Hypergraphs With a Unique Perfect Matching

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Under the direction of
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This presentation discusses the paper “On the maximum number of edges in a hypergraph with a unique perfect matching” written by:

Deepak Bal  
Andrzej Dudek  
Zelealem B. Yilma
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A **graph** \( G \) is a finite set of vertices \( \mathcal{V} \) along with a set of edges \( \mathcal{E} \) where every edge is a set containing exactly two vertices.
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A **hypergraph** $G$ is a finite set of vertices $\mathcal{V}$ along with a set of edges $\mathcal{E} \subseteq \mathcal{P}\mathcal{V} \setminus \{\emptyset\}$ (where $\mathcal{P}\mathcal{V}$ denotes the power set of $\mathcal{V}$) such that no two edges in $\mathcal{E}$ are equal as sets. A hypergraph is **$k$-uniform** if every $E \in \mathcal{E}$ has cardinality $k$. 
A matching in a hypergraph $G = (\mathcal{V}, \mathcal{E})$ is a set of pairwise disjoint edges $\{M_1, \ldots, M_m\}$.
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A **perfect matching** is a matching $\{M_1, \ldots, M_m\}$ such that $\mathbb{V} = \bigcup_{i=1}^{m} M_i$. In other words, a perfect matching is a collection of edges that partition the vertex set.
For $k \geq 2$ and $m \geq 1$, let

$$b_{k,\ell} = \frac{\ell - 1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell - i)}{k}.$$
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**Theorem**

Let $\mathcal{H}_m = (\mathcal{V}_m, \mathcal{E}_m)$ be a $k$-uniform hypergraph with $km$ vertices and unique perfect matching. Then

$$|\mathcal{E}_m| \leq f(k, m)$$

where

$$f(k, m) = m + b_{k,2} \binom{m}{2} + b_{k,3} \binom{m}{3} + \cdots + b_{k,k} \binom{m}{k}.$$ 

Moreover, this bound is tight.
Values of $f(k, m)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>2</td>
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<td>1</td>
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<td>1</td>
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<td>1878</td>
<td>10504</td>
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Outline

1 Construction

2 Proof of the Upper Bound

3 Application
We create a sequence of $k$-uniform hypergraphs attaining the claimed bound through an iterative construction.

First, define $H^*_{1}$ as the hypergraph with $k$ vertices and one edge containing all the vertices. To create $H^*_{m}$, start with $H^*_{m-1}$ and...

- Add $k-1$ new vertices
- Add every edge that intersects at least one of these new vertices
- Add 1 new pendant vertex
- Add the edge that contains the $k$ new vertices including the pendant.
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- Add the edge that contains the \( k \) new vertices including the pendant.
Example $k = 2$

Start with $\mathcal{H}_1^*$. 
Example \( k = 2 \)

Add \( k - 1 = 1 \) new vertex and connect it to all previous vertices.
Example \( k = 2 \)

Add a pendant vertex and an edge containing "new" vertices. This is now \( \mathcal{H}_2^* \).
Repeat the process again to create the next hypergraph. Add \( k - 1 = 1 \) new vertex and connect it to all previous vertices.
Example $k = 2$

Add a pendant vertex and an edge containing “new” vertices. This is now $\mathcal{H}_3^*$. 
Example $k = 3$

Start with $\mathcal{H}_1^*$. 
Example $k = 3$

Add $k - 1 = 2$ new vertices.
Example $k = 3$

Add all edges that contain some “new” vertex. Add edge $\{2, 3, 4\}$. 
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Example $k = 3$
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Add a pendant vertex.
Example $k = 3$

Add edge \( \{4, 5, 6\} \) containing the “new” vertices. This is now \( \mathcal{H}_2^* \).
Theorem

There is a unique perfect matching in $\mathcal{H}_m^*$. 
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[Pictures depict the proof in the 2-uniform case.]

- Proof by induction
Theorem

There is a unique perfect matching in $H^*_m$.

Proof by induction

Base Case: Trivial. This hypergraph only has one edge which contains every vertex.
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Inductive Step: The edge $E$ incident with the pendant vertex in $\mathcal{H}_m^*$ must be included in any perfect matching.
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- **Inductive Step**: The edge $E$ incident with the pendant vertex in $\mathcal{H}_m^*$ must be included in any perfect matching.
- No other edge in a perfect matching can intersect $E$. This excludes all edges incident with some “new” vertex.
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Inductive Step: The edge $E$ incident with the pendant vertex in $\mathcal{H}_m^*$ must be included in any perfect matching.

No other edge in a perfect matching can intersect $E$. This excludes all edges incident with some "new" vertex.

After eliminating such edges, we are left with a hypergraph isomorphic to $\mathcal{H}_{m-1}^*$ which has a unique perfect matching by induction hypothesis.
Theorem

The \( k \)-uniform hypergraph \( \mathcal{H}^*_m \) attains the edge bound presented in the main corollary.
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- There are two methods to count the number of edges in $\mathcal{H}_m^*$.
- Method 1: Directly track the number of edges in the hypergraphs based upon how they were constructed.
- Set up a recurrence relation:

  $$ (\# \text{Edges in } \mathcal{H}_m^*) = (\# \text{New Edges}) + (\# \text{Edges in } \mathcal{H}_{m-1}^*) $$

  This is reminiscent of

  $$ a_n = \text{(stuff)} + a_{n-1} $$
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- This equation can be solved by induction or by using recurrence relation solving strategies.
The $k$-uniform hypergraph $\mathcal{H}^*_m$ attains the edge bound presented in the main corollary.

Unfortunately, the most obvious way to solve this equation yields

$$f(k, m) = m + \sum_{i=1}^{m-1} \left[ \binom{k(i + 1) - 1}{k} - \binom{ki}{k} \right]$$
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- This is a true but different formula. We need to show this is equivalent to the formula presented in the main theorem. Algebraically showing the equivalence of these two formulas is difficult because they use summations and include binomial coefficients.
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- This is a true but different formula. We need to show this is equivalent to the formula presented in the main theorem. Algebraically showing the equivalence of these two formulas is difficult because they use summations and include binomial coefficients.

- Additionally, induction-based proofs are rarely enlightening as to the true meaning of formulas.
Theorem

The $k$-uniform hypergraph $\mathcal{H}_m^*$ attains the edge bound presented in the main corollary.

- Method 2: Count edges in $\mathcal{H}_m^*$ directly based upon the structure of the hypergraph without comparing it to $\mathcal{H}_{m-1}^*$. 
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- Method 2: Count edges in $\mathcal{H}_m^*$ directly based upon the structure of the hypergraph without comparing it to $\mathcal{H}_{m-1}^*$.
- This requires some clever counting techniques such as the inclusion-exclusion principle. However, it does properly establish the correct formula for the number of edges in this hypergraph.
Theorem (Lovász)

For $k = 2$ any graph with a unique perfect matching attaining the edge bound is isomorphic to $H^*_m$. 
Uniqueness of Construction?

**Theorem (Lovász)**

For $k = 2$ any graph with a unique perfect matching attaining the edge bound is isomorphic to $\mathcal{H}_m^*$.  

**Theorem**

For $k \geq 3$ and $m \geq 2$, there exist hypergraphs which have a unique perfect matching and attain the edge bound that are not isomorphic to $\mathcal{H}_m^*$.  

Uniqueness of Construction?

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Suppose the above depicts a portion of a graph that has a unique perfect matching. The solid edges represent matching edges.
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Two-Switch Example:

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- We cannot include both of the dashed edges in the graph. Otherwise, the perfect matching would not be unique:
- Start with the original perfect matching and discard the solid edges. Instead, trade them for the dashed edges to create a distinct perfect matching.
- Since we are not allowed to have both of the dashed edges in the graph, the total number of edges becomes constrained.
Hypergraph Generalization Example:

The top left image depicts part of a hypergraph with a perfect matching. The edges shown are part of the perfect matching. The top right image depicts the same vertices. Suppose the edges in this image were also present in the hypergraph. Start with the perfect matching. Remove the “matching edges” and include the “covering edges.” This creates a distinct perfect matching.

By uniqueness of the perfect matching, no such covering is allowed in the hypergraph, constraining the total number of possible edges.
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Suppose $\mathcal{L} = \{E_1, \ldots, E_\ell\}$ with $1 \leq \ell \leq k$ is a collection of disjoint edges. A collection of $k$-sets $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ is a covering of $\mathcal{L}$ if
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- Both $\mathcal{L}$ and $\mathcal{C}$ partition the same set of vertices.
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Definition

Define $\mathcal{L}$ as above and let $F \subseteq \bigcup \mathcal{L}$ be a $k$-set that intersects every edge of $\mathcal{L}$. The **ordered type** of $F$ is $\vec{b} = (b_1, \ldots, b_\ell)$ where $b_i = |F \cap E_i|$ for $1 \leq i \leq \ell$. The **unordered type** (abbreviated type) of $F$ is the unique rearrangement of the ordered type $(b_1, \ldots, b_\ell)$ of $F$ such that the entries appear in nonincreasing order.
Suppose $\mathcal{L} = \{E_1, \ldots, E_\ell\}$ with $1 \leq \ell \leq k$ is a collection of disjoint edges. A collection of $k$-sets $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ is a covering of $\mathcal{L}$ if

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Coverings Example

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<tr>
<th>$k$-set</th>
<th>Ordered Type</th>
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</tr>
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<tbody>
<tr>
<td>$F_1$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
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Coverings Example

\[ E_1 \quad E_2 \quad E_3 \quad E_4 \]

\[ F_1 \quad F_2 \]

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Coverings Example

\begin{tabular}{|c|c|c|}
\hline
\textbf{$k$-set} & \textbf{Ordered Type} & \textbf{(Unordered) Type} \\
\hline
$F_1$ & (1, 3, 2, 2) & (3, 2, 2, 1) \\
$F_2$ & & \\
\hline
\end{tabular}
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Coverings Example

\[ \begin{align*}
E_1 & \quad E_2 & \quad E_3 & \quad E_4 \\
\end{align*} \]

\[ \begin{array}{c}
F_1 \\
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Consider a 3-uniform complete hypergraph on 9 vertices. Let \( \mathcal{M} = \{M_1, M_2, M_3\} \) be a perfect matching as depicted above.
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To organize our search, we start by considering the possible types of edges in the hypergraph:
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$$\begin{array}{|c|c|c|}
\hline
(1,1,1) & & \\
\hline
\end{array}$$
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| (1,1,1) | (2,1,0) |
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(1,1,1) & (2,1,0) & (3,0,0) \\
\end{array}
\]

Since edges in a covering must intersect every matching edge, we only consider edges of type \((1,1,1)\).
We count the number of coverings of \( \{M_1, M_2, M_3\} \) that only use edges of type \((1, 1, 1)\). Suppose \( \{A, B, C\} \) is such a covering.
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We label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in \(M_1\) are labeled as above.
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We label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in \( M_1 \) are labeled as above.

There are 6 ways to assign labels to \( M_2 \) and 6 ways to assign labels to \( M_3 \), giving a total of 36 coverings.
Let $C$ be a fixed edge of type $(1, 1, 1)$ as depicted above.
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Again, we label each vertex by the covering edge that contains it. After possibly renaming the covering edges, we assume the vertices in $M_1$ are labeled as above.

There are 2 ways to assign labels to $M_2$ and 2 ways to assign labels to $M_3$, giving a total of 4 coverings that contain edge $C$. 
By symmetry, every edge of type \((1,1,1)\) is contained in 4 coverings.
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Removing 1 edge from the hypergraph breaks at most 4 coverings.

Removing 9 edges from the hypergraph breaks at most 36 coverings.

In order to remove all 36 coverings from the hypergraph, we must remove at least 9 edges of type \((1, 1, 1)\).
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Main Theorem Proof Sketch

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- Removing 2 edges from the hypergraph breaks at most 8 coverings.
- Removing 9 edges from the hypergraph breaks at most 36 coverings.
- In order to remove all 36 coverings from the hypergraph, we must remove at least 9 edges of type \((1, 1, 1)\).
We must also remove coverings of \( \{ M_1, M_2 \} \).
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- Every edge in a covering of \( \{M_1, M_2\} \) is of type \((2,1)\).
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To specify an edge \( E \) with \(|E \cap M_1| = 1\) and \(|E \cap M_2| = 2\), pick one vertex from \( M_1 \) and 2 vertices from \( M_2 \). There are \( 3 \cdot \binom{3}{2} = 9 \) such edges.
Main Theorem Proof Sketch

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- All coverings are of the form \( \{E, \overline{E}\} \) for an edge as previously described. Hence, there are also 9 coverings.
Main Theorem Proof Sketch

- Given any edge $F$ of type $(2, 1)$, $F$ lies on exactly one covering $\{F, \overline{F}\}$. Caution: we may have $|F \cap M_1| = 1$ or $|F \cap M_1| = 2$.

![Diagram of two sets $M_1$ and $M_2$ with edges](attachment:diagram.png)
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In order to break all 9 coverings, we must remove at least 9 edges of type $(2, 1)$.

A symmetric situation occurs for any pair of 2 matching edges ($\{M_1, M_2\}, \{M_1, M_3\}$, or $\{M_2, M_3\}$). Hence we must remove at least $\binom{3}{2} \cdot 9 = 27$ edges of type $(2, 1)$ from the hypergraph.
The complete hypergraph has \( \binom{9}{3} = 84 \) edges.

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There are at most \(84 - 9 - 27 = 48\) edges remaining in the hypergraph.
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For \( k \geq 2 \) and \( m \geq 1 \), let

\[
b_{k,\ell} = \frac{\ell - 1}{\ell} \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i} \binom{k(\ell - i)}{k}.
\]

**Theorem**

Let \( \mathcal{H}_m = (\mathcal{V}_m, \mathcal{E}_m) \) be a \( k \)-uniform hypergraph with \( km \) vertices and unique perfect matching. Then

\[
|\mathcal{E}_m| \leq f(k, m)
\]

where

\[
f(k, m) = m + b_{k,2} \binom{m}{2} + b_{k,3} \binom{m}{3} + \cdots + b_{k,k} \binom{m}{k}.
\]

Moreover, this bound is tight.
Main Theorem Proof Sketch

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\[ f(3, 3) = 3 + 9\binom{3}{2} + 18\binom{3}{3} = 48. \]
Outline

1 Construction

2 Proof of the Upper Bound

3 Application
Benzene consists of 6 carbon atoms arranged in a hexagon.
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Electrons in Benzene Molecules

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- This graph has two perfect matchings. By symmetry, both of these have the same energy. Hence, the molecule resonates in between the two configurations.

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Thank You

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