated with the sequence of coefficients found in the rows of A. In a similar way, we can also define a sequence of polynomials from a three-term recurrence using the columns of A as coefficients. This amounts to a change in normalization of the u_i, following [50, Lem. 3.13].

Definition 2.6.2. With reference to Definition 2.1.1, and (2.4,2.5). let

$$v_{-1}(\lambda), v_0(\lambda), v_1(\lambda), \ldots, v_d(\lambda)$$

be the unique sequence of polynomials in $\mathbb{K}[\lambda]$ satisfying:

$$v_{-1}(\lambda) = 0, \qquad v_0(\lambda) = 1,$$

$$\lambda v_i(\lambda) = c_{i+1} v_{i+1}(\lambda) + a_i v_i(\lambda) + b_{i-1} v_{i-1}(\lambda) \qquad (0 \leq i \leq d-1). \tag{2.18}$$

For each i ($0 \leq i \leq d$), the polynomials $u_i(\lambda)$ and $v_i(\lambda)$ have degree i. By [50, Lem. 3.13], we can relate the two sequences of polynomials to each other as follows:

$$v_i(\lambda) = k_i u_i(\lambda) \qquad (0 \leq i \leq d). \tag{2.19}$$

These polynomials also satisfy the following orthogonality relations, as shown in [50, Thms. 4.6, 4.7]:

$$\sum_{j=0}^{d} k_j^* u_n(\theta_j) u_m(\theta_j) = \frac{\nu}{k_n} \delta_{nm}, \tag{2.20}$$

$$\sum_{j=0}^{d} k_j u_j(\theta_n) u_j(\theta_m) = \frac{\nu}{k_n^*} \delta_{nm}, \tag{2.21}$$

$$\sum_{j=0}^{d} k_j^* v_n(\theta_j) v_m(\theta_j) = k_n \nu \delta_{nm}, \tag{2.22}$$

$$\sum_{j=0}^{d} \frac{v_j(\theta_n) v_j(\theta_m)}{k_j} = \frac{\nu}{k_n^*} \delta_{nm}. \tag{2.23}$$

Definition 2.6.3. With reference to Definition 2.1.1, 2.6.1, and 2.6.2 suppose we are given a Leonard pair A, A^* of diameter d, and the corresponding polynomials u_j and v_j. Let P be the matrix with entries $P_{i,j}$, where

$$P_{i,j} = v_j(\theta_i) = k_j u_j(\theta_i) \tag{2.24}$$

The matrix P is called the *character table* of the Leonard pair (A, A^*).

Note that P^*, u_i^*, v_i^* can be found using the definitions above by starting with the Leonard pair (A^*, A). By [50, Thm. 4.1, 4.2]) we also have, for $0 \leq i, j \leq d$,

$$u_i(\theta_j) = u_j^*(\theta_i^*), \qquad \frac{v_i(\theta_j)}{k_i} = \frac{v_j^*(\theta_i^*)}{k_j^*}. \tag{2.25}$$

From these equations and the orthogonality relations, we see that

$$P^*P = \nu I = \left(\sum_{i=0}^{d} k_i \right) I. \tag{2.26}$$

Hence $P^{-1} = \frac{1}{\nu} P^*$. This and (2.17), (2.18), and (2.19), imply that the columns of $P^* = \nu P^{-1}$ form a basis of right eigenvectors of A and the rows of P form left eigenvectors of A, hence P diagonalizes A and the columns of $P*$ form the basis where A^* is irreducible tridiagonal. Hence we have the following theorem.

Theorem 2.6.4. *[48, Thm. 4.10] Let (A, A^*) be a Leonard pair, let P and P^* be as in Def. 2.6.3, and let $\sharp$ be the map from Def. 2.1.3 applied to the Leonard pair (A^*, A). Then*

$$\begin{aligned} PA &= A^\sharp P, \\ P^*(A^*)^\sharp &= A^* P^*. \end{aligned} \tag{2.27}$$

Recall from Definition 2.1.3 that this means $A^\sharp = \mathrm{diag}(\theta_0, \ldots, \theta_d)$, and similarly $(A^*)^\sharp$ is irreducible tridiagonal. Note that this theorem tells us that the ith row of P is the eigenvector of A with eigenvalue θ_i. Similarly the ith row of $P*$ is an eigenvector of $(A^*)^\sharp$ with eigenvalue θ^*_i.

2.7 P for the Leonard pair in Def. 2.3.1 and self-duality

Recall the Leonard pair in Definition 2.3.1. By [49, Ex. 5.10], we have that

$$u_i(\theta_j) = {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ d+2,\ -d \end{matrix}\ ;1\right] \qquad (0 \le i, j \le d). \tag{2.28}$$

By this and 2.13, the matrix P in 2.3.2 is indeed the character table from Definition 2.6.3 for this Leonard pair. And from

Lemma 2.3.3 implies that $A = \frac{1}{\nu} P A^* P = P^{-1} A^* P$ and $A^* = \frac{1}{\nu} PAP = P^{-1}AP$. Hence the map $\sigma(X) = P^{-1}XP$ is an automorphism of $\mathrm{End}(V)$ that swaps A and A^*. By Lemma 2.4.8, σ is the unique automorphism that performs this swap.

Recalling Definition 2.4.6, the Leonard pair (A, A^*) is isomorphic to (A^*, A).

Definition 2.7.1. [40, p.5] A Leonard pair (A, A^*) is *self-dual* if it is isomorphic to (A^*, A).

Hence the Leonard pair in Definition 2.3.1 is self-dual with duality map σ.

2.8 Two commutative subalgebras of $\mathrm{Mat}_{d+1}(\mathbb{K})$

In this section, we discuss the algebras generated by the matrices A, A^* of a Leonard pair. Let $\sharp$ be the map defined in 2.1.3 for the Leonard pair (A^*, A).

Definition 2.8.1. Let $\mathcal{M}$ denote the subalgebra of $\mathrm{Mat}_{d+1}(\mathbb{K})$ generated by A. Let $\mathcal{M}^*$ denote the subalgebra of $\mathrm{Mat}_{d+1}(\mathbb{K})$ generated by A^*.

We describe a basis for $\mathcal{M}$ and a basis for $\mathcal{M}^*$.

Definition 2.8.2. For $0 \leq i \leq d$ define

$$B_i = v_i(A), \qquad\qquad B_i^* = v_i^*(A^*),$$

where $v_i(\lambda)$ and $v_i^*(\lambda)$ is from (2.6.2).

These matrices are called the *intersection matrices* of (A, A^*).

Lemma 2.8.3. *For $0 \leq i \leq d$ we have*

$$PB_i = B_i^\sharp P, \qquad\qquad P^*(B_i^*)^\sharp = B_i^* P^*.$$

Proof. These follow immediately by Lemma 2.6.4, Definition 2.8.2, and linear algebra. □

Lemma 2.8.3 tells us that, for integers $0 \leq i, j \leq d$, column j of $P^* = \nu P^{-1}$ is an eigenvector of B_i with eigenvalue $v_i(\theta_j)$. A similar statement can be said about $(B_i^*)^\sharp$ and the columns of $P = \nu^{-1}(P^*)^{-1}$ and $v_i^*(\theta_j^*)$. We emphasize one special case. Let

$\mathbb{1}$ denote the vector in $\mathbb{K}^{D+1}$ that has all entries 1.

Lemma 2.8.4. *For each integer $0 \leq i \leq d$, the vector $\mathbb{1}$ is an eigenvector for B_i with eigenvalue k_i.*

Proof. Immediate from Definitions 2.6.2, 2.6.3 applied to Leonard pair (A^*, A) and Lemma 2.8.3. □

Lemma 2.8.5. *The matrices $\{B_i\}_{i=0}^d$ form a basis for $\mathcal{M}$. The matrices $\{B_i^*\}_{i=0}^d$ form a basis for $\mathcal{M}^*$.*

Proof. By Lemma 2.4.2, the matrices A and A^* have $d+1$ distinct eigenvalues. Hence $\mathcal{M}^*$ has dimension $d+1$. By Definition 2.8.2, the matrices $\{B_i^*\}_{i=0}^d$ belong to $\mathcal{M}^*$. By these comments, the matrices $\{B_i^*\}_{i=0}^d$ form a basis for $\mathcal{M}^*$. We have now verified the second assertion. For the first, note that the matrix $B_i^\sharp = \text{diag}(v_i(\theta_0), \ldots, v_i(\theta_d))$. By the same argument for $\mathcal{M}^*$, the $B_i^\sharp$ generate the algebra generated by $A^\sharp$. Since $\sharp$ is a $\mathbb{K}$- algebra isomorphism between $\text{End}(V)$ and $\text{Mat}_{d+1}(\mathbf{K})$ with basis $\mathcal{B}^*$ the result follows. □

We will refer to the algebra $\mathcal{M}$ as the *intersection algebra* of (A, A^*). Next we discuss the entries of the matrices $\{B_i\}_{i=0}^d$. The following definition will be convenient.

Definition 2.8.6. For $0 \leq h, i, j \leq d$ let $p_{i,j}^h$ denote the (h, j)-entry of B_i. In other words,

$$p_{i,j}^h = (B_i)_{h,j}. \tag{2.29}$$

The $p_{i,j}^h$ are called the *intersection parameters* of Leonard pair (A, A^*).

Definition 2.8.7. For $0 \leq h, i, j \leq d$, let $q^h_{i,j}$ denote the (h, j)-entry of $(B^*_i)^\sharp$. In other words,

$$q^h_{i,j} = ((B^*_i)^\sharp)_{h,j}. \tag{2.30}$$

The $q^h_{i,j}$ are called the *Krein parameters* of Leonard pair (A, A^*).

We have a comment about the scalars $p^h_{i,j}$ and $q^h_{i,j}$.

Lemma 2.8.8. *[40, Lem. 4.19] For $0 \leq i, j \leq d$ we have*

$$B_i B_j = \sum_{h=0}^{D} p^h_{i,j} B_h, \qquad B^*_i B^*_j = \sum_{h=0}^{D} q^h_{i,j} B^*_h. \tag{2.31}$$

The scalars $p^h_{i,j}$ can be computed using the following result. This result is from [39]; we include a proof for the sake of completeness.

Proposition 2.8.9. *[39, Lem. 12.12] For $0 \leq h, i, j \leq d$ we have*

$$p^h_{i,j} = \frac{k_i k_j}{\nu} \sum_{t=0}^{D} k^*_t u^*_t(\theta^*_h) u_i(\theta_t) u_j(\theta_t). \tag{2.32}$$

Proof. We invoke Equation (2.29). By (2.26) and Lemma 2.8.3 we have that $B_i = \nu^{-1} P^* B^\sharp_i P$. Recall that the matrix P has entries $P_{i,j} = k_j u_j(\theta_i)$ and P^* has entries $P^*_{i,j} = k^*_j u^*_j(\theta^*_i)$. We also have $B^\sharp_i = v_i(A^\sharp) = k_i u_i(A^\sharp)$ and $A^\sharp = \mathrm{diag}(\theta_0, \theta_1, \ldots, \theta_d)$. Evaluating (2.29) using these comments, we obtain the result. □

The corresponding result for the $q^h_{i,j}$ is similar, and can be seen by considering the Leonard pair (A^*, A).

We end this section with a comment about Proposition 2.8.9.

Lemma 2.8.10. *For* $0 \leq h, i, j \leq d$ *we have*

$$p^h_{i,j} = p^h_{j,i}, \qquad\qquad k_h p^h_{i,j} = k_j p^j_{h,i} = k_i p^i_{j,h}. \tag{2.33}$$

Proof. Immediate from (2.32). □

3 Association Schemes and Leonard Pairs

3.1 Association schemes

A closely related structure to Leonard pairs and an important structure of algebraic combinatorics and coding theory is the concept of an association scheme.

Definition 3.1.1. Let d denote a positive integer. A d-class *association scheme* $\mathscr{X}$ is a pair $(X, \{R_i\}_{i=0}^{d})$, where X is a finite set and R_i is a relation on X for each i, with the following properties.

(i) $R_0 = \{(x, x) \mid x \in X\}$.

(ii) $\bigcup_{i=0}^{d} R_i = X \times X$ and $R_i \cap R_j = \emptyset$ for $i \neq j$.

(iii) For all $i \in \{0, 1, \ldots, d\}$, ${}^tR_i = \{(y, x) \mid (x, y) \in R_i\} = R_j$ for some $j \in \{0, 1, \ldots, d\}$.

(iv) For any $i, j, k \in \{0, 1, \ldots, d\}$ and $(x, y) \in R_k$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant $p^k_{i,j}$ independent of which $(x, y) \in R_k$ was chosen.

If ${}^tR_i = R_i$ for all $i \in \{0, 1, \ldots, d\}$, then $\mathscr{X}$ is said to be *symmetric*. If $p^k_{i,j} = p^k_{j,i}$ for all $i, j, k \in \{0, 1, \ldots, d\}$, then $\mathscr{X}$ is said to be *commutative*.

We may sometimes call an association scheme simply a scheme for brevity. Some sources use the term association scheme to mean a symmetric association scheme, and unless stated otherwise, our schemes will be assumed to be symmetric schemes. We also note that symmetric schemes are commutative.

Proposition 3.1.2. *A symmetric scheme is commutative*

Proof. In a symmetric scheme, for any h, $(x, y) \in R_h$ if and only if $(y, x) \in R_h$.

In particular, for any fixed i, j, h and $(x, y) \in R_h$, we have that z is such that $(x, z) \in R_i$ and $(z, y) \in R_j$ if and only if $(z, x) \in R_i$ and $(y, z) \in R_j$. Since $(y, x) \in R_h$ we have $p^h_{i,j} = p^h_{j,i}$. □

If we associate an adjacency matrix A_i with each of the relations R_i, so that

$$(A_i)_{uv} = \begin{cases} 1 & \text{if } (u, v) \in R_i, \\ 0 & \text{otherwise,} \end{cases} \tag{3.34}$$

then we can equivalently define an association scheme by the following.

Definition 3.1.3. Let d denote a positive integer. A d-class *association scheme* $\mathscr{X}$ is a pair $(X, \{A_i\}_{i=0}^d)$, with X a finite set, $|X| = n$, and for each i, A_i is an $n \times n$ adjacency matrix on X with the following properties.

$$A_0 = I \tag{3.35}$$

$$\sum_{i=0}^{d} A_i = J_n \tag{3.36}$$

$$A_i^\top = A_j \quad \text{for some } j \in \{0, 1, \ldots, d\} \tag{3.37}$$

$$A_i A_j = \sum_{h=0}^{d} p^h_{i,j} A_h \quad \text{for all } i, j. \tag{3.38}$$

In the commutative case we have, for all i,j, the extra condition,

$$A_i A_j = A_j A_i,$$

hence $p^h_{i,j} = p^h_{j,i}$. In the symmetric case we have, for all i, the extra condition,

$$A_i^\top = A_i.$$

Definition 3.1.4. With reference to Definition 3.1.3. The subalgebra $\mathfrak{A}$ of $\mathrm{Mat}_n(\mathbb{K})$, which is generated by $\langle A_0, A_1, \ldots, A_d\rangle$ for a commutative association scheme $\mathscr{X} = (X, \{A_i\}_{i=0}^d)$, is called the *Bose-Mesner* algebra of $\mathscr{X}$. The algebra $\mathfrak{A}$ is $d+1$ dimensional. The $p^h_{i,j}$ are called the *structure constants* of $\mathfrak{A}$.

We now define an important family of association schemes.

Definition 3.1.5. Let $\mathscr{X} = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. Then $\mathscr{X}$ is called a *P-polynomial association scheme*, if for some ordering of the $R_0, \ldots, R_d$ and for each $i (0 \leq i \leq d)$ there is a polynomial $v_i(x)$ of degree i, in indeterminate x, such that the adjacency matrix $A_i = v_i(A_1)$.

In particular, this means that the Bose-Mesner algebra, and hence the A_i, are generated by A_1.

3.2 Distance-regular graphs, P-polynomial association schemes

An association scheme is P-polynomial exactly when the graph with adjacency matrix A_1 has a property known as distance-regularity (see [1, Prop. 1.1]), which we now define.

Definition 3.2.1. A graph X of diameter d is called *distance regular* if it is connected and given any integers $0 \leq h, i, j \leq d$, there are constant values $p^h_{i,j}$, such that for any vertices of u, v of distance $d(u, v) = h$, there are exactly $p^h_{i,j}$ vertices w with

$d(u,w) = i$ and $d(w,v) = j$.

In this case, we can define an association scheme $\mathscr{X} = (X, \{A_i\}_{i=0}^d)$, where A_i is the distance-i matrix, and this scheme will have the P-polynomial property.

Given any vertex u and integer i, we will denote by $X_i(u)$ the set of vertices at distance i from u. Hence for any vertex u, we can see that the collection $\{X_i(u) : 0 \leq i \leq d\}$ partitions the vertices of X. We call this the *distance partition* from vertex u. This notation allows us to formulate the following alternative to Definition (3.2.1) [1, p. 192].

Definition 3.2.2. A connected graph X of diameter d is *distance regular* if there are parameters

$$\left\{\begin{matrix} - & c_1 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & \cdots & b_{d-1} & - \end{matrix}\right\}, \tag{3.39}$$

with $c_{i+1}, b_i \neq 0$ for $(0 \leq i \leq d-1)$, such that, given any vertex u, the corresponding distance partition $\{X_i(u) : 0 \leq i \leq d\}$ satisfies the following property. For any i $(0 \leq i \leq d)$, any vertex x in $X_i(u)$ has exactly c_i neighbors in $X_{i-1}(u)$, a_i neighbors in $X_i(u)$, and b_i neighbors in $X_{i+1}(u)$. The parameters (3.39) are called the *distance parameters.*

Figure 3.2 depicts the distance partition with the $X_i(u)$ circled by solid lines. The dashed bubbles comprise the neighborhood of a vertex v in $X_i(u)$. The cardinalities of each dashed bubble in $X_{i-1}(u)$, $X_i(u)$, and $X_{i+1}(u)$ are shown as c_i, a_i, and b_i respectively.

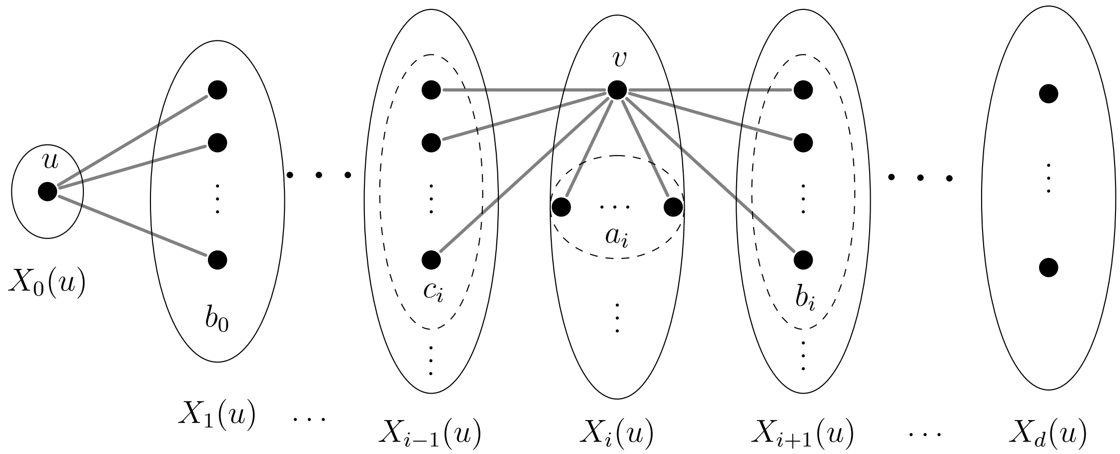

Figure 3.2: The distance partition of a distance-regular graph.

We saw an example of a distance regular graph in Fig. 2.1, along with its associated distance partition from a vertex u.

Because we have limited our discussion to simple graphs (hence no loops), it follows that

$$a_0 = 0, \quad \text{and} \quad c_1 = 1. \tag{3.40}$$

The regularity around every vertex implies that the graph is regular with valency b_0, and hence we have

$$b_0 = c_i + a_i + b_i, \quad \text{for } 0 \leq i \leq d. \tag{3.41}$$

This also means that only two rows of the distance parameters are needed, so the distance parameters are sometimes given as

$$(b_0, b_1, \cdots, b_{d-1}; c_1, \cdots, c_{d-1}, c_d). \tag{3.42}$$

We often list the parameters in a tridiagonal matrix, called the *intersection* matrix, which is the following:

$$B_1 = \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1} & a_{d-1} & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}. \tag{3.43}$$

Despite its tridiagonal appearance, the B_1 from a distance regular graph does not always come from a Leonard pair (A, A^*), with B_1 being the matrix for A in the basis where it is irreducible tridiagonal. For example, the graph of the vertices of a dodecahedron is distance-regular, but the tridiagonal matrix does not come from any Leonard pair [34]. In essence, we can define a polynomial using the a_i, b_i, c_i in a manner analogous to Definition 2.6.2, but that does not guarantee us a way to define the dual-polynomials v_i^*, and hence the θ_i^*. For the intersection matrix of a distance regular graph to form part of a Leonard pair, we need our graph to have another property, called the Q-polynomial property, that we will discuss in the next section.

3.3 Primitive idempotents and Q-polynomial association schemes

Recall that the Bose-Mesner algebra $\mathfrak{A}$ of commutative association scheme $\mathscr{X} = (X, \{A_i\}_{i=0}^d)$ is generated by $\langle A_0, A_1, \ldots, A_d \rangle$. Because this algebra is commutative and closed under transposes, the A_i are normal matrices. Therefore, by the spectral theorem, the A_i are unitarily diagonalizable. Since they commute, each A_i preserves the eigenspaces of all other A_j. So it follows that these matrices are all simultaneously diagonalizable by a unitary matrix U [25, Thm. 1.3.21]. As a result, we can decompose

$V = \mathbb{K}^n$, where $n = |X|$, into the direct sum of common eigenspaces

$$V = V_0 + \cdots + V_r.$$

By (3.36) we have that the all ones matrix J corresponds to the common eigenspace spanned by $(1, 1, \ldots, 1)^\top = \mathbb{1}$. We relabel to let V_0 correspond to this eigenspace. Let E_i be the projection matrices from $V \to V_i$. As a consequence $r = d$ and the $\{E_0, \ldots, E_d\}$ form another basis of $\mathfrak{A}$ with the following properties:

$$nE_0 = J_n, \tag{3.44}$$

$$E_0 + E_1 + \cdots + E_d = I_n, \tag{3.45}$$

$$E_i E_j = \delta_{i,j} E_i. \tag{3.46}$$

If the association scheme is symmetric, so that $A_i^\top = A_i$ $(0 \leq i \leq d)$, then we have the additional condition:

$$E_i^\top = E_i. \tag{3.47}$$

The $\{E_0, E_1, \ldots, E_d\}$ are called the *primitive idempotents* of $\mathscr{X}$, and E_0 is the *trivial* idempotent of $\mathscr{X}$.

The binary operation $\circ$, the entry-wise product for matrices M, N of the same dimension (also known as the Hadamard or Schur product), produces the matrix $M \circ N$ having entries:

$$(M \circ N)_{i,j} = M_{i,j} N_{i,j}. \tag{3.48}$$

From the definition of association scheme, we have that $A_i \circ A_j = \delta_{i,j} A_i$, hence

$\mathfrak{A}$ is closed under $\circ$. Since $\{E_i\}_{i=0}^d$ is another basis of $\mathfrak{A}$, there are also parameters $q_{i,j}^h \in \mathbb{K}$ such that:

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q_{i,j}^h E_h \quad \text{for all } i, j. \tag{3.49}$$

The $q_{i,j}^h$ are known as the *Krein parameters* of $\mathscr{X}$. For commutative association schemes, the Krein parameters are known to be non-negative real numbers [1, Thm. 3.8].

Definition 3.3.1. Let $\mathscr{X} = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. Then $\mathscr{X}$ is called *Q-polynomial* if, for some ordering of the primitive idempotents $E_0, \dots, E_d$ and for each $i (0 \leq i \leq d)$ there is a polynomial $v_i^*(x)$ of degree i, in indeterminate x, such that the adjacency matrix $E_i = v_i^*(E_1)$ under the Hadamard product.

Or equivalently, we can define the Q-polynomial property as in [1, p. 193].

Definition 3.3.2. A symmetric association scheme is *Q-polynomial* if for some ordering of the primitive idempotents, the matrix B_1^* with entries $(B_1^*)_{h,j} = q_{1,j}^h$, where $q_{i,j}^h$ are the Krein parameters, is irreducible tridiagonal. In particular, the matrix $(B_1)^*$ has the form

$$B_1^* = \begin{pmatrix} a_0^* & b_0^* & & & \mathbf{0} \\ c_1^* & a_1^* & b_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ \mathbf{0} & & & c_d^* & a_d^* \end{pmatrix}, \tag{3.50}$$

with nonzero super- and sub-diagonals (see Lemma A.1.3).

From (3.38) we see that, for a distance regular graph, the distance matrices $\{A_i\}_{i=0}^d$ satisfy the same recurrence that the intersection matrices $\{B_i\}_{i=0}^d$ satisfy from a Leonard pair, as given in Definitions 2.8.2, 2.6.2. In other words, the polynomials in Definition 3.1.5 are defined by taking the a_i, b_i, c_i from the intersection matrix (3.43), and defining $v_i(\lambda)$ in an identical way to Definition 2.6.2:

$$v_{-1}(\lambda) = 0, \qquad v_0(\lambda) = 1,$$

$$\lambda v_i(\lambda) = c_{i+1}v_{i+1}(\lambda) + a_i v_i(\lambda) + b_{i-1}v_{i-1}(\lambda) \qquad (0 \leq i \leq d-1). \tag{3.51}$$

And similar to what we saw with Leonard pairs, the matrices $(B_i)_{h,j} = p^h_{i,j}$, whose entries are given by the intersection numbers $p^h_{i,j}$ of a P-polynomial association scheme also satisfy the same recurrence as the A_i. This follows from the equivalence of Def. 3.1.5 and Def. 3.2.2 ([1, Prop. 1.1]) and linear algebra. In particular, the matrices B_i can be expressed as $B_i = v_i(B_1)$, where the v_i are the same polynomials that generate A_i in terms of A_1, but here they are evaluated at B_1. Hence the algebra generated by $\{A_i\}_{i=0}^d$ and the algebra generated by $\{B_i\}_{i=0}^d$ are isomorphic as $\mathbb{K}$-algebras.

In a similar manner, when an association scheme is Q-polynomial, then, under some ordering of the idempotents, the entries of matrix B_1^* can be used to define the polynomials v_i^* such that $v_i^*(E_1) = E_i$ under the Hadamard product. Furthermore, a basis of $\mathrm{Mat}_{d+1}(\mathbb{K})$ exists where B_1 is irreducible tridiagonal and B_1^* is diagonal, and another basis where B_1^* is irreducible tridiagonal and B_1 is diagonal [51, Lem. 16.1]. Hence (B_1, B_1^*) is a Leonard pair. So under this ordering of the idempotents, we can similarly define matrices $(B_i^*)_{h,j} = q^h_{i,j}$ whose entries are the Krein parameters of a P- and Q-polynomial association scheme, and these matrices will satisfy $B_i^* = v^*(B_1^*)$

under normal matrix multiplication.

When the matrices A, A^* of a Leonard pair come from a symmetric association scheme in this way, then the $p^h_{i,j}$ and $q^h_{i,j}$ are nonnegative. However, this is not necessarily the case for Leonard pairs in general. Nonetheless nonnegativity is a very useful property, with applications to open problems [50, prob. 11.3, 11.5] and [1, p. 205], and we will use one such example to prove the conjecture in [32, Conj. 2] in Chapter 4.

3.4 Intersection matrices of a Leonard pair

Since the intersection matrices $\{B_i\}_{i=0}^d$ of a Q-polynomial distance regular graph act (multiplicatively) like the distance matrices $\{A_i\}_{i=0}^d$, we can think of the $\{B_i\}_{i=0}^d$ as algebraic proxies for the distance matrices. Not all Leonard pairs come from an association scheme; nonetheless, the $\{B_i\}_{i=0}^d$ from a Leonard pair A, A^* can be thought of as a generalization of the distance matrices of a Q- polynomial distance regular graph. In this spirit, we occasionally refer to the $\{B_i\}_{i=0}^d$ as *pseudo-distance matrices*, and we refer to the algebra generated by B_1, which by Lemma 2.8.5 has basis $\{B_i\}_{i=0}^d$, as the *pseudo-distance algebra* of the Leonard pair A, A^*.

For a given Leonard pair, it may be unknown whether there is a combinatorial interpretation that shows the intersection or Krein parameters are nonnegative. We would like to study this nonnegativity regardless. One method will be discussed in a later section where we prove the Kresch-Tamvakis conjecture.

Another more speculative method of doing this could be to look for other combinatorial connections. If an association scheme cannot be found, perhaps a weaker connection to a combinatorial structure would still allow us to prove nonnegativity.

Though we have yet to find such an example, we show an algebraic connection between a certain family of Leonard pairs with rational intersection numbers and Krein parameters and a family of strongly regular graphs. We hope that our results may perhaps give clues on a possible direction to search for other combinatorial connections. In any case, we are of the opinion that this connection is of interest in its own right.

Regardless of their ability to prove the nonnegativity of these parameters, both methods allow us to prove some hypergeometric identities related to these parameters, at least one of which (4.5) appears to have been unknown prior to our recent paper [6]. Next we show an example to illustrate this second idea.

3.5 Subalgebra example from Def. 2.3.1

In this section, we return to our consideration of the Leonard pair of diameter $d = 3$ that was given in Definition 2.3.1.

We will define a subalgebra $\mathcal{B}$ of the algebra $\mathcal{M}$. Recall that $\mathcal{M}$ is generated by the pseudo-distance matrices $\{B_i\}_{i=0}^{3}$ for this Leonard pair. So in the case of this specific Leonard pair, $\mathcal{M}$ is generated by the following matrices:

$$B_0 = I, \quad B_1 = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & \frac{2}{5} & \frac{8}{5} & 0 \\ 0 & \frac{24}{25} & \frac{6}{5} & \frac{21}{25} \\ 0 & 0 & \frac{3}{5} & \frac{12}{5} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & \frac{8}{5} & 2 & \frac{7}{5} \\ 1 & \frac{6}{5} & 0 & \frac{14}{5} \\ 0 & \frac{3}{5} & 2 & \frac{12}{5} \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 7 \\ 0 & 0 & \frac{7}{5} & \frac{28}{5} \\ 0 & \frac{21}{25} & \frac{14}{5} & \frac{84}{25} \\ 1 & \frac{12}{5} & \frac{12}{5} & \frac{6}{5} \end{pmatrix}. \tag{3.52}$$

Note that, in general, the character table matrix P enjoys the property of being a $(d+1) \times (d+1)$ matrix, whose entries in the ith column are the the right eigenvalues of matrix B_i. For our specific Leonard pair, the matrix P is given as follows:

$$P = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 1 & \frac{11}{5} & 1 & -\frac{21}{5} \\ 1 & \frac{3}{5} & -3 & \frac{7}{5} \\ 1 & -\frac{9}{5} & 1 & -\frac{1}{5} \end{pmatrix}. \tag{3.53}$$

Let us define a subalgebra $\mathcal{B}$ of the algebra $\mathcal{M}$ to be the subalgebra generated by

$$\tilde{A}_0 = I, \quad \tilde{A}_1 = B_2, \quad \tilde{A}_2 = B_1 + B_3.$$

We will call a subalgebra of this type, where the basis consists of sums of disjoint sets of basis elements of the parent algebra a *fusion algebra* or simply a *fusion*. (We note that, in this case, we could just as easily have chosen to study the fusion algebra generated by $\tilde{A}_1 = B_1 + B_3$ and $\tilde{A}_2 = B_2$. But, as we will see later, this alternate choice would simply yield the complement structure.)

When defining a fusion algebra, we can, for each i, let S_i denote the set of indices such that $\tilde{A}_i = \sum_{j \in S_i} B_j$. Since the B_i are mutually diagonalizable, we can keep the eigenvectors in the same order and list the eigenvalues of $\tilde{A}_i$ in column i of a matrix by summing the corresponding columns of P for all indices in S_i. Viewing a character table as a matrix whose ith column contains the eigenvalues of the ith basis matrix, this gives us something resembling a character table for the fusion subalgebra. For our example, this matrix (which we denote $\hat{P}$) is given as follows:

$$\hat{P} = \begin{pmatrix} 1 & 5 & 10 \\ 1 & 1 & -2 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix}. \tag{3.54}$$

Note the repeated row formed in $\hat{P}$. From direct computation, one can check that $\mathcal{B}$ has structure constants, $\tilde{p}_{i,j}^{k}$, such that $\tilde{A}_i\tilde{A}_j = \sum_{h=0}^{2} \tilde{p}_{i,j}^{h}\tilde{A}_h$. These values $\tilde{p}_{i,j}^{h}$ can be given as the entries in the following matrices $(\tilde{B}_i)_{h,j} = \tilde{p}_{i,j}^{h}$, where:

$$\tilde{B}_0 = I, \quad \tilde{B}_1 = \begin{pmatrix} 0 & 5 & 0 \\ 1 & 0 & 4 \\ 0 & 2 & 3 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 10 \\ 0 & 4 & 6 \\ 1 & 3 & 6 \end{pmatrix}. \tag{3.55}$$

These matrices generate a matrix algebra that has the same structure constants as $\mathcal{B}$ and has the matrix of eigenvectors:

$$\tilde{P} = \begin{pmatrix} 1 & 5 & 10 \\ 1 & 1 & -2 \\ 1 & -3 & 2 \end{pmatrix}. \tag{3.56}$$

In other words, the columns (rows) are the right (left) eigenvectors of the $\tilde{B}_i$. Also the (i, j) entry is the eigenvalue of B_j (B_i) of the ith column (jth row) vector, of $\tilde{P}$. We can see $\tilde{P}$ matches the matrix $\hat{P}$, but with the redundant row removed.

An interesting observation here is that these are exactly the intersection matrices and character table of a certain distance regular graph of diameter 2 (also called a connected *strongly regular graph*). Strongly regular graphs are accompanied by parameters (v, k, λ, μ) where:

- v denotes the number of vertices,
- k is the valency,
- every pair of adjacent vertices has λ common neighbors,
- every pair of nonadjacent vertices has μ common neighbors.

The matrices in (3.55), (3.56) are the intersection matrices and character table of a strongly regular graph with parameters $(v, k, \lambda, \mu) = (16, 5, 0, 2)$. It is known that there is only one strongly regular graph with these parameters; this graph is called the Clebsch graph [4, 19]. The graph is shown in Figure 3.3

One way to define the Clebsch graph is as the graph whose vertices are the even size subsets of [5], and where two vertices are adjacent whenever their symmetric difference has cardinality 4.

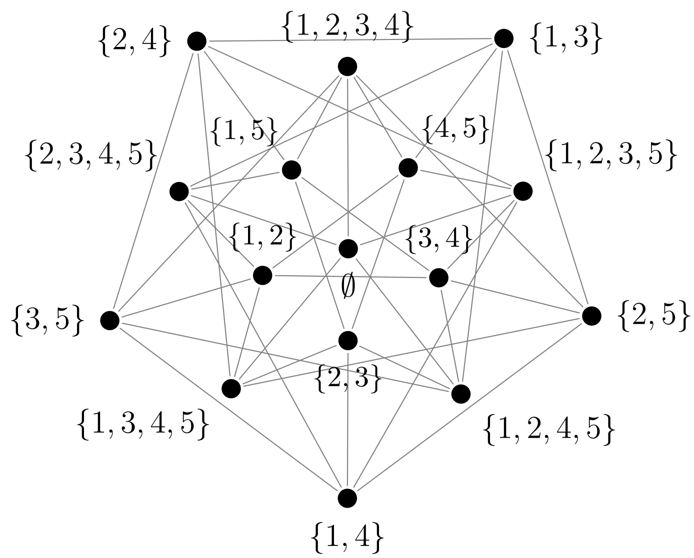

Figure 3.3: Clebsch graph

The unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (16, 5, 0, 2)$.

Deleting any vertex and its neighborhood results in a Petersen graph.

If we instead consider the fusion algebra in $\mathcal{M}$ with basis $\tilde{A}_1 = B_1 + B_3$ and $\tilde{A}_2 = B_2$, we get intersection matrices and character table that correspond to the complement of the Clebsch graph. This graph is also strongly regular, with parameters (16,10,6,6).

Later in Section 5.6, we will consider a more general version of this construction.

4 Proof of the Kresch-Tamvakis Conjecture

In this chapter, we present an application of the Leonard pairs and pseudo-distance matrices introduced above. In particular, we present the proof of a conjecture from special functions theory that has been open for 22 years. The matter was recently settled by Caughman and the present author in [6] and we include the details of our proof here.

Before we do so, we mention some background about the problem. This conjecture has its origins in algebraic geometry. To prove the so-called arithmetic standard conjectures for the Grassmanian $G(2, N)$, Kresch and Tamvakis (in [32]) proved a bound on certain ${}_4F_3$ hypergeometric series. Along the way, they conjectured that a stronger bound than the one they needed appeared to hold. Their conjecture can be stated as the following theorem.

Theorem 4.0.1. *[32, Conjecture 2] For any positive integer D and any integers i, j $(0 \leq i, j \leq D)$, the absolute value of the following hypergeometric series is at most 1:*

$$ {}_4F_3\left[\begin{array}{c} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{array} ;1\right]. \tag{4.57}$$

Note 4.0.2. Theorem 4.0.1 is taken from [32, Conjecture 2] with

$$n = i, \qquad s = j, \qquad T = D + 1.$$

Notice that (4.57) is the same expression as (2.28) (and hence the ${}_4F_3$ component in (2.12)), with d relabeled by D. For this chapter, we will use variables a-g often, so for convenience, we will relabel the diameter of a Leonard pair from d to D.

Next we discuss some of the evidence for Conjecture 4.0.1 that had previously been offered by Kresch, Tamvakis, and others. In [32, Proposition 2], Kresch and Tamvakis proved that the absolute value of (4.57) is at most 1, provided that $i \leq 3$ or $i = D$. In [26, p. 863], Ismail and Simeonov proved that the absolute value of (4.57) is at most 1, provided that $i = D - 1$ and $D \geq 6$. They also gave asymptotic estimates to further support the conjecture. In [36], Mishev obtained several relations satisfied by the ${}_4F_3$ hypergeometric series in question.

Our proof of Theorem 4.0.1 will not rely on any of the partial results mentioned above.

In this chapter, we will consider all of our vector spaces to be over $\mathbb{R}$

4.1 Outline of proof

To prove Theorem 4.0.1 we use the following approach. For $0 \leq i \leq D$ we take the matrices $B_i \in \mathrm{Mat}_{D+1}(\mathbb{R})$ from Definition 2.8.2 applied to the Leonard pair from Def. 2.3.1. Using the Biedenharn-Elliott identity [2, p. 356], we show that the entries of B_i are nonnegative. Using the theory of Leonard pairs [39, 40, 48, 49, 50], we saw by Thm. 2.6.4 and 2.28 that the eigenvalues of B_i are $2i + 1$ times

$$
{}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix};1\right] \qquad (0 \leq j \leq D).
$$

We also showed in Lem. 2.8.4 that $\mathbb{1}$, the all 1's vector in $\mathbb{R}^{D+1}$, is an eigenvector for B_i with eigenvalue $2i + 1$. Applying the Perron-Frobenius theorem [25, p. 529], we show that the eigenvalues of B_i have absolute value at most $2i + 1$. Using these results, we obtain the proof of Theorem 4.0.1.

4.2 The $p^h_{i,j}$ in the self-dual case

As mentioned in Section 2.7 the Leonard pairs in Def. 2.3.1 are self-dual.

In the self-dual case (2.32) becomes

$$p^h_{i,j} = \frac{k_i k_j}{\nu} \sum_{t=0}^{D} k_t u_t(\theta_h) u_t(\theta_i) u_t(\theta_j). \tag{4.58}$$

We will take advantage of this formula when proving that, for the Leonard pairs in Def. 2.3.1, the $p^h_{i,j}$ are nonnegative.

4.3 The nonnegativity of the $p^h_{i,j}$

Our goal for this section is to show that $p^h_{i,j} \geq 0$ for $0 \leq h, i, j \leq D$. To obtain this inequality, we use the Biedenharn-Elliott identity [2, p. 356].

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Define $\frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$.

Definition 4.3.1. Given $a, b, c \in \frac{1}{2}\mathbb{N}$, we say that the triple (a, b, c) is *admissible* whenever $a + b + c \in \mathbb{N}$ and

$$a \leq b + c, \qquad b \leq c + a, \qquad c \leq a + b. \tag{4.59}$$

Definition 4.3.2. Referring to Definition 4.3.1, assume that (a, b, c) is admissible. Define

$$\Delta(a, b, c) = \left(\frac{(a + b - c)!(b + c - a)!(c + a - b)!}{(a + b + c + 1)!} \right)^{\frac{1}{2}}. \tag{4.60}$$

Next, we recall the Racah coefficients.

Definition 4.3.3. ([2, Eq. 5.11.4] and [35, p. 1063]) For $a, b, c, d, e, f \in \frac{1}{2}\mathbb{N}$, we define a real number $W(a, b, c, d; e, f)$ as follows.

First assume that each of (a, b, e), (c, d, e), (a, c, f), (b, d, f) is admissible. Then

$$W(a,b,c,d;e,f) = \frac{\Delta(a,b,e)\Delta(c,d,e)\Delta(a,c,f)\Delta(b,d,f)(\beta_1+1)!(-1)^{\beta_1-(a+b+c+d)}}{(\beta_2-\beta_1)!(\beta_3-\beta_1)!(\beta_1-\alpha_1)!(\beta_1-\alpha_2)!(\beta_1-\alpha_3)!(\beta_1-\alpha_4)!} \times {}_4F_3\left[\begin{matrix}\alpha_1-\beta_1,\ \alpha_2-\beta_1,\ \alpha_3-\beta_1,\ \alpha_4-\beta_1 \\ -\beta_1-1,\ \beta_2-\beta_1+1,\ \beta_3-\beta_1+1\end{matrix};1\right], \tag{4.61}$$

where

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \text{any permutation of } (a+b+e,\ c+d+e,\ a+c+f,\ b+d+f),$$

and where

$$\beta_1 = \min(a+b+c+d,\ a+d+e+f,\ b+c+e+f),$$

and β_2, β_3 are the other two values in the triple $(a+b+c+d, a+d+e+f, b+c+e+f)$ in either order.

Next assume that (a, b, e), (c, d, e), (a, c, f), (b, d, f), are not all admissible. Then

$$W(a,b,c,d;e,f) = 0. \tag{4.62}$$

We call $W(a, b, c, d; e, f)$ the *Racah coefficient* associated with a, b, c, d, e, f.

Let $0 \le h, i, j \le D$. In order to show that $p^h_{i,j} \ge 0$, we will show that

$$p^h_{i,j} = (2i+1)(2j+1)(D+1)\Big(W\left(\tfrac{D}{2}, \tfrac{D}{2}, i, h; j, \tfrac{D}{2}\right)\Big)^2.$$

We will use the Biedenharn-Elliott identity.

Proposition 4.3.4. (Biedenharn-Elliott [2, p. 356]) *Let $a, a', b, b', c, c', e, f, g \in \frac{1}{2}\mathbb{N}$. Then*

$$\begin{aligned} \sum_{d \in \frac{1}{2}\mathbb{N}} (-1)^{c+c'-d}(2d+1)W(b, b', c, c'; d, e)W(a, a', c, c'; d, f)W(a, a', b, b'; d, g) \\ = (-1)^{e+f-g}W(a, b, f, e; g, c)W(a', b', f, e; g, c'). \end{aligned} \tag{4.63}$$

In order to evaluate the Racah coefficients in the Biedenharn-Elliott identity, we will use the following transformation formula of Whipple.

Proposition 4.3.5. (Whipple [18, p. 49]) *For integers $p, q, a_1, a_2, r, b_1, b_2$ we have*

$$\begin{aligned} {}_4F_3\left[\begin{matrix} -p,\ q,\ a_1,\ a_2 \\ r,\ b_1,\ b_2 \end{matrix}\ ;1\right] = & \frac{(b_1-q)_p(b_2-q)_p}{(b_1)_p(b_2)_p} \\ & \times {}_4F_3\left[\begin{matrix} -p,\ q,\ r-a_1,\ r-a_2 \\ r,\ 1+q-b_1-p,\ 1+q-b_2-p \end{matrix}\ ;1\right], \end{aligned} \tag{4.64}$$

provided that $p \ge 0$ and $q + a_1 + a_2 + 1 = r + b_1 + b_2 + p$.

We are interested in the following Racah coefficient. For $0 \le i, j \le D$ consider

$$W\left(\tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}; i, j\right).$$

Evaluating this Racah coefficient using Definition 4.3.3 we get a scalar multiple of a certain ${}_4F_3$ hypergeometric series. Applying several Whipple transformations to this hypergeometric series, we get the following result as we will see.

Proposition 4.3.6. *For integers $0 \le i, j \le D$ we have*

$$W\left(\tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}, \tfrac{D}{2}; i, j\right) = \frac{(-1)^{i+j-D}}{D+1} {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ; 1\right]. \tag{4.65}$$

Proof. To evaluate $W\left(\frac{D}{2}, \frac{D}{2}, \frac{D}{2}, \frac{D}{2}; i, j\right)$, we will consider two cases: $i + j \le D$ and $i + j > D$.

Case $i + j \le D$. In this case, from (4.61) we get $\beta_1 = D + i + j$, $\beta_2 = 2D$, $\beta_3 = D + i + j$, $\alpha_1 = \alpha_2 = D + i$, $\alpha_3 = \alpha_4 = D + j$. The hypergeometric term in (4.61), after rearranging the upper indices, becomes

$${}_4F_3\left[\begin{matrix} -i,\ -i,\ -j,\ -j \\ -D-i-j-1,\ D-i-j+1,\ 1 \end{matrix} ; 1\right]. \tag{4.66}$$

The coefficient in (4.61) is

$$\begin{aligned} &\frac{\left(\Delta\left(\frac{D}{2}, \frac{D}{2}, i\right)\right)^2 \left(\Delta\left(\frac{D}{2}, \frac{D}{2}, j\right)\right)^2 (D+i+j+1)!(-1)^{i+j-D}}{(D-i-j)!(j!)^2(i!)^2} \\ &\quad = \frac{(D-i)!(i!)^2(D-j)!(j!)^2(D+i+j+1)!(-1)^{i+j-D}}{(D+i+1)!(D+j+1)!(D-i-j)!(j!)^2(i!)^2}. \end{aligned} \tag{4.67}$$

The expression (4.67) is equal to

$$\frac{(D-i)!(D-j)!(D+i+j+1)!(-1)^{i+j-D}}{(D+i+1)!(D+j+1)!(D-i-j)!}. \tag{4.68}$$

Performing a Whipple transformation (4.64) with the substitutions $-p = -i$, $q = -j$, $a_1 = -i$, $a_2 = -j$, $r = 1$, $b_1 = -D - i - j - 1$, $b_2 = D - i - j + 1$, the hypergeometric component in (4.66), after rearranging lower indices, becomes

$$ {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ; 1\right]. \tag{4.69} $$

The coefficient contribution from the Whipple transformation is

$$ \begin{aligned} &\frac{(-D-i-1)_i(D-i+1)_i}{(-D-i-j-1)_i(D-i-j+1)_i} \\ &\quad = \frac{(-1)^i(D+i+1)!}{(D+1)!}\frac{D!}{(D-i)!}\frac{(D+j+1)!}{(-1)^i(D+i+j+1)!}\frac{(D-i-j)!}{(D-j)!}. \end{aligned} \tag{4.70} $$

We see that coefficients (4.68) and (4.70) multiply to $\frac{(-1)^{i+j-D}}{D+1}$, as desired.

Case $i+j > D$. In this case, from (4.61) we get $\beta_1 = 2D$, $\beta_2 = D+i+j$, $\beta_3 = D+i+j$, $\alpha_1 = \alpha_2 = D+i$, $\alpha_3 = \alpha_4 = D+j$. The hypergeometric term in (4.61) becomes

$$ {}_4F_3\left[\begin{matrix} i-D,\ i-D,\ j-D,\ j-D \\ -2D-1,\ i+j-D+1,\ i+j-D+1 \end{matrix} ; 1\right]. \tag{4.71} $$

The coefficient in (4.61) is

$$ \begin{aligned} &\frac{\left(\Delta\left(\frac{D}{2},\frac{D}{2},i\right)\right)^2\left(\Delta\left(\frac{D}{2},\frac{D}{2},j\right)\right)^2(2D+1)!}{\big((i+j-D)!\big)^2\big((D-i)!\big)^2\big((D-j)!\big)^2} \\ &\quad = \frac{(D-i)!(i!)^2(D-j)!(j!)^2(2D+1)!}{(D+i+1)!(D+j+1)!\big((i+j-D)!(D-i)!(D-j)!\big)^2}. \end{aligned} \tag{4.72} $$

The expression (4.72) is equal to

$$C_0 = \frac{(i!)^2 (j!)^2 (2D+1)!}{(D+i+1)!(D+j+1)!\big((i+j-D)!\big)^2 (D-i)!(D-j)!}. \tag{4.73}$$

Now we will perform three Whipple transformations. For each one we list the indices chosen $-p,\ q,\ a_1,\ a_2,\ r,\ b_1,\ b_2$, the resulting hypergeometric term (with possible rearranging of some upper indices), and the coefficient contribution, C_i, from the corresponding Whipple transformation.

1. Using $-p = i-D$, $q = j-D$, $a_1 = i-D$, $a_2 = j-D$, $r = i+j-D+1$, $b_1 = -2D-1$, $b_2 = i+j-D+1$:

$${}_4F_3\left[\begin{matrix} i-D,\ i+1,\ j-D,\ j+1 \\ i+j+2,\ -D,\ i+j-D+1 \end{matrix} ;1\right], \tag{4.74}$$

$$\begin{aligned} C_1 &= \frac{(-D-j-1)_{D-i}(i+1)_{D-i}}{(-2D-1)_{D-i}(i+j-D+1)_{D-i}} \\ &= \frac{(-1)^{D-i}(D+j+1)!}{(i+j+1)!}\frac{D!}{i!}\frac{(D+i+1)!}{(-1)^{D-i}(2D+1)!}\frac{(i+j-D)!}{j!}. \end{aligned} \tag{4.75}$$

2. Using $-p = i-D$, $q = j+1$, $a_1 = i+1$, $a_2 = j-D$, $r = -D$, $b_1 = i+j+2$, $b_2 = i+j-D+1$:

$${}_4F_3\left[\begin{matrix} i-D,\ -D-i-1,\ -j,\ j+1 \\ -D,\ -D,\ 1 \end{matrix} ;1\right], \tag{4.76}$$

$$C_2 = \frac{(i+1)_{D-i}(i-D)_{D-i}}{(i+j+2)_{D-i}(i+j-D+1)_{D-i}}$$
$$= \frac{D!}{i!}(-1)^{D-i}(D-i)!\frac{(i+j+1)!}{(D+j+1)!}\frac{(i+j-D)!}{j!}. \tag{4.77}$$

3. Using $-p = -j$, $q = j+1$, $a_1 = i-D$, $a_2 = -D-i-1$, $r = -D$, $b_1 = -D$, $b_2 = 1$:

$${}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ -D,\ D+2,\ 1 \end{matrix} ;1\right] = {}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ D+2,\ -D \end{matrix} ;1\right], \tag{4.78}$$

$$C_3 = \frac{(-D-j-1)_j(-j)_j}{(-D)_j(1)_j}$$
$$= \frac{(-1)^j(D+j+1)!}{(D+1)!}(-1)^j j!\frac{(D-j)!}{(-1)^j D!}\frac{1}{j!}. \tag{4.79}$$

Combining coefficients we see that $C_0C_1C_2C_3 = \frac{(-1)^{D-i+j}}{D+1} = \frac{(-1)^{i+j-D}}{D+1}$, since i, j, D are integers. □

We now evaluate the Biedenharn-Elliott identity using Proposition 4.3.6.

Proposition 4.3.7. *For integers $0 \leq h, i, j \leq D$ we have*

$$\sum_{t=0}^{D}(2t+1)u_t(\theta_h)u_t(\theta_i)u_t(\theta_j) = (D+1)^3\Big(W\left(\tfrac{D}{2}, \tfrac{D}{2}, i, h; j, \tfrac{D}{2}\right)\Big)^2. \tag{4.80}$$

Proof. First we apply Proposition 4.3.4 with $a = a' = b = b' = c = c' = \frac{D}{2}$, $e = h$,

$f = i$, $g = j$, and $d = t$ to obtain

$$\sum_{t\in\frac{1}{2}\mathbb{N}} (-1)^{D-t}(2t+1)W(\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2};t,h)W(\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2};t,i)W(\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2},\tfrac{D}{2};t,j)$$
$$= (-1)^{h+i-j} W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right) W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right). \quad (4.81)$$

Note that $\frac{D}{2}+\frac{D}{2}+t$ is an integer if and only if t is an integer. So by (4.62), the terms of the sum vanish in which t is not an integer or $t > D$. By Proposition 4.3.6 and (2.28), the left hand side of (4.81) becomes

$$\sum_{t=0}^{D} (-1)^{D-t}(2t+1)\frac{(-1)^{t+h-D}u_t(\theta_h)}{D+1}\,\frac{(-1)^{t+i-D}u_t(\theta_i)}{D+1}\,\frac{(-1)^{t+j-D}u_t(\theta_j)}{D+1},$$

which simplifies to

$$\frac{(-1)^{i+j+h}}{(D+1)^3}\sum_{t=0}^{D}(2t+1)u_t(\theta_h)u_t(\theta_i)u_t(\theta_j). \quad (4.82)$$

Setting (4.82) equal to the right hand side of (4.81) and dividing by the coefficients completes the proof. □

Corollary 4.3.8. *For $0 \le h, i, j \le D$ we have*

$$p^h_{i,j} = (2i+1)(2j+1)(D+1)\Big(W\left(\tfrac{D}{2},\tfrac{D}{2},i,h;j,\tfrac{D}{2}\right)\Big)^2. \quad (4.83)$$

Proof. Using Propositions 2.8.9, 4.3.7 and substituting (2.13),(2.14) we have

$$
\begin{aligned}
p^h_{i,j} &= \frac{k_i k_j}{\nu} \sum_{t=0}^{D} k_t u_t(\theta_i) u_t(\theta_j) u_t(\theta_h) \\
&= \frac{(2i+1)(2j+1)}{(D+1)^2} \left((D+1)^3 \Big(W\left(\tfrac{D}{2}, \tfrac{D}{2}, i, h; j, \tfrac{D}{2}\right) \Big)^2 \right) \\
&= (2i+1)(2j+1)(D+1) \Big(W\left(\tfrac{D}{2}, \tfrac{D}{2}, i, h; j, \tfrac{D}{2}\right) \Big)^2 .
\end{aligned}
$$

□

Corollary 4.3.9. *For $0 \le h, i, j \le D$ we have*

$$p^h_{i,j} \ge 0.$$

Proof. Immediate from Corollary 4.3.8. □

4.4 Proof of the Kresch-Tamvakis conjecture

We are now ready to prove our main result of this chapter. We will use the Perron-Frobenius theorem [25, p. 529].

Proposition 4.4.1. *For $0 \le i, j \le D$ we have*

$$|u_i(\theta_j)| \le 1.$$

Proof. By Lemma 2.8.4, the vector $\mathbb{1}$ is an eigenvector for B_i with eigenvalue k_i. By Corollary 4.3.9, the entries of B_i are all nonnegative. By Lemma 2.8.3 the scalar $v_i(\theta_j)$ is an eigenvalue of B_i. By the Perron-Frobenius theorem [25, p. 529], we have $|v_i(\theta_j)| \le k_i$. The result follows from this and (2.19). □

Equation (2.28) and Proposition 4.4.1 imply Theorem 4.0.1.

4.5 New $p_{i,j}^h$ formula for Def. 2.3.1

We finish this chapter with some more details about the formula for $p_{i,j}^h$ in Corollary 4.3.8. By Lemma 2.8.10, without loss of generality we assume $i \leq j \leq h$. Also, in order to avoid trivialities, we assume that h, i, j satisfy the triangle inequalities, which in this case become $h \leq i + j$. As we evaluate $p_{i,j}^h$ in line (4.83), we consider the last factor. We evaluate that factor using Definition 4.3.3 with

$$a = \frac{D}{2}, \qquad b = \frac{D}{2}, \qquad c = i, \qquad d = h, \qquad e = j, \qquad f = \frac{D}{2}.$$

For these values, we have:

$$\alpha_1 = D + i, \qquad \alpha_2 = D + j, \qquad \alpha_3 = D + h, \qquad \alpha_4 = h + i + j,$$

$$\beta_1 = D + i + j, \qquad \beta_2 = D + h + i, \qquad \beta_3 = D + h + j.$$

Note that we then have:

$$\alpha_1 - \beta_1 = -j, \qquad \alpha_2 - \beta_1 = -i, \qquad \alpha_3 - \beta_1 = h - i - j, \qquad \alpha_4 - \beta_1 = h - D$$

$$-\beta_1 - 1 = -D - i - j - 1, \qquad \beta_2 - \beta_1 + 1 = h - j + 1, \qquad \beta_3 - \beta_1 + 1 = h - i + 1.$$

For the above data, (4.83) becomes

$$p_{i,j}^h = C_{i,j}^h (2i+1)(2j+1)(D+1) \left({}_4F_3 \left[\begin{matrix} -j,\ -i,\ h-i-j,\ h-D \\ -D-i-j-1,\ h-j+1,\ h-i+1 \end{matrix} ; 1 \right] \right)^2,$$

where

$$C^h_{i,j} = \left(\frac{\Delta(\frac{D}{2}, \frac{D}{2}, i)\Delta(\frac{D}{2}, \frac{D}{2}, j)\Delta(\frac{D}{2}, \frac{D}{2}, h)\Delta(i, j, h))(D + i + j + 1)!}{(h - i)!(h - j)!i!j!(i + j - h)!(D - h)!} \right)^2$$

$$= \frac{(D - i)!(D - j)!(D - h)!(j + h - i)!(h + i - j)!}{(D + i + 1)!(D + j + 1)!(D + h + 1)!(i + j + h + 1)!(i + j - h)!}$$

$$\times \left(\frac{h!(D + i + j + 1)!}{(h - i)!(h - j)!(D - h)!} \right)^2 .$$

5 Spin Leonard pairs and fusion algebras

A spin model is a type of statistical mechanical model that was used by Vaughan Jones to construct topological link invariants in his 1989 paper *On Knot Invariants Related to Some Statistical Mechanical Models* [30]. Every spin model produces a link invariant, although in some circumstances the invariant produced may be trivial. The Jones polynomial is an example of a nontrivial knot invariant that is constructed using a spin model called the Potts model [29]. Spin models have connections to association schemes and Leonard pairs. In particular, Jaeger used topology to prove that every spin model is contained in the Bose-Mesner algebra of an association scheme [28], and, shortly after, Nomura proved the same result using linear algebraic techniques [38].

Curtin in [11] defined the notion of so called spin Leonard pairs. By the work of Caughman and Wolff in [7], and Curtin [9], if a Bose-Mesner algebra supports a spin model, then the intersection matrix B_1 and Krein parameter matrix B_1^*, as mentioned in Sec. 3.3, form a spin Leonard pair. Nomura and Terwilliger showed that, in certain instances, it may be possible to construct a spin model given a spin Leonard pair and a distance regular graph of a certain form [40, 41].

In reference to the subalgebras given in Sections 3.5 and 5.6, there are some cases when the corresponding strongly regular graphs exist, and there is an associated spin model [27], [13]. These, in turn, give rise to a link invariant which is an evaluation of the well-known Kauffman polynomials. However, by the work of De La Harpe [13], it appears that the specific evaluations that appear in these cases result only in trivial link invariants. In the example of the Clebsch graph, which we saw in (3.55) and Figure 3.3, the Bose-Mesner algebra is known to afford a spin model (see [12]). As mentioned by De La Harpe, despite the triviality of the link invariant in some of these cases, such spin models may still be of interest to combinatorialists [13].

In this section, we will use the theory of spin Leonard pairs to prove a generalization of the correspondence in Section 3.5, specifically Thm. 5.6.1 and Cor. 5.6.2. In fact, we will prove a more general result that applies to all spin Leonard pairs of Racah type. By work of Curtin ([11] and [10]), the same result holds for the spin Leonard pairs of the following types:

(1) Racah type (with $h^* = h$, $s^* = s$, $r_1 = s/2$, $r_2 = 3s/2 + d + 1$ in the notation of [49, Ex. 5.10])

(2) Krawtchouk type with $v = -1$ (or $r = \frac{3ss^*}{4}$ and $s = s^*$ in the notation of [49, Ex. 5.13])

(3) Bannai-Ito type (with $h^* = h$, $s^* = s$, $r_1 = -3s/2 + d + 1$, $r_2 = -s/2$ in the notation of [49, Ex. 5.14])

We are primarily interested in the Lenoard pairs from Def. 2.3.1 and so we will only consider the Racah type spin-Leonard pair case, of which the Leonard pairs in Def. 2.3.1 are a special case.

5.1 Spin model definition

We now review the definition of a spin model.

Definition 5.1.1. Let n be a positive integer. A *spin model* is a triple $S = (X, W^+, W^-)$ where $X = [n]$ and W^+, W^- are symmetric $n \times n$ complex matrices satisfying the following properties.

$$W^{+}_{a,b} W^{-}_{a,b} = 1 \quad \text{for all } a, b \in X, \tag{5.84}$$

$$\sum_{x \in X} W^{-}_{a,x} W^{+}_{x,c} = n \delta_{a,c} \quad \text{for all } a, b, c \in X, \tag{5.85}$$

$$\sum_{x \in X} W^{+}_{a,x} W^{+}_{b,x} W^{-}_{c,x} = \sqrt{n} W^{+}_{a,b} W^{-}_{b,c} W^{-}_{c,a} \quad \text{for all } a, b, c \in X. \tag{5.86}$$

The elements of X are called the *spins* of S. Matrices that satisfy 5.85 are called *type II*, and matrices that satisfy 5.86 are called *type III*.

Jaeger and Nomura showed that a symmetric spin model is contained in a Bose-Mesner algebra of some symmetric association scheme [28], [38]. Hence, if a spin matrix W^{+} exists in the Bose-Mesner algebra generated by $\{A_i\}_{i=0}^{d}$, then there exist scalars $\{t_0, t_1, \ldots, t_d\}$ in $\mathbb{K}$, such that

$$W^{+} = \sum_{i=0}^{d} t_i A_i. \tag{5.87}$$

The coefficients $t_0, t_1, \ldots, t_d$, are called the *Boltzmann coefficients* of the spin model.

From Definition 5.1.1 we have that

$$W^{-} = \sum_{i=0}^{d} t_i^{-1} A_i. \tag{5.88}$$

5.2 Spin Leonard pairs

In this section we recall the definition of a spin Leonard pair.

Definition 5.2.1. A Leonard pair A, A^* on the vector space V is called a *spin Leonard pair* when there exist invertible linear transformations W, W^* on V that satisfy the following:

$$WA = AW, \tag{5.89}$$

$$W^*A^* = A^*W^*, \tag{5.90}$$

$$WA^*W^{-1} = W^{*-1}AW^*. \tag{5.91}$$

The matrices W, W^* are called a *Boltzmann pair* for (A, A^*).

Since the A and A^* of a Leonard pair are multiplicity free, Lem. 2.4.2 and conditions (5.89) and (5.90) are equivalent to W being in $\langle A \rangle$, and W^* being in $\langle A^* \rangle$, the algebras generated by A and A^* respectively (see [11, Lem. 3.2]).

Definition 5.2.2. Let $A, A^*, A^\diamond$ be linear transformations on a finite-dimensional vector space V. We say $A, A^*, A^\diamond$ is a *Leonard triple* on V if, for any $B \in \{A, A^*, A^\diamond\}$, there is a basis for V such that the matrix representing B is diagonal and the matrices representing the other two transformations are irreducible tridiagonal.

Recall τ is an *antiautomorphism* on the transformations of V if it is $\mathbb{K}$-linear and $\tau(XY) = \tau(Y)\tau(X)$ for any two linear transformations X, Y on V.

Definition 5.2.3. Let $A, A^*, A^\diamond$ be a Leonard triple on V. We say $A, A^*, A^\diamond$ is a *modular Leonard triple* if, for any $B \in \{A, A^*, A^\diamond\}$, there exists an antiautomorphism that fixes B, and swaps the other two transformations in the Leonard triple.

Curtin [11] classified all spin Leonard pairs and showed that if $A, A^*, A^\diamond$ is a modular Leonard triple, then A, A^* is a spin Leonard pair. Conversely, Curtin also showed that if A, A^* is a spin Leonard pair, with Boltzmann pair W, W^*, then

$(A, A^*, A^\diamond)$ is a modular Leonard triple, where $A^\diamond = WA^*W^{-1} = W^{*-1}AW^*$. For completeness, we present the proof of the direction that modular Leonard triples can be used to form spin Leonard pairs. This direction will suffice for our needs. For more details on the other direction, and for the classification, we point the reader to [10] and [11].

Definition 5.2.4. Let $A, A^*, A^\diamond$ and $B, B^*, B^\diamond$ denote Leonard triples on V and V' respectively. We say $A, A^*, A^\diamond$ is *isomorphic* (as Leonard triples) to $B, B^*, B^\diamond$ if there is a $\mathbb{K}$-algebra isomorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$ such that $\sigma(A) = B$, $\sigma(A^*) = B^*$, and $\sigma(A^\diamond) = B^\diamond$.

Definition 5.2.5. A *canonical modular Leonard triple* of diameter d, is an ordered triple of matrices $(A, A^*, A^\diamond)$ in $\mathrm{Mat}_{d+1}(\mathbb{K})$ which form a modular Leonard triple on $\mathbb{K}^{d+1}$, where A, $A^\diamond$ are irreducible tridiagonal, and A^* is diagonal, and where the row sums of A are equal to $(A^*)_{0,0}$.

Curtin in [10] proved the following theorem and lemma.

Theorem 5.2.6. *[1.6, [10]] Let $A, A^*, A^\diamond$ be a modular Leonard triple, and let $\theta_0, \ldots, \theta_d$ be an eigenvalue sequence of $A, A^*, A^\diamond$. Then $A, A^*, A^\diamond$ is isomorphic to a unique canonical modular Leonard triple $B, B^*, B^\diamond$ such that $B^* = diag(\theta_0, \ldots, \theta_d)$.*

Lemma 5.2.7. *[10, Lem. 1.7] Let $A, A^*, A^\diamond$ be a canonical modular Leonard triple of diameter d. Then*

$$A = tridiag \begin{pmatrix} b_0 & b_1 & \ldots & b_{d-1} & * \\ a_0 & a_1 & \ldots & a_{d-1} & a_d \\ * & c_1 & \ldots & c_{d-1} & c_d \end{pmatrix} \tag{5.92}$$

$$A^* = diag(\theta_0, \ldots, \theta_d,) \tag{5.93}$$

$$A^\diamond = tridiag\begin{pmatrix} b_0\nu_1 & b_1\nu_2 & \ldots & b_{d-1}\nu_d & * \\ a_0 & a_1 & \ldots & a_{d-1} & a_d \\ * & c_1/\nu_1 & \ldots & c_{d-1}/\nu_{d-1} & c_d/\nu_d \end{pmatrix} \tag{5.94}$$

with $c_0 = b_d = 0$*, where* $b_{i-1}, c_i, \nu_i \neq 0$ *for* $1 \leq i \leq d$*, where the eigenvalues* $\theta_i, \ldots, \theta_d$ *are distinct, and where* $c_i + a_i + d_i = \theta_0$ $(0 \leq i \leq d)$*.*

The full theorem and lemma stated above are not needed for our goals, but we direct the reader to [10] and [11] for more details. In particular, the uniqueness of the canonical form is not necessary for us. The Leonard pairs in Def. 2.3.1 belong to the family of Leonard pairs of Racah type. We will identify a particular subfamily of Racah type Leonard pairs that contain these examples and that always form spin Leonard pairs. This is stated in the next lemma and will be proved over the next two sections.

It is worth noting that Curtin showed that the family mentioned in the next lemma in fact constitues all of the spin Leonard pairs of Racah type [11, Theorem 1.13]. As a result, we will refer to them as the *spin Leonard pairs of Racah type.* We will not present the necessary condition of this classification, but only the sufficiency condition, and we again direct the interested reader to the work of Curtin.

Lemma 5.2.8. *(Lemma 1.8 [11]) Fix a nonnegative integer* d*, a field* $\mathbb{K}$*, and assume* $char(\mathbb{K}) = 0$ *or* $char(\mathbb{K})$ *is an odd prime greater than* d*. Take* θ_0*,* h, s *in* $\mathbb{K}$ *such that* $h \neq 0$*,* $s \neq -i$ $(2 \leq i \leq 2d)$*, and* $3s \neq -2i$ $(d+2 \leq i \leq 2d+1)$*. Define the following* *matrices*

$$A = tridiag\begin{pmatrix} b_0 & b_1 & \dots & b_{d-1} & * \\ a_0 & a_1 & \dots & a_{d-1} & a_d \\ * & c_1 & \dots & c_{d-1} & c_d \end{pmatrix} \tag{5.95}$$

and

$$A^* = diag(\theta_0, \dots, \theta_d,), \tag{5.96}$$

where

$$\theta_i = \theta_0 + hi(i+s+1) \quad (0 \leq i \leq d), \tag{5.97}$$

$$b_0 = -\frac{hd(3s+2d+4)}{4}, \tag{5.98}$$

$$b_i = \frac{h(i+s+1)(i-d)(2i+3s+2d+4)}{4(2i+s+1)} \quad (1 \leq i \leq d-1), \tag{5.99}$$

$$c_i = \frac{hi(i+s+d+1)(2i-s-2d-2)}{4(2i+s+1)} \quad (1 \leq i \leq d-1), \tag{5.100}$$

$$c_d = -\frac{hd(s+2)}{4}, \tag{5.101}$$

$$a_i = \theta_0 - b_i - c_i \quad (0 \leq i \leq d)(c_0 = 0, b_d = 0). \tag{5.102}$$

Then (A, A^*) *form a spin Leonard pair.*

The fact that the matrices A, A^* given above define a Leonard pair on V is by the work of Terwilliger (see [48, Ex. 5.10], and also [11, Lem. 1.8], and [10, Lem. 1.10]). We also note that, in these works, it is shown that the A, A^*, have the same spectrum, and hence these Leonard pairs are all self-dual. We will show these are spin Leonard pairs in Section 5.4. We will do this by first showing that a modular Leonard triple is a spin Leonard pair in Theorem 5.3.4, and then we show the Leonard pairs in Lemma 5.2.8 are modular Leonard triples in Theorem 5.4.3.

5.3 Modular Leonard triples are spin Leonard pairs

Lemma 5.3.1. *Let $A, A^*, A^\diamond$ be a modular Leonard triple on V. For each $B \in \{A, A^*, A^\diamond\}$, the antiautomorphism that fixes B and swaps the other two is unique and an involution.*

Proof. Removing B from the set $\{A, A^*, A^\diamond\}$, the remaining two elements form a Leonard pair. Call the two elements A' and A''. By Lemma 2.4.8, the A', A'' generate $\mathrm{End}(V)$. Hence an antiautomorphism is uniquely determined by what its action this generating set. Also, this antiautomorphism swaps A' and A''; hence, it is an involution. □

Lemma 5.3.2. *[2.5, [11],10.1, [10]] Let $A, A^*, A^\diamond$ be a modular Leonard triple on V. Then there exist automorphisms ν, ν^* of $\mathrm{End}(V)$ such that*

$$\begin{aligned} \nu(A) = A, &\quad \nu(A^*) = A^\diamond, \\ \nu^*(A^*) = A^*, &\quad \nu^*(A^\diamond) = A, \\ \nu^\diamond(A^\diamond) = A^\diamond, &\quad \nu^\diamond(A) = A^*. \end{aligned} \tag{5.103}$$

Proof. Given distinct $X, Y, Z \in \{A, A^*, A^\diamond\}$, let $\alpha_{X,Y}$ be the antiautomorphism in Theorem 2.5.3 that fixes X and Y. Let $\mu_{X,Y}$ be the antiautomorphism as in Definition 5.2.3 that swaps X and Y and fixes Z. Then let $\nu = \alpha_{A^\diamond,A}\mu_{A^*,A^\diamond}$, let $\nu^* = \alpha_{A^*,A}\mu_{A^\diamond,A}$, and let $\nu^\diamond = \alpha_{A^\diamond,A^*}\mu_{A,A^*}$. The result follows. □

Corollary 5.3.3. *[[11, Cor. 2.6] [10, Lem. 10.2]] Let $A, A^*, A^\diamond$ be a modular Leonard* *triple on V. Then there exist U and U^* such that:*

$$\begin{aligned} UA &= AU \\ UA^* &= A^\diamond U \\ AU^* &= U^* A^\diamond \\ A^* U^* &= U^* A^*. \end{aligned} \tag{5.104}$$

Proof. We have ν and ν^* as given in Lemma 5.3.2. By Lemma 2.5.1 there exist invertible U and U^* in $\mathrm{End}(V)$ that represent ν and ν^* respectively. Then, by Lemma 5.3.2, we have:

$$A = \nu(A) = UAU^{-1}$$

$$A^\diamond = \nu(A^*) = UA^*U^{-1}$$

$$A^* = \nu^*(A^*) = U^*A^*(U^*)^{-1}$$

$$A = \nu(A^\diamond) = U^*A^\diamond(U^*)^{-1}.$$

The result follows. □

Theorem 5.3.4. *[11, Thm. 1.5] Let $A, A^*, A^\diamond$ be a modular Leonard triple on V. Then A, A^* is a spin Leonard pair.*

Proof. Take U and U^* from Corollary 5.3.3. We have that

$$UA = AU,$$

$$U^*A^* = A^*U^*, \quad \text{and}$$

$$UA^*U^{-1} = A^\diamond = (U^*)^{-1}AU^*.$$

The result follows. □

5.4 Racah type spin Leonard pairs

Our next task is to prove that the Leonard pairs given by the matrices defined in (5.95, 5.96, 5.97-5.102) in Lemma 5.2.8 are spin Leonard pairs. We do this by taking such a Leonard pair, A, A^*, and showing that there is a transformation $\diamond$ such that $A, A^*, A^\diamond$ is a modular Leonard triple. The result then follows from Theorem 5.3.4. We will be able to use this result to explicitly construct the matrices U and U^*. Throughout, we will work in a basis where A is irreducible tridiagonal, and A^* is diagonal, hence we can assume they are in the form as given in (5.95, 5.96).

Recall the antiautomorphism † given in Theorem 2.5.3. Note that composing an automorphism σ with an antiautomorphism τ yields $\sigma \circ \tau$, which is another antiautomorphism.

Definition 5.4.1. Let A, A^* be a Leonard pair of diameter d on vector space V. Suppose A, A^* is of the type given in Lemma 5.2.8, represented in the basis where A is irreducible tridiagonal and A^* is diagonal. Let P be the corresponding matrix from Definition 2.6.3. Let † be the antiautomorphism given in 2.5.3.

Define the following matrix $N \in \mathbb{K}^{d+1}$:

$$N = \text{diag}(1, -1, 1, \ldots, (-1)^d)). \qquad (5.105)$$

Define the following automorphisms:

$\sigma : \mathrm{End}(V) \to \mathrm{End}(V)$

$$\sigma(X) = P^{-1}XP = PXP^{-1}, \tag{5.106}$$

$\delta : \mathrm{End}(V) \to \mathrm{End}(V)$

$$\delta(X) = N^{-1}XN = NXN^{-1}. \tag{5.107}$$

Define the following antiautomorphisms,

$\mu : \mathrm{End}(V) \to \mathrm{End}(V)$

$$\mu(X) = \delta(\sigma(\delta(X^\dagger))) = (NPN)^{-1}X^\dagger NPN = (NP^{-1}N)^{-1}X^\dagger NP^{-1}N, \tag{5.108}$$

$\mu^* : \mathrm{End}(V) \to \mathrm{End}(V)$

$$\mu^*(X) = \delta(X^\dagger) = N^{-1}X^\dagger N = NX^\dagger N^{-1}, \tag{5.109}$$

$\mu^\diamond : \mathrm{End}(V) \to \mathrm{End}(V)$

$$\mu^\diamond(X) = \sigma(X^\dagger) = P^{-1}X^\dagger P = PX^\dagger P^{-1}. \tag{5.110}$$

Note that, since the inverses of N and P (by self-duality) in the previous definition differ from the original matrix by a constant, i.e. $N^{-1} = N$ and $P^{-1} = \nu^{-1}P$, where $\nu = \sum_{i=0}^{d} k_i$, all the automorphisms and antiautomorphisms of Definition 5.4.1 are involutions. Also recall that self-duality implies $u_i(\theta_j) = u_j^*(\theta_i^*) = u_j(\theta_i)$. We also need one more basic result.

Proposition 5.4.2. *Fix a Leonard pair A, A^* on a vector space V of the type given* *in Lemma 5.2.8. Then*

$$\theta_i - 2a_j = \theta_j - 2a_i. \tag{5.111}$$

Proof. Expanding both sides individually, we find that they both equal $-d^2h + hi^2 + hj^2 - \frac{3}{2}dhs + his + hjs - 2dh + hi + hj - \theta_0$. $\square$

We are now ready to prove that the Leonard pairs in Lemma 5.2.8 are modular Leonard triples, and hence spin Leonard pairs.

Theorem 5.4.3. *With reference to Definition 5.4.1, let A, A^* be a Leonard pair of diameter d on vector space V. Suppose A, A^* is of the type given in Lemma 5.2.8, represented in the standard basis, where A is irreducible tridiagonal and A^* is diagonal. Define $A^\diamond = \delta(A) = N^{-1}AN$. Then $A, A^*, A^\diamond$ is a modular Leonard triple.*

Proof. Recall from Theorem 2.5.3 that $\dagger$ fixes A and A^*, and $X^\dagger = KXK^{-1}$, where $K = \mathrm{diag}(k_0, k_1, \ldots, k_d)$. Since N and K are diagonal, they commute, hence $(A^\diamond)^\dagger = K^{-1}(N^{-1}AN)^\top K = K^{-1}N^\top A^\top (N^{-1})^\top K = N^\top(K^{-1}A^\top K)(N^{-1})^\top = NA^\dagger N^{-1} = NAN^{-1} = A^\diamond$.

Recall that $\sigma(A) = A^*$ and $\sigma(A^*) = A$ by Lem. 2.8.3. Also note that $A, A^*, A^\diamond$ have the form given in (5.92), (5.93), and (5.94) with $\nu_i = -1$ for $(1 \leq i \leq d)$.

Our proof will be completed by the following two steps, each with three sub-steps.

(1) We show that $\mu, \mu^*, \mu^\diamond$ are antiautomorphisms satisfying the definitions for modular Leonard triples:

  (i) μ fixes A and swaps $A^*, A^\diamond$.

  Proof of (i). To see that μ fixes A, we must show $ANPNK^{-1}$ equals

$(NP^{-1}N)^{-1}K^{-1}A^\top = NPNK^{-1}A^\top$.

First note that $NPNK^{-1} = NPK^{-1}N$, and the i, j entry of PK^{-1} is $v_j(\theta_i)/k_j = u_j(\theta_i) = u_i(\theta_j)$.

By matrix multiplication, the i, j entry of $NPNK^{-1}A^\top$ is:

$$
\begin{aligned}
&(-1)^{i+j+1}(c_j u_{j-1}(\theta_i) - a_j u_j(\theta_i) + b_j u_{j+1}(\theta_i)) \\
&= (-1)^{i+j+1}(c_j u_{j-1}(\theta_i) + a_j u_j(\theta_i) + b_j u_{j+1}(\theta_i) - 2a_j u_j(\theta_i)) \\
&= (-1)^{i+j+1}(\theta_i u_j(\theta_i) - 2a_j u_j(\theta_i)) && \text{(by Def. 2.6.1)} \\
&= (-1)^{i+j+1}u_j(\theta_i)(\theta_i - 2a_j)
\end{aligned}
$$

By matrix multiplication, the i, j entry of $ANPNK^{-1}$ is:

$$
\begin{aligned}
&(-1)^{i+j+1}(c_i u_j(\theta_{i-1}) - a_i u_j(\theta_i) + b_i u_j(\theta_{i+1})) \\
&= (-1)^{i+j+1}(c_i u_j(\theta_{i-1}) + a_i u_j(\theta_i) + b_i u_j(\theta_{i+1}) - 2a_i u_j(\theta_i)) \\
&= (-1)^{i+j+1}(c_i u_{i-1}(\theta_j) + a_i u_i(\theta_j) + b_i u_{i+1}(\theta_j) - 2a_i u_i(\theta_j)) \\
& && \text{(since } u_i(\theta_j) = u_j(\theta_i)\text{)} \\
&= (-1)^{i+j+1}(\theta_j u_i(\theta_j) - 2a_i u_i(\theta_j)) && \text{(by Def. 2.6.1)} \\
&= (-1)^{i+j+1}u_i(\theta_j)(\theta_j - 2a_i).
\end{aligned}
$$

And, by Propoisition 5.4.2, we know that $\theta_i - 2a_j = \theta_j - 2a_i$. Hence $ANPNK^{-1} = (NP^{-1}N)^{-1}K^{-1}A^\top$.

To see μ swaps A^* and $A^\diamond$, we have:

$$
\begin{aligned}
\mu(A^\diamond) &= \delta(\sigma(\delta((A^\diamond)^\dagger))) \\
&= \delta(\sigma(\delta(A^\diamond))) \\
&= \delta(\sigma(\delta(N^{-1}AN))) \\
&= \delta(\sigma(A)) \\
&= \delta(A^*) \\
&= A^*. && (A^* \text{ and } N \text{ are diagonal})
\end{aligned}
$$

(ii) μ^* fixes A^* and swaps $A, A^\diamond$.

Proof of (ii). Since $\mu^*(X) = N^{-1}X^\dagger N$, and N and A^* are diagonal, we see that μ^* fixes A^*. We also have $\mu^*(A) = N^{-1}A^\dagger N = N^{-1}AN = A^\diamond$, and μ^* is an involution. Hence $\mu^*(A^\diamond) = A$.

(iii) $\mu^\diamond$ fixes $A^\diamond$ and swaps A, A^*.

Proof of (iii). Recall $\dagger$ fixes A and A^*. By Lemma 2.8.3 and self-duality, we see that $\mu^\diamond$ swaps A and A^*. Also $\dagger$ fixes $A^\diamond$ and $\mu^\diamond(A^\diamond) = \sigma((A^\diamond)^\dagger) = \sigma(A^\diamond) = \sigma(\delta(A)) = \sigma(\delta(A^\dagger))$, and recall $\delta(\sigma(\delta(A))) = \mu(A) = A$. It follows that $\delta(\mu^\diamond(A^\diamond)) = A$. Since δ is an involution and $\delta(A) = A^\diamond$, we have $\mu^\diamond(A^\diamond) = A^\diamond$.

(2) We show that $A, A^*, A^\diamond$ is a Leonard triple.

(i) There is a basis of V where A is diagonal and $A^*, A^\diamond$ are irreducible tridiagonal.

Proof of (i). We see that $\mu^\diamond\mu$ is the composistion of two antiautomorphisms and is therefore an automorphism. And $\mu^\diamond\mu$ maps $A, A^*, A^\diamond$ to $A^*, A^\diamond, A$, respectively. Now by the Noether-Skolem Lemma 2.5.1, There is a invertible Q such that $\mu^\diamond(\mu(X)) = QXQ^{-1}$. Hence, in the basis with the columns of Q, the matrices representing $A, A^*, A^\diamond$ are the same as the matrices $A^*, A^\diamond, A$ represented in the standard basis, as desired.

(ii) There is a basis of V where A^* is diagonal and $A, A^\diamond$ are irreducible tridiagonal.

Proof of (ii). This is satisfied by the standard basis in the assumption of the problem.

(iii) There is a basis of V where $A^\diamond$ is diagonal and A, A^* are irreducible tridiagonal.

Proof of (iii). Similar to (i), we can take the basis consisting of the columns of matrix Q representing the automorphism $\mu(\mu^\diamond(X)) = QXQ^{-1}$. Hence, the matrices representing $A, A^*, A^\diamond$ in this basis are the same as the matrices $A^\diamond, A, A^*$ in the standard basis.

The proof is now complete. □

By Theorems 5.3.4 and 5.4.3, we have the following corollary.

Corollary 5.4.4. *With reference to Definition 5.4.1. Let A, A^* be a Leonard pair of type as given in Definition 2.3.1. Then A, A^* is a spin Leonard pair, with Boltzmann pair $U^* = N$, $U = NPN$.*

Proof. The only thing left to show is that $U = NPN$ and $U^* = N$. By the construction of U,U^* from ν,ν^* given in Lemma 5.3.2, and by the proof of Theorem 5.4.3, we have that $U = K^{-1}NP^{\top}NK = NK^{-1}P^{\top}KN$, and $U^* = K^{-1}NK = N$.

And finally, we know that $K^{-1}P^{\top}K = P$, since $P_{i,j} = v_j(\theta_i) = k_j u_j(\theta_i) = k_j u_i(\theta_j) = \frac{k_j}{k_i} v_i(\theta_j) = \frac{k_j}{k_i} P_{j,i}$. □

Lemma 5.4.5. *[11, Lem. 5.1] Let A, A^* be a spin Leonard pair on vector space V, with Boltzmann pair U, U^*. Then UU^*U and U^*UU^* are nonzero scalar multiples of each other.*

Proof. Define automorphisms on $\mathrm{End}(V)$ by letting $\nu(X) = UXU^{-1}$, and $\nu^*(X) = U^*X(U^*)^{-1}$. We see from Definition 5.2.1 and (5.91) that we have $UA^*U^{-1} = U^{*-1}AU^*$. By (5.89), $U \in \langle A \rangle$, the algebra generated by A, and by (5.90), $U^* \in \langle A^* \rangle$, the algebra generated by A^*. Hence by Theorem 2.5.3, the map † fixes U and U^*. Applying the antiautomorphism † to (5.91) we get $U^{-1}A^*U = U^*AU^{*-1}$. Let $T = UA^*U^{-1} = U^{*-1}AU^*$, and let $T^* = U^{-1}A^*U = U^*AU^{*-1}$. We see that $\nu(A) = A$, $\nu^*(A) = T^*$, $\nu^*(A^*) = A^*$, $\nu(A^*) = T$, and we see that both $\nu\nu^*\nu$ and $\nu^*\nu\nu^*$ fix both A and A^*. By Cor. 2.4.8, A and A^* generate $\mathrm{End}(V)$, hence we have that $\nu\nu^*\nu$ and $\nu^*\nu\nu^*$ agree on $\mathrm{End}(V)$ and are hence equal. The result follows from Lem. 2.5.2. □

An implication of this lemma for the Leonard pairs from Definition 2.3.1 is that PNP and NPN are scalar multiples of each other. In this case, comparing the (0,0) entry of NPN and PNP, we find that they differ by the constant $\tilde{\nu} = \sum_{i=0}^{d}(-1)^i k_i = (-1)^d(d+1)$. Therefore, we have the following result.

Corollary 5.4.6. *With reference to Definition 5.4.1, let A, A^* be the Leonard pair given in Definition 2.3.1. Let $N = diag(1, -1, \ldots, (-1)^d)$, let P be as in Definition*

2.6.3, and let $\tilde{\nu} = \sum_{i=0}^{d}(-1)^i k_i = (-1)^d(d+1)$. *Then*

$$PNP = \tilde{\nu}NPN = (-1)^d(d+1)NPN. \tag{5.112}$$

We will see in the next section (Section 5.5) that this result can be used to prove Corollary 5.5.2, which will imply a generalization of the construction made in Section 3.5.

5.5 Alternating sum of products formula for Def. 2.3.1

Note that Corollary 5.4.6 also tells us that the columns of P are eigenvectors of $U = U^*PU^*$. Specifically the mth column of P, which we denote by $\vec{v}_m$, has eigenvalue $\tilde{\nu}(-1)^m$:

$$U\vec{v}_m = NPN\vec{v}_m = \tilde{\nu}(-1)^m\vec{v}_m = (-1)^{d+m}(d+1)\vec{v}_m. \tag{5.113}$$

Dividing (5.113) by k_m, and finding the nth entry in the vector on both sides, we have the following corollary.

Corollary 5.5.1. *For any* n, m $(0 \leq n, m \leq d)$,

$$(-1)^{d+n+m}(d+1)u_m(\theta_n) = \sum_{j=0}^{d}(-1)^j(2j+1)u_n(\theta_j)u_m(\theta_j).$$

Equivalently,

$$(-1)^{d+n+m}(d+1)_4F_3\left[\begin{matrix}-n,\, n+1,\, -m,\, m+1\\ 1,\, d+2,\, -d\end{matrix};1\right]$$

$$=\sum_{j=0}^{d}(-1)^j(2j+1)_4F_3\left[\begin{matrix}-n,\, n+1,\, -j,\, j+1\\ 1,\, d+2,\, -d\end{matrix};1\right]$$

$$\times_4F_3\left[\begin{matrix}-m,\, m+1,\, -j,\, j+1\\ 1,\, d+2,\, -d\end{matrix};1\right].$$

By taking $m = 0$ in the previous corollary we get the next corollary, which we will use in Section 5.6 to prove a generalization of the correspondence observed in Section 3.5. Specifically, we will show an algebraic connection between the feasible strongly regular graph parameters

$$(4n^2, 2n^2 - n - 1, , n^2 - n - 2, n^2 - n)$$

and the Racah-type orthogonal polynomials (and Leonard pairs) given in Definition 2.3.1.

Corollary 5.5.2. *For all* k $(0 \le k \le d)$

$$\sum_{n=0}^{d}(-1)^n(2n+1)_4F_3\left[\begin{matrix}-n,\, n+1,\, -k,\, k+1\\ 1,\, d+2,\, -d\end{matrix};1\right] = (-1)^{d+k}(d+1). \tag{5.114}$$

Proof. If we take A, A^* to be the Racah type spin Leonard pair with $s = 0$ and $h = -\frac{6}{d(d+2)}$, then $k_i = \prod_{j=0}^{i}\frac{b_{j-1}}{c_j} = 2i+1$, hence $\tilde{\nu} = \sum_{i=0}^{d}(-1)^i k_i = (-1)^d(d+1)$.

From Def. 2.6.3 and (2.28) the matrix P has entries:

$$P_{i,j} = k_j u_i(\theta_j) = (2j+1)\,{}_4F_3\left[\begin{matrix} -i,\ i+1,\ -j,\ j+1 \\ 1,\ d+2,\ -d \end{matrix}\ ;\ 1\right].$$

By evaluating the hypergeometric series with $j = 0$, we see that first column of P is $(1, 1, \ldots, 1)^{\top}$. Hence the first column of U^*P is $(1, -1, \ldots, (-1)^d)^{\top}$.

Therefore, we get that the $(i, 0)$-entry of PU^*P is:

$$\sum_{n=0}^{d} (-1)^n (2n+1)\,{}_4F_3\left[\begin{matrix} -n,\ n+1,\ -i,\ i+1 \\ 1,\ d+2,\ -d \end{matrix}\ ;\ 1\right]$$

By Corollaries 5.4.6 and 5.4.4 we have that $PU^*P = \tilde{\nu}U^*PU^*$, and hence, this same $(i, 0)$-entry is equal to $\tilde{\nu}(-1)^{i+0}P_{i,0} = (d+1)(-1)^{d+i}$. □

It is worth noting that Corollary 5.5.2 can actually be proved by expressing the hypergeometric series as a sum, swapping the order of the summation, and then using the following sum formulas:

$$\sum_{n=0}^{d} (-1)^n (2n+1) \binom{n+h}{2h} = (-1)^d (d-h+1) \binom{d+h+1}{2h}, \tag{5.115}$$

$$\sum_{h=0}^{d} (-1)^h \binom{k}{h} \binom{k+h}{h} = (-1)^k. \tag{5.116}$$

This alternate proof is given in Appendix B.

5.6 Odd diameter fusion of Def. 2.3.1

In this section we show the following generalization from the example in Section 3.5 for all odd d (A similar formula follows for even values of d, but the $\tilde{p}^h_{i,j}$ values are not integers, and so they do not correspond to feasible parameters of a strongly regular graph).

Theorem 5.6.1. *Let $d = 2n - 1$, for some integer $n > 1$. Let $\mathcal{B}$ be the matrix algebra generated by the $\{B_i\}_{i=0}^d$ for the Leonard pair in Definition 2.3.1. Let $\tilde{\mathcal{B}}$ be the subalgebra generated by the matrices $\tilde{A}_0 = I$, $\tilde{A}_1 = \sum_{m=1}^{n} B_{2m}, \tilde{A}_2 = \sum_{m=1}^{n} B_{2m-1}$. Then the following hold.*

(i) There are values $\tilde{p}^h_{i,j}$ such that for all i, j, $\tilde{A}_i\tilde{A}_j = \sum_{h=0}^{2} \tilde{p}^h_{i,j}\tilde{A}_h$.

(ii) The matrices $(\tilde{B}_i)_{h,j} = \tilde{p}^h_{i,j}$, are given by

$$\tilde{B}_0 = I, \quad \tilde{B}_1 = \begin{pmatrix} 0 & 2n^2-n-1 & 0 \\ 1 & n^2-n-2 & n^2 \\ 0 & n^2-n & n^2-1 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 2n^2+n \\ 0 & n^2 & n^2+n \\ 1 & n^2-1 & n^2+n \end{pmatrix}. \tag{5.117}$$

(iii) These matrices have the matrix of eigenvectors (and eigenvalues),

$$\tilde{P} = \begin{pmatrix} 1 & 2n^2-n-1 & 2n^2+n \\ 1 & n-1 & -n \\ 1 & -(n+1) & n \end{pmatrix}. \tag{5.118}$$

In particular the matrix in (5.118) has columns (rows) that are the right (left) eigenvectors of the $\tilde{B}_i$. Also the (i, j) entry is the eigenvalue of B_j (B_i) of the ith column (jth row) vector, of $\tilde{P}$.

Proof. Since the Leonard pairs in Def. 2.3.1 are self-dual and the B_i are simultaneously diagonalizable, we can work with the B_i^*. Let $\tilde{A}_0^* = B_0^* = I$, $\tilde{A}_1^* = \sum_{m=1}^{n} B_{2m}^*, \tilde{A}_2^* = \sum_{m=1}^{n} B_{2m-1}^*$, i.e. the corresponding duals of the $\tilde{A}_i$. And since the diagonal of B_i^* is the ith column of P, we can think of them as these columns if we like, and hence the diagonal of $\tilde{A}_i^*$ as the corresponding sums of these columns. By (2.14) and the orthogonality relation (2.20) the sum of the columns of P has the 0th entry $\nu = (d+1)^2$ and all other entries 0. Hence all for $j > 0$ we have $(\tilde{A}_0^*)_{j,j} + (\tilde{A}_2^*)_{j,j} + (\tilde{A}_1^*)_{j,j} = 0$. Cor. 5.5.2, Def. 2.3.2, and (2.28) tell us $(\tilde{A}_0^*)_{j,j} + (\tilde{A}_2^*)_{j,j} - (\tilde{A}_1^*)_{j,j} = (-1)^{d+j}(d+1)$. The 0th entry of $\tilde{A}_2^*$ can be found by evaluating the sum of every other odd integer starting with 3 to $2(d+1)+1$, and similarly for $\tilde{A}_1^*$ the odd integers starting with 5 to $2d+1$.

Hence it is straightforward to check that for odd d the first diagonal entries of $\tilde{A}_1^*$ and $\tilde{A}_2^*$ are the entries $(0,1)$ and $(0,2)$ of the matrix $\tilde{P}$ in (5.118), and the other diagonal entries repeat with period 2, specifically with the entries in $(1,1),(2,1)$, and $(2,1),(2,2)$ of $\tilde{P}$ respectively.

Note that the B_i^* are the images of the B_i under a $\mathbb{K}$-algebra isomorphism, they form a commutative $\mathbb{K}$-algebra, and $\tilde{A}_0 = I$. So, to work out the $\tilde{p}_{i,j}^h$, one only needs to compute the products $(\tilde{A}_1^*)^2$, $(\tilde{A}_2^*)^2$, and $\tilde{A}_1^*\tilde{A}_2^*$. Because of the repeated entries, it suffices to look at the entry wise products of the columns of $\tilde{P}$. It is a straightforward calculation to see they come out to the entries in the matrices $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2$ from (5.117). $\square$

Note, this technique also allows one to compute the $\tilde{p}_{i,j}^h$, and the $\tilde{P}$ in the even diameter case, and $\tilde{B}_1$ will still be irreducible tridiagonal. However, some values will be half integers, and so will not correspond to intersection matrices, eigenvectors, and eigenvalues of feasible strongly regular graphs.

The intersection parameters and eigenvalues of a strongly regular graph are determined by their parameters (v, k, λ, μ) [5, 19]. It is straightforward to check that if an SRG exists with parameters $(4n^2, 2n^2 - n - 1, n^2 - n - 2, n^2 - n)$, then it will have the the intersection matrices (5.117), and the distance matrices will have the eigenvalues in (5.118). Hence as a consequence of the previous theorem, we have the following corollary.

Corollary 5.6.2. *Suppose $d = 2n - 1$, where $n > 1$ is an integer, and recall the intersection matrices $B_0 = I, B_1, \ldots, B_d$ from the Leonard pair in Def. 2.3.1. Let $\tilde{\mathcal{M}}$ be the subalgebra generated by the following matrices, $\tilde{A}_0 = I$, $\tilde{A}_1 = \sum_{m=1}^{n} B_{2m}$, $\tilde{A}_2 = \sum_{m=1}^{n} B_{2m-1}$. Then the following hold.*

(i). There are structure constants $\tilde{p}^h_{i,j}$ such that for all i, j,

$$\tilde{A}_i \tilde{A}_j = \sum_{h=0}^{2} \tilde{p}^h_{i,j} \tilde{A}_h,$$

(ii). The matrices $\{\tilde{B}_i\}_{i=0}^{d}$ with entries $(\tilde{B}_i)_{h,j} = \tilde{p}^h_{i,j}$, are the feasible intersection matrices of the feasible strongly regular graph parameters $(v, k, \lambda, \mu) = (4n^2, 2n^2 - n - 1, n^2 - n - 2, n^2 - n)$.

(iii). The matrix $\tilde{B}_1$ is irreducible tridiagonal.

For odd diameters satisfying:

$$d \in \{1, 3, 5, 7, 9, 11, 15, 17, 19, 23, 27, 31, 35\}, \tag{5.119}$$

the intersection matrices and $\tilde{P}$ in (5.117) and (5.118) correspond to known strongly regular graphs [3],[8]. For a given $d = 2n-1$, if the $\tilde{A}_1$ corresponds to an intersection matrix of a distance regular graph, then any such graph is strongly regular with parameters:

$$(4n^2, 2n^2 - n - 1, n^2 - n - 2, n^2 - n). \tag{5.120}$$

Or, similarly, we could swap B_1 and B_2 and get the parameters of the complement:

$$(4n^2, 2n^2 + n, n^2 + n, n^2 + n). \tag{5.121}$$

These graphs are also known as the *maximal energy graphs*, which means that the sum of magnitudes of the eigenvalues equal the maximum value possible for a given number of vertices [31]. Haemers [22] conjectured that these graphs exist for all n. The existence of a strungly regular graph for a given n is also equivalent to the existence of a regular graphical Hadamard matrix of negative type of order $4n^2$, and in [23] Haemers proved they exist whenever n is a perfect square.

Other infinite families are known. Fickus et.al., in [16], show that for $d = 2n-1$ and $n = 2^{j-1}$ for some $j \geq 2$, there exist strongly regular graphs with the parameters given in equations (5.120) and (5.121). Odd values of n in general appear to be open. In particular, it is unknown [3] whether or not such a graph exists when

$$2n - 1 = d \in \{13, 21, 25, 29, 33\}. \tag{5.122}$$

As far as we know, this connection between the pseudo-distance matrices of this set of Leonard pairs and this family of feasible strongly regular graph parameters was unobserved.

In the next section, we prove a generalization of Corollary 5.4.6 for other families of spin Leonard pairs.

5.7 Signed sum of products for certain Boltzmann pairs

In Section 5.4 we proved that, for the spin Leonard pairs of Racah type (as given in Lemma 5.2.8), we have $PNP = \tilde{\nu}NPN$, where P is from Definition 2.6.3, N is from (5.105), and $\tilde{\nu} = \sum_{i=0}^{d}(-1)^i k_i$. If we assume the work of Curtin [11], a similar formula can be shown to hold for a larger class of Leonard pairs. We direct the interested reader to [11] for details. In [11], Curtin gave formulas for the Boltzmann pairs of all spin Leonard pairs. We will show that, under certain conditions, we can derive a simplified formula for these Boltzmann pairs. The equivalence of these matrices (up to a scalar) implies a formula for the eigenvectors of the matrix P from Def. 2.6.3. It also gives us a proof of an identity involving a signed sum of products of certain hypergeometric series given in Corollary 5.7.3.

The condition needed for this simplification is valid for all known spin Leonard pairs that have Racah type, Krawtchouk type (when parameter $v = -1$), or Bannai-Ito type [11, Lems. 1.8-1.11].

Theorem 5.7.1. *Let A, A^* be a spin Leonard pair, with character table P, and with Boltzmann pair W, W^* as given in [11, Theorem 1.17]. If $(W^*)^{-1} = W^*$, then the pair:*

$$U^* = W^*, \tag{5.123}$$

$$U = W^*PW^* = U^*PU^*, \tag{5.124}$$

forms a Boltzmann pair for A, A^*. Note: Lemma [11, 1.17] tells us that $W^* = U^*$ is diagonal.

Proof. We can take $A, A^*, A^\diamond$ to be a canonical modular Leonard triple as in Lemma 5.2.7 and [10, Lemmas 1.6, 1.7]. By [40, Theorem 5.10], spin Leonard pairs are self-dual. Hence the matrix P with entries $P_{i,j} = v_j(\theta_i)$ diagonalizes the matrix of A into A^*. This is since $v_j(\theta_i) = k_j u_j(\theta_i)$, $\theta_j v_i(\theta_j) = c_{i+1}v_{i+1}(\theta_j) + a_i v_i(\theta_j) + b_{i-1}v_{i-1}(\theta_j)$, and $\theta_j u_i(\theta_j) = b_i u_{i+1}(\theta_j) + a_i u_i(\theta_j) + c_i u_{i-1}(\theta_j)$. Hence, P is a matrix of left and right eigenvectors of A. The orthogonality relations in this case give $PP = \nu I$, where $\nu = \sum_{i=0}^{d} k_i$, hence $P^{-1} = \frac{1}{\nu}P$.

So we have $A^* = \frac{1}{\nu}PAP$ and $A = \frac{1}{\nu}PA^*P$, and so the duality σ from $\langle A, A^*\rangle \to \langle A^*, A\rangle$ is defined by $\sigma(X) = \frac{1}{\nu}PXP$.

(i) Since U^* and A^* are diagonal, we have $U^*A^* = A^*U^*$.

(ii) If it is the case that $(U^*)^{-1} = U^*$, then

$$\begin{aligned} U^{-1}(U^*)^{-1}AU^*U &= (U^*P^{-1}U^*)U^*AU^*(U^*PU^*) = U^*P^{-1}IAIPU^* \\ &= U^*P^{-1}APU^* = U^*A^*U^* = A^*, \end{aligned}$$

since A^* and U^* are diagonal. Hence, we have that $(U^*)^{-1}AU^* = UA^*U^{-1}$.

(iii) By [11, Lemma 1.6] $A^\diamond = (U^*)^{-1}AU^*$, and $(A^\diamond)^* = U^*A(U^*)^{-1}$.

Hence we have

$$U^*A(U^*)^{-1} = (A^\diamond)^* = \sigma(A^\diamond) = \sigma((U^*)^{-1}AU^*) = \frac{1}{\nu}P(U^*)^{-1}AU^*P.$$

This means $A = \frac{1}{\nu}(U^*)^{-1}P(U^*)^{-1}AU^*PU^*$, so finally

$$UA = U^*PU^*A = AU^*PU^* = AU.$$

Therefore, U^*, U is a Boltzmann pair for A, A^*. □

The assumption of the theorem above holds in the case that we have a spin Leonard pair of what Curtin refers to as Type II, IV, or V. Specifically, this result applies to the spin Leonard pairs of

(1) Racah type (with $h^* = h$, $s^* = s$, $r_1 = s/2$, $r_2 = 3s/2 + d + 1$ in the notation of [49, Ex. 5.10]),

(2) Krawtchouk type with $v = -1$ (or $r = \frac{3ss^*}{4}$ and $s = s^*$ in the notation of [49, Ex. 5.13]),

(3) Bannai-Ito type (with $h^* = h$, $s^* = s$, $r_1 = -3s/2 + d + 1$, $r_2 = -s/2$ in the notation of [49, Ex. 5.14]).

In contrast, the formula given in [11, Theorem 1.17] gives the Boltzmann pair $W^* = U^*, W = PU^*P$. However, as shown in [11, Theorem 1.18], for any other Boltzmann pair U^*, U, there exist scalars a and b such that $U^* = aW^*$, and $U = bW$. (There is an exception in the Bannai-Ito case of [11, Lemma 1.11], where there are two choices for W and W^*, as chosen in [11, Lemma 1.17]. However, they still satisfy $(W^*)^{-1} = W^*$, and the previous theorem and the next argument still hold for finding the coefficients a and b.) We defined U^* to be W^*, hence $a = 1$. The (0,0) entry of U is 1, and the $(0, 0)$ entry of W is $\tilde{\nu} = \sum_{i=0}^{d}(U^*)_{ii}k_i$, as in [50, Lemma 3.10]. Hence, we get $PU^*P = \tilde{\nu}U^*PU^*$, so $b = \frac{1}{\tilde{\nu}}$. Therefore, we have the following corollary.

Corollary 5.7.2. *Let A, A^* be a spin Leonard pair with character table P. Suppose P satisfies the condition to have Boltzmann pair $U^*, U = U^*PU^*$ as given in Theorem (5.7.1). Let $\tilde{\nu} = \sum_{i=0}^{d}(U^*)_{i,i}k_i$. Then*

$$PU^*P = \tilde{\nu}U^*PU^*. \tag{5.125}$$

Furthermore, the eigenvectors of P are given by the columns of U^*P, and the ith entry of the jth eigenvector v_j under the ordering implied by U^*P is given by

$$(U^*P)_{i,j} = U^*_{i,i}P_{i,j}, \tag{5.126}$$

which has eigenvalue $\lambda_j = \tilde{\nu}U^*_{j,j}$. □

Proof. We already proved (5.125), and the eigenvector result is proven from noticing that (5.125) gives the diagonalization of $P = \frac{1}{\tilde{\nu}}(U^*)^{-1}PU^*P(U^*)^{-1}$. □

Note that the above corollary also tells us that the eigenvectors of $U = U^*PU^*$ are the columns of P, and for the jth column of P, which we denote by v_j, we have

$$Uv_j = \tilde{\nu}U^*_{j,j}v_j. \tag{5.127}$$

We have the following immediate consequence, which is a generalization of Corollary 5.5.1.

Corollary 5.7.3. *Let u_i be the orthogonal polynomials associated with spin Leonard pairs of Racah type, Krawtchouk type with $v = -1$ ($r = 3$ and $s = s^* = 2$), or Bannai-Ito type, [11, Lems. 1.8-1.11]. Let U^* be defined as in Theorem 5.7.1 and let $\tilde{\nu} = \sum_{i=0}^{d} U^*_{i,i}k_i$. Then*

$$\sum_{j=0}^{d} U_{jj}^{*} k_j u_j(\theta_n) u_j(\theta_m) = \tilde{\nu} U_{n,n}^{*} U_{m,m}^{*} u_m(\theta_n).$$

6 Directions for further research

As we mentioned in Section 3.4, for a Leonard pair coming from the intersection parameters of a distance regular graph, the intersection parameters are nonnegative integers and the Krein parameters are nonnegative real numbers. Although we were not able to find a direct combinatorial interpretation of the Leonard pairs in Def. 2.3.1, we were able to prove nonnegativity of the intersection numbers. Since these Leonard pairs are self-dual, the intersection numbers and Krein parameters are the same. As mentioned, this was used to solve a previously unknown special case of [50, Problems 11.3, 11.5]. We hope this might serve as a possible direction to follow for further understanding of these two problems.

In the future, we also hope to further explore the general Racah-type Leonard pairs and learn what can be said about the nonnegativity or factorization of the intersection and Krein parameters. We would also like to explore the q-Racah type Leonard pairs and see if some of the techniques we have presented can be used in combination with some of the fundamental formulas associated with the q-Racah coefficients and $U_q(sl(2))$.

We would also like to further understand the algebraic connection between the fusion matrices and the strongly regular graphs mentioned in Section 5.6, and to further explore the fusions in the general Racah, Krawtchouk, and Bannai-Ito type spin-Leonard pairs. It would be interesting to learn if any algebras isomorphic to adjacency algebras of other graphs arise.

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Appendix A Irreducible matrices

The following theorem is proved in [25], and the history and origins of the following definition are discussed in [44]. The following references are also of interest: [[17], [20], [24], [45]].

Theorem A.1.1. *Let A be an $n \times n$ matrix. The following are equivalent:*

(1) There exists a permutation of A to block lower-triangular form with more than one block.

(2) There exists a permutation of A to block upper-triangular form with more than one block.

(3) The digraph with adjacency matrix $\mathbb{1}_A$ (the nonzero indicator matrix of A, See [25, Page 399]), is not strongly connected.

Definition A.1.2. An $n \times n$ matrix is called *reducible* if it satisfies any of the three equivalent conditions in Theorem A.1.1.

A matrix is *irreducible* if it is not reducible.

Lemma A.1.3. *A $n \times n$ tridiagonal matrix A is irreducible if and only if it has nonzero super- and sub-diagonal.*

Proof. If A has nonzero super- and sub-diagonal, the directed graph D with adjacency matrix $\mathbb{1}_A$ contains a two way directed path through all vertices, and hence D is connected. Hence A is irreducible by Theorem A.1.1, which proves one direction.

Suppose some $b_i = 0$ (the $c_i = 0$ case is symmetrical). Then we have

$$
A = \left(\begin{array}{ccccc|ccccc}
a_0 & b_0 & 0 & \cdots & 0 & 0 & & \cdots & & 0 \\
c_1 & a_1 & b_1 & \ddots & \vdots & & & & & \\
0 & \ddots & \ddots & \ddots & 0 & \vdots & & \iddots & & \vdots \\
\vdots & \ddots & c_{i-1} & a_{i-1} & b_{i-1} & & & & & \\
0 & \cdots & 0 & c_i & a_i & 0 & & \cdots & & 0 \\
\hline
0 & & \cdots & 0 & c_{i+1} & a_{i+1} & b_{i+1} & 0 & \cdots & 0 \\
 & & & 0 & 0 & c_{i+2} & a_{i+2} & b_{i+2} & \ddots & \vdots \\
\vdots & & \iddots & & \vdots & 0 & \ddots & \ddots & \ddots & 0 \\
 & & & & & \vdots & \ddots & c_{d-1} & a_{d-1} & b_{d-1} \\
0 & & \cdots & & 0 & 0 & \cdots & 0 & c_d & a_d
\end{array}\right)
$$

$$
= \left(\begin{array}{c|c} X & 0 \\ \hline Z & Y \end{array}\right).
$$

Hence we have that A is reducible by Theorem A.1.1, which completes the other direction. ∎

Appendix B Alternate proof of Cor. 5.5.1

First we evaluate two sums (5.115), (5.116) mentioned at the end of Sec. 5.5.

Proposition B.1.1.

$$\sum_{n=0}^{d}(-1)^n(2n+1)\binom{n+h}{2h} = (-1)^d(d-h+1)\binom{d+h+1}{2h} \tag{B.128}$$

Proof. Looking at a summand we have,

$$\begin{aligned}
(-1)^n(2n+1)\binom{n+h}{2h} &= (-1)^n\binom{n+h}{2h}(2n+1-h+h)\\
&= -(-1)^n\binom{n+h}{2h}(-h-n-1+h-n)\\
&= -(-1)^n\binom{n+h}{2h}\left(\frac{-(h-n-1)(h+n+1)}{(h-n-1)}+h-n\right)\\
&= -(-1)^n\binom{n+h}{2h}\left((h-(n+1))\frac{(n+h+1)}{(n+h-2h+1)}+h-n\right)\\
&= -(-1)^n\left((h-(n+1))\frac{(n+h+1)}{(n+h-2h+1)}\binom{n+h}{2h}+(h-n)\binom{n+h}{2h}\right)\\
&= -(-1)^n\left((h-(n+1))\binom{n+h+1}{2h}+(h-n)\binom{n+h}{2h}\right)\\
&\qquad\qquad \left(\text{since } \tbinom{n+1}{k}=\tfrac{(n+1)}{(n-k+1)}\tbinom{n}{k}\right)\\
&= (-1)^{n+1}(h-(n+1))\binom{(n+1)+h}{2h}-(-1)^n(h-n)\binom{n+h}{2h}.
\end{aligned}$$

This expression telescopes in n, hence summing from $n=0$ to d, evaluates to $(-1)^{d+1}(h-(d+1))\binom{(d+1)+h}{2h} = (-1)^d(d-h+1\binom{d+h+1}{2h}$, as desired. □

Proposition B.1.2.

$$\sum_{h=0}^{d}(-1)^h\binom{k}{h}\binom{k+h}{h} = (-1)^k \tag{B.129}$$

Proof.

First note that $\sum_{h=0}^{d}(-1)^h\binom{k}{h}\binom{k+h}{h} = \sum_{h=0}^{k}(-1)^h\binom{k}{h}\binom{k+h}{h}$. For a base case $k=0$

we have

$$\sum_{h=0}^{0}(-1)^h\binom{0}{h}\binom{0+h}{h} = 1 = (-1)^0.$$

We will show that the sum of interest of value k and $k+1$ are negatives of each other. We will do this by showing that the summand,

$$(-1)^h\binom{k+1}{h}\binom{k+1+h}{h} + (-1)^h\binom{k}{h}\binom{k+h}{h} \tag{B.130}$$

is equivalent to the telescoping sum,

$$\frac{(-1)^{h+1}2(h+1)^2}{(k+1)(h+1-k-1)}\binom{k}{h+1}\binom{k+h+1}{h+1} - \frac{(-1)^h 2h^2}{(k+1)(h-k-1)}\binom{k}{h}\binom{k+h}{h}. \tag{B.131}$$

Hence summing (B.130) from $h=0$ to d, will evaluate to 0, which implies the result.

Towards this we have,

$$\begin{aligned}
(B.130) &= (-1)^h\left(\binom{k+1}{h}\binom{k+1+h}{h} + \binom{k}{h}\binom{k+h}{h}\right)\\
&= (-1)^h\binom{k}{h}\binom{k+h}{h}\left(\frac{(k+1)}{(k-h+1)}\frac{(k+h+1)}{(k+1)}+1\right)\\
&\qquad \text{(by } \tbinom{n+1}{k} = \tfrac{(n+1)}{(n-k+1)}\tbinom{n}{k}\text{)}\\
&= (-1)^h\binom{k}{h}\binom{k+h}{h}\left(\frac{(k+h+1)}{(k-h+1)}+1\right)\\
&= (-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{(k-h+1)}((k+h+1)+(k-h+1))\\
&= -(-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{(h-k-1)}(2(k+1))\\
&= -(-1)^h\frac{2\binom{k}{h}\binom{k+h}{h}}{(h-k-1)(k+1)}(k+1)^2\\
&= -(-1)^h\frac{2\binom{k}{h}\binom{k+h}{h}}{(h-k-1)(k+1)}\big(-(h-k-1)(k+1+h)+h^2\big)\\
&= -(-1)^h\frac{2\binom{k}{h}\binom{k+h}{h}}{(k+1)}\left(-(k+h+1)+\frac{h^2}{(h-k-1)}\right)\\
&= -(-1)^h\frac{2\binom{k}{h}\binom{k+h}{h}}{(k+1)}\left(\frac{(h+1)^2}{(h-k)}\frac{(k-h)}{(h+1)}\frac{(k+h+1)}{(h+1)}+\frac{h^2}{(h-k-1)}\right)
\end{aligned}$$

$$= \frac{(-1)^{h+1}2(h+1)^2}{(k+1)(h+1-k-1)}\binom{k}{h+1}\binom{k+h+1}{h+1}$$
$$- \frac{(-1)^h 2h^2}{(k+1)(h-k-1)}\binom{k}{h}\binom{k+h}{h}$$
(using $\binom{n}{k+1} = \frac{(n-k)}{(k+1)}\binom{n}{k}$, and $\binom{n+1}{k+1} = \frac{(n+1)}{(k+1)}\binom{n}{k}$)
$$= (B.131).$$

This completes the proof. □

We now give an alternate proof of Cor. 5.5.2, hence for $d \geq 1$, $0 \leq n, k, \leq d$,

$$\sum_{n=0}^{d}(-1)^n(2n+1)_4F_3\left[\begin{matrix}-n,\ n+1,\ -k,\ k+1 \\ 1,\ d+2,\ -d\end{matrix}\ ;1\right] = (-1)^{d+k}(d+1). \qquad (5.114)$$

Proof.

$$\sum_{n=0}^{d}(-1)^n(2n+1)_4F_3\left[\begin{matrix}-n,\ n+1,\ -k,\ k+1 \\ 1,\ d+2,\ -d\end{matrix}\ ;1\right]$$
$$= \sum_{n=0}^{d}(-1)^n(2n+1)\sum_{h=0}^{n}\frac{(-n)_h(n+1)_h(-k)_h(k+1)_h}{h!(1)_h(d+2)_h(-d)_h}$$
$$= \sum_{n=0}^{d}(-1)^n(2n+1)\sum_{h=0}^{n}(-1)^h\frac{\binom{n}{h}\binom{n+h}{h}\binom{k}{h}\binom{k+h}{h}}{\binom{d}{h}\binom{d+1+h}{h}}$$
$$= \sum_{h=0}^{n}(-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{\binom{d}{h}\binom{d+1+h}{h}}\sum_{n=0}^{d}(-1)^n(2n+1)\binom{n}{h}\binom{n+h}{h}$$
$$= \sum_{h=0}^{n}(-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{\binom{d}{h}\binom{d+h}{h}\frac{d+h+1}{d+1}}\sum_{n=0}^{d}(-1)^n(2n+1)\binom{n}{h}\binom{n+h}{h} \quad \left(\binom{n+1}{k} = \frac{n+1}{n-k+1}\binom{n}{k}\right)$$
$$= \sum_{h=0}^{n}(-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{\binom{2h}{h}\binom{d+h}{2h}\frac{d+h+1}{d+1}}\binom{2h}{h}\sum_{n=0}^{d}(-1)^n(2n+1)\binom{n+h}{2h}$$
$$\left(\binom{n}{k}\binom{n+k}{k} = \binom{2k}{k}\binom{n+k}{2k}\right)$$
$$= (d+1)\sum_{h=0}^{n}(-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{\binom{d+h}{2h}(d+h+1)}\sum_{n=0}^{d}(-1)^n(2n+1)\binom{n+h}{2h}$$
$$= (d+1)\sum_{h=0}^{n}(-1)^h\frac{\binom{k}{h}\binom{k+h}{h}}{\binom{d+h}{2h}(d+h+1)}\left((-1)^d(d-h+1)\binom{d+h+1}{2h}\right)$$
(by Prop. B.1.1)

$$= (-1)^d(d+1)\sum_{h=0}^{n}(-1)^h \frac{\binom{k}{h}\binom{k+h}{h}}{\binom{d+h}{2h}(d+h+1)}\left(\frac{(d-h+1)(d+h+1)}{(d-h+1)}\binom{d+h}{2h}\right)$$

$$\left(\binom{n+1}{k} = \tfrac{n+1}{n-k+1}\binom{n}{k}\right)$$

$$= (-1)^d(d+1)\sum_{h=0}^{n}(-1)^h\binom{k}{h}\binom{k+h}{h}$$

$$= (-1)^{d+k}(d+1) \qquad \text{(by Prop. B.1.2)}$$

□