Fractional Colorings and Zykov Products of graphs

Who? Nichole Schimanski

When? July 27, 2011

Graphs

A **graph**, G, consists of a vertex set, $V(G)$, and an edge set , $E(G)$.

- $V(G)$ is any finite set
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Example

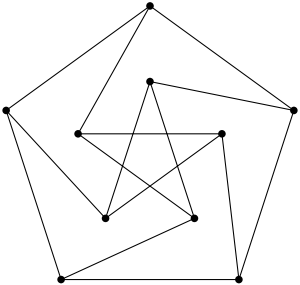

Figure: Peterson graph

Subgraphs

A **subgraph** H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

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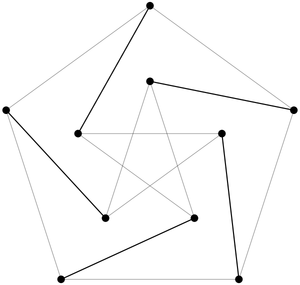

Figure: Subgraph of the Peterson graph

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An **induced subgraph**, H, of G is a subgraph with property that any two vertices are adjacent in H if and only if they are adjacent in G.

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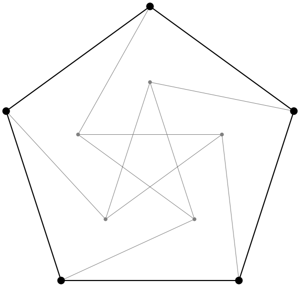

Figure: Induced Subgraph of Peterson graph

Independent Sets

A set of vertices, S, is said to be **independent** if those vertices induce a graph with no edges.

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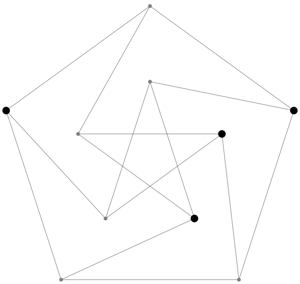

Figure: Independent set

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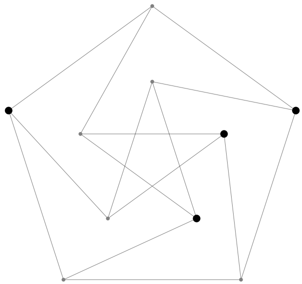

Figure: Independent set

- The set of all independent sets of a graph G is denoted $\mathscr{I}(G)$.

Weighting $\mathscr{I}(S)$

A **weighting** of $\mathscr{I}(G)$ is a function $w : \mathscr{I}(G) \to \mathbb{R}^{\geq 0}$.

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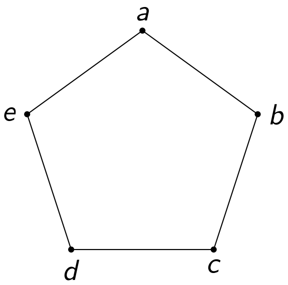

S	$w(S)$
$\{a\}$	1/3
$\{b\}$	1/3
$\{c\}$	1/3
$\{d\}$	1/3
$\{e\}$	1/3
$\{a,c\}$	1/3
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Figure: C_5 and a corresponding weighting

Fractional k-coloring

A **fractional k-coloring** of a graph, G, is a weighting of $\mathscr{I}(G)$ such that

- $\sum_{S \in \mathscr{I}(G)} w(S) = k$; and

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- $\sum_{S \in \mathscr{I}(G)} w(S) = k$; and
- For every $v \in V(G)$,

$$\sum_{\substack{S \in \mathscr{I}(G) \\ v \in S}} w(S) = w[v] \geq 1$$

Fractional k-coloring

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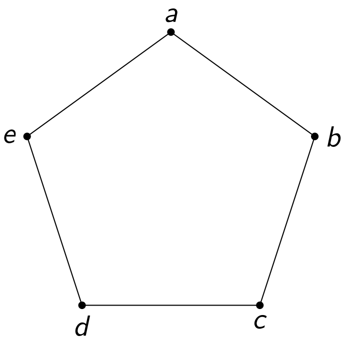

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Figure: A fractional coloring of C_5 with weight $10/3$

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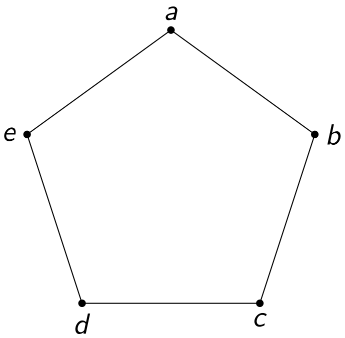

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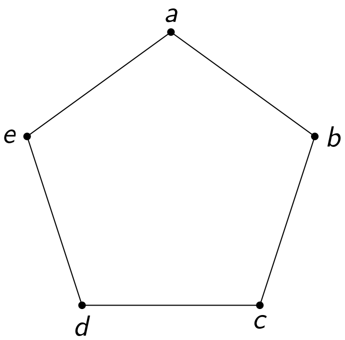

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- $\sum_{S \in \mathscr{I}(G)} w(S) = 10/3$
- $w[v] = 1$ *for every* $v \in V(G)$

Fractional Chromatic Number

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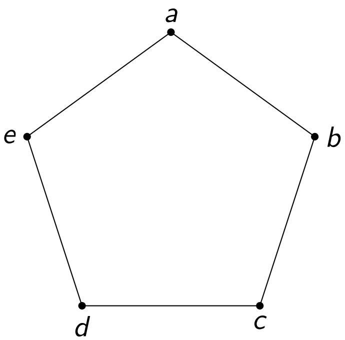

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Figure: A weighting C_5

Fractional Chromatic Number

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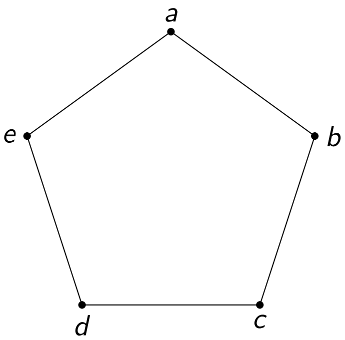

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Figure: A fractional $5/2$-coloring of C_5

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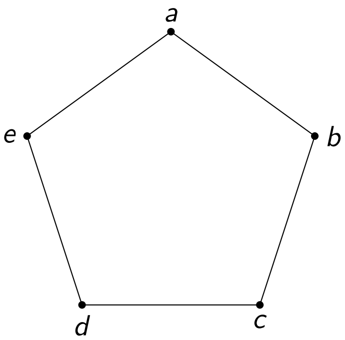

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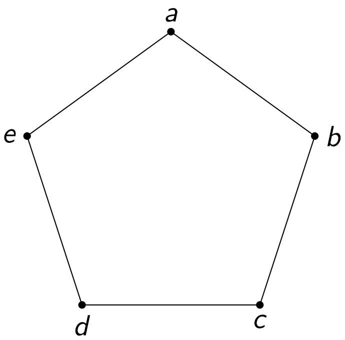

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- $\sum_{S \in \mathscr{I}(G)} w(S) = 5/2$
- $w[v] = 1$ for every $v \in V(G)$

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How do we know what the minimum is?

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- Linear Programming

Fractional Chromatic Number

How do we know what the minimum is?

- Linear Programming
- Formulas

Zykov Product of Graphs

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Figure: Drawings of P_2 and P_3

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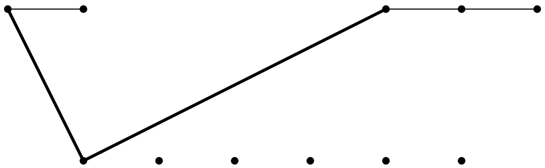

Figure: Constructing $\mathcal{Z}(P_2, P_3)$

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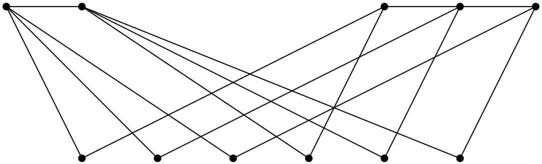

Figure: $\mathcal{Z}(P_2, P_3)$

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Figure: Drawing of $\mathcal{Z}_1$

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Figure: Drawings of $\mathcal{Z}_1$ and $\mathcal{Z}_2$

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Figure: Drawings of $\mathcal{Z}_1$, $\mathcal{Z}_2$, and $\mathcal{Z}_3$

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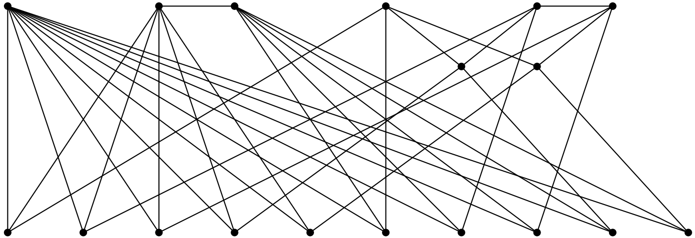

Figure: Drawing of $\mathcal{Z}_4$

Jacobs' Conjecture

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Corollary *For $n \geq 1$,*

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Example

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Example

- $\chi_f(\mathcal{Z}_1) = 1$
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Example

- $\chi_f(\mathcal{Z}_1) = 1$
- $\chi_f(\mathcal{Z}_2) = 1 + \frac{1}{1} = 2$
- $\chi_f(\mathcal{Z}_3) = 2 + \frac{1}{2} = \frac{5}{2}$

Verifying $\chi_f(C_5)$

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Notice that

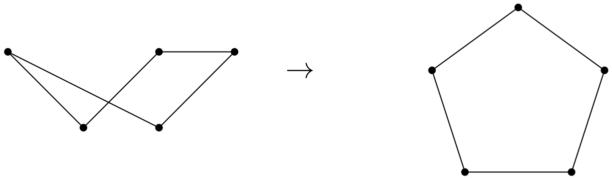

Figure: $\mathcal{Z}_3$ and C_5

Verifying $\chi_f(C_5)$

Notice that

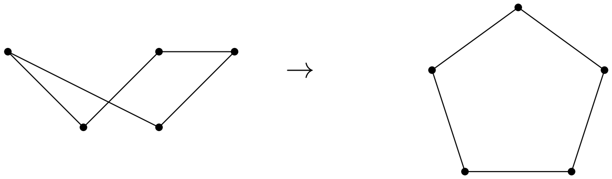

Figure: $\mathcal{Z}_3$ and C_5

- So, $\chi_f(\mathcal{Z}_3) = \chi_f(C_5) = 5/2$

The Main Result: Theorem 1

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Theorem *For $n \geq 2$, let $G_1, \ldots, G_n$ be graphs. Set $G := \mathcal{Z}(G_1, \ldots, G_n)$ and $\chi_i = \chi_f(G_i)$. Suppose also that the graphs G_i are numbered such that $\chi_i \leq \chi_{i+1}$. Then*

$$\chi_f(G) = \max\left(\chi_n, 2 + \sum_{i=2}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right)\right)$$

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Example

$$\begin{aligned} \chi_f(\mathcal{Z}(P_2, P_3)) &= \max\left(2, 2 + \left(1 - \frac{1}{2}\right)\right) \\ &= \max(2, \frac{5}{2}) \\ &= \frac{5}{2}. \end{aligned}$$

Lower Bound: $\chi_f(G) \geq \max(\chi_n, f(n))$

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Conclusion $\chi_f(G) \geq \chi_n$

Lower Bound: $\chi_f(G) \geq \max(\chi_n, f(n))$

Lemma *Let G be a graph and w a weighting of $\mathscr{X} \subseteq \mathscr{I}(G)$. Then, for every induced subgraph H of G, there exists $x \in V(H)$ such that*

$$w[x] \leq \frac{1}{\chi_f(H)} \sum_{S \in \mathscr{X}} w(S).$$

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- Construct $\mathcal{F}_2 = \{S \in \mathscr{I}(G) : S \cap \{x_1, x_2\} \neq \emptyset\}$ with the property $\sum_{S \in \mathcal{F}_2} w(S) \geq 1 + \left(1 - \frac{1}{\chi_2}\right) \sum_{S \in \mathcal{F}_1} w(S)$.

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Continue this process so that for all $k \in \{1, ..., n\}$,

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It follows that

$$\sum_{S \in \mathcal{F}_n} w(S) \geq 1 + \sum_{i=2}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k}\right) = f(n) - 1.$$

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Conclusion $\chi_f(G) \geq f(n)$

Upper Bound: $\chi_f(G) \leq \max(\chi_n, f(n))$

Special Sets and Cool Weightings

Special Sets *Let $\mathscr{M}(G) \subset \mathscr{I}(G)$ be the set of all maximal independent sets of G*

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$$\mathcal{F}_i := \{S \in \mathscr{M}(G) | S \cap V(G_j) = \emptyset \text{ if and only if } j < i\}$$

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- $w_i : \mathscr{I}(G_i) \to \mathbb{R}^{\geq 0}$, *a $\chi_f(G_i)$-coloring of each G_i*

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- $p_i : \mathscr{I}(G_i) \to \mathbb{R}^{\geq 0}$ *where* $p_i := w_i(S)/\chi_i$
- $p : \cup_{i=1}^{n} \mathcal{F}_i \to \mathbb{R}^{\geq 0}$ *where* $p(S) := \prod_{i=1}^{n} p_i(S \cap V(G_i))$

Upper Bound: $\chi_f(G) \leq \max(\chi_n, f(n))$

The Final Weighting

Final Weighting *We construct a fractional* $\max(\chi_n, f(n))$*-coloring of G defined by the weighting*

$$w(S) = \begin{cases} (\chi_i - \chi_{i-1})p(S), & S \in \mathcal{F}_i \\ \max(0, f(n) - \chi_n), & S = V_0 \\ 0, & \textit{otherwise} \end{cases}$$

Upper Bound: $\chi_f(G) \leq \max(\chi_n, f(n))$

The Final Weighting works!

We can show,

- $\sum_{S \in \mathscr{I}(G)} w(S) = \max(\chi_n, f(n))$
- $w[x] \geq 1$ for all $x \in V(G)$

So, w is a fractional $\max(\chi_n, f(n))$-coloring of G.

Conclusion $\chi_f(G) \leq \max(\chi_n, f(n))$

Results

Theorem *For $n \geq 2$, let $G_1, \ldots, G_n$ be graph. Suppose also that the graphs G_i are numbered such that $\chi_i \leq \chi_{i+1}$. Then*

$$\chi_f(\mathcal{Z}(G_1, \ldots, G_n)) = \max \left(\chi_n, 2 + \sum_{i=2}^{n} \prod_{k=i}^{n} \left(1 - \frac{1}{\chi_k} \right) \right)$$

Corollary *For every $n \geq 2$,*

$$\chi_f(\mathcal{Z}_{n+1}) = \chi_f(\mathcal{Z}_n) + \frac{1}{\chi_f(\mathcal{Z}_n)}$$

Jacobs' Conjecture - Proved!

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- Inductive Hypothesis: Suppose $\chi_n = f(n-1)$.

□

Jacobs' Conjecture - Proved!

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$$f(n) = 2 + \sum_{i=2}^{n} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right)$$

$$= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot (f(n-1) - 1)$$

$$= \chi_n + \frac{1}{\chi_n}$$

Jacobs' Conjecture - Proved!

Proof.

- Inductive Hypothesis: Suppose $\chi_n = f(n-1)$. Then

$$\begin{aligned} f(n) &= 2 + \sum_{i=2}^{n} \prod_{k \geq i} \left(1 - \frac{1}{\chi_k}\right) \\ &= 2 + \left(1 - \frac{1}{\chi_n}\right) \cdot (f(n-1) - 1) \\ &= \chi_n + \frac{1}{\chi_n} \end{aligned}$$

Since $\chi_{n+1} = \max(\chi_n, f(n))$, we have $\chi_{n+1} = \chi_n + \frac{1}{\chi_n}$. □

Questions?