SPIN MODELS AND BOSE-MESNER ALGEBRAS

By

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Chapter 1

Introduction

1.1 History

In 1990 Vaughan Jones received the Fields medal partly for his work connecting statistical mechanical methods to the study of link invariants. In particular, the paper on knot invariants [15] introduced the famous Jones polynomial and a related concept that uses matrices known as spin models. Since then, many researchers in the field of algebraic combinatorics have continued this work - first by generalizing the Jones polynomial, and second by studying the existence of spin models and constructing several infinite families. The complete classification of spin models is a hard problem, similar in spirit to other simply stated but difficult matrix problems - for example, the classification of Hadamard matrices, or the search for finite projective planes of various orders.

In 1996, however, Francois Jaeger discovered that the matrices of spin models are always contained in a certain kind of matrix algebra known as the Bose-Mesner algebra of an association scheme [12]. This result was surprising and significant because association schemes are well-studied combinatorial objects for which many examples are known. The implication was that if one wanted to find and classify spin models, one could narrow the search by answering the question as to which association schemes give rise to spin models. This important question remains open, although many new examples of spin models have been found using this approach. See [1], [11], [18].

One drawback to Jaeger's original paper was that it used subtle topological methods to prove what is essentially a linear algebraic result. But in 1995, Kazumasa Nomura found a beautiful proof of Jaeger's result using only linear algebra [20].

1.2 Summary

In this paper we offer a unified presentation of spin models along the lines of Nomura's proof. This represents a contribution to the mathematical literature by collecting results that are scattered among several papers (primarily [15] and [20], Jones' and Nomura's original papers) and giving them a complete and consolidated treatment.

The organization of this paper is as follows. In the first two sections of Chapter 2, we give the basic definitions of links, link diagrams and link invariants. Furthermore, we introduce spin models and a link invariant that is constructed from these spin models, called the partition function. In the last two sections of Chapter 2, we present the concept of the Bose-Mesner algebra of an association scheme and we give an explicit example of such an algebra constructed using distance-regular graphs.

The main discussion of the partition function is given in Chapter 3. We look at Reidemeister moves II and III in great detail to see why they leave partition function invariant. Moreover, we consider the behavior of the link invariant under Reidemeister move I.

Finally, Chapter 4 gives an explicit version of Nomura's linear algebraic proof of Jaeger's result. In particular, we discuss how spin models give rise to the Bose-Mesner algebra of an association scheme and how invariance under Reidemeister moves II and III guarantees that the matrices of a spin model are contained in this algebra.

Chapter 2

Preliminaries

2.1 Link Invariants

A **graph** Γ is a pair (V, E), where V is a finite set, whose elements are called **vertices** and E is a set of unordered pairs of distinct vertices, the **edges**. We say that a vertex v is **adjacent** to vertex v' when there is an edge between v and v'. A **path** is a sequence of distinct vertices such that consecutive vertices are adjacent. Furthermore, Γ is **connected** if there exists a path between every pair of vertices, otherwise we say Γ is **disconnected**. A more complete introduction to graphs can be found in [23].

A **knot** is a continuous simple piece-wise linear closed curve in Euclidean 3-space, $\mathbb{R}^3$. The assumption that a knot is piece-wise linear ensures that it consists of a finite number of straight line segments, thus avoiding knots of infinite length. We will draw our knots with enough segments so as to make them appear smooth. A **link** is a finite collection of pair-wise disjoint knots. The knots constituting a link are called the **components** of the link. In particular, a knot is said to be a one component link. We consider two links, L_1 and L_2, to be equivalent if there exists an orientation-preserving homeomorphism from $\mathbb{R}^3 \cup \infty$ to itself mapping L_1 onto L_2. It is most useful to work with a planar representation of a link, i.e. a projection of the link onto a plane. Such a projection is called a **link diagram**. Figure 1 is a projection of a link known as the figure-eight knot. It is clear that some of the spatial information of a link can be lost in a link diagram. To ensure that the projection retains the important information, we require that it satisfies the following conditions:

(i) No three points on the link project to the same point in the diagram.

(ii) Only finitely many points in the diagram correspond to more than one point on the link.

The link diagram is drawn in such a way that each crossing point indicates the spatial structure of the link.

Note that a link has many different link diagrams. For example, a second projection of the figure-eight knot is also given in Figure 1. This raises the question of how to determine whether two given link diagrams represent equivalent links. To address this issue we observe that there exist three simple operations that can be applied to a link diagram without changing the link it represents. These operations are called the **Reidemeister moves** and are shown in Figure 2. The following well-known theorem by Alexander and Briggs indicates the significance of the Reidemeister moves.

2.1.1 Theorem. *[16, p.31] Two link diagrams determine equivalent links if and only if one can be obtained from the other by a sequence of Reidemeister moves.*

In view of this theorem, observe that in order to show that two links L_1 and L_2 are equivalent, we have to prove that there exists a sequence of Reidemeister moves changing the link diagram for L_1 into the link diagram for L_2. Conversely, if we want to show that the two links are not equivalent, then we need to show that there does not exist a sequence of Reidemeister moves changing the link diagram of L_1 into the link diagram of L_2. But since there is no known bound on the length of such a sequence, an exhaustive search is not possible. Thus, we try to find characteristics of link diagrams that remain unchanged by the Reidemeister moves.

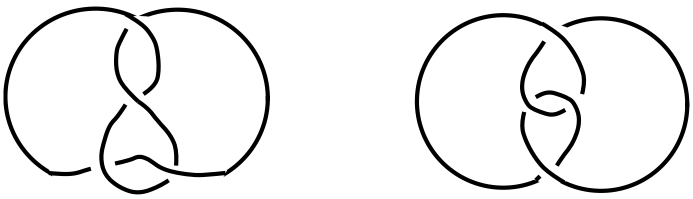

Figure 1: Two different projections of the figure-eight knot

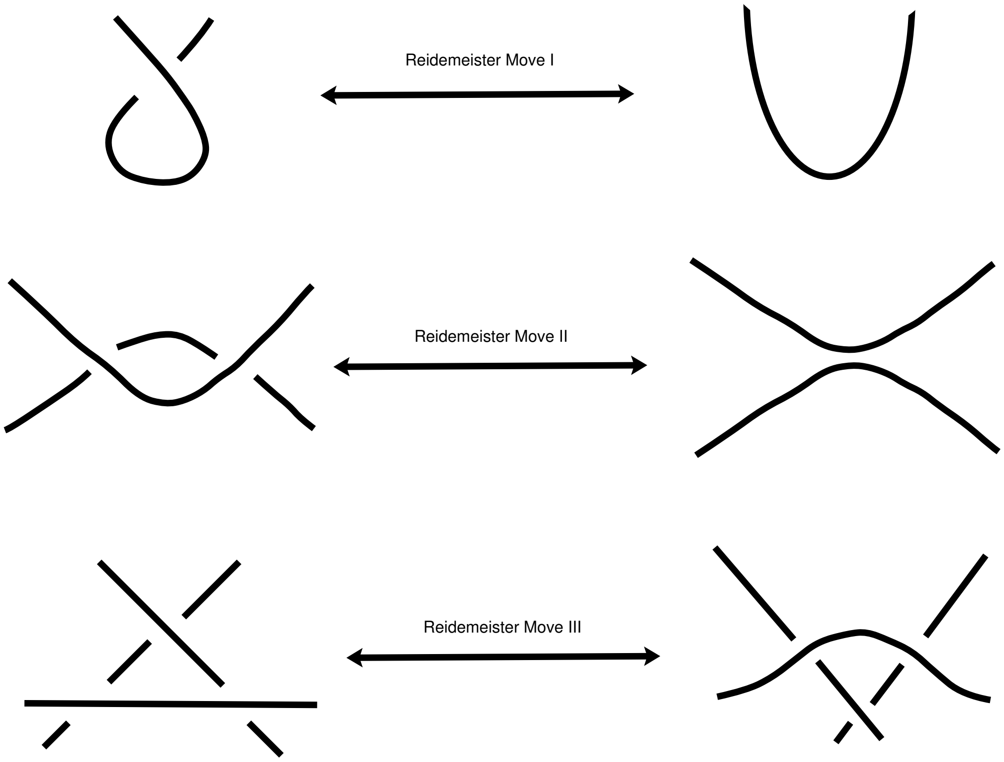

Figure 2: Reidemeister moves I, II, III

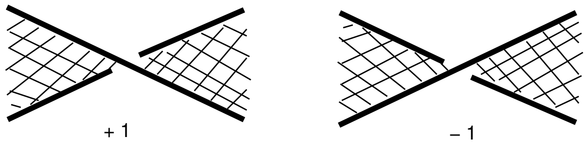

Figure 3: Sign convention

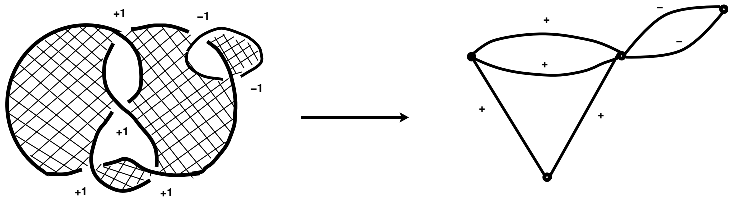

Figure 4: Forming the Tait graph

We call such characteristics **link invariants**. In particular, if a link invariant takes on different values for two link diagrams, then they represent different links.

Often it is the case that a characteristic of a link diagram stays invariant only under one or two of the Reidemeister moves. In particular, a characteristic of link diagrams that is invariant under Reidemeister moves II and III is called a **partial link invariant**. Two link diagrams related by just Reidemeister move I and II are said to be **regularly isotopic**.

Some link and partial link invariants are constructed using a graphical representation of a link diagram. An example of such a graphical representation is a **Tait graph**. A Tait graph is a signed graph, that is a graph with labels + and - on each edge. In general, the Tait graph will also have loops and multiple edges. We will explain how such a graph is formed from a link diagram as well as how the Reidemeister moves translate to moves in the Tait graph.

The regions of any link diagram can always be colored using two colors such that adjacent regions receive opposite colors. This is called the **two-color theorem** of knot theory. (To see this, note that by construction, the Tait graph is always the planar dual of a 4-regular planar graph, and hence is two-colorable. See [23, p.239] and [9, p.379] for details.) A two coloring with the unbounded region colored white is sometimes referred to as the **checkerboard coloring** of the diagram. To form a Tait graph we associate a vertex with each black region and we associate an edge with each crossing. This edge is labeled + or - according to the type of crossing as indicated in Figure 3. See Figure 4 for an example of the Tait graph of a link diagram.

The Reidemeister moves can be translated to graphical moves on the Tait graph as shown in Figure 5 . Note that for the Reidemeister move III, the Tait graph forms a star and a triangle, it is therefore called the star-triangle relation. The next section will introduce a partial link invariant defined using the Tait graph of a link diagram. Chapter 3 will give a detailed explanation, using this graphical representation, as to why it is a partial link invariant. A more complete introduction to knots can be found in [16] or [17].

2.2 Spin Models

In this section we will introduce spin models, which are used to construct link invariants. Vaughan Jones [15] introduced the construction of this invariant based on the statistical mechanical concept of a spin model, hence the name.

2.2.1 Definition. A **spin model** is a triple $S = (X, W^+, W^-)$ where $X = \{1, \ldots, n\}$, and $W^{\pm}$ are symmetric $n \times n$ matrices with complex number entries that satisfy the following equation:

$$W^+_{a,b} W^-_{a,b} = 1 \qquad (a, b \in X) \tag{2.1}$$

The elements of X are called the **spins** of S.

In order for a spin model to be invariant under the Reidemeister moves II and

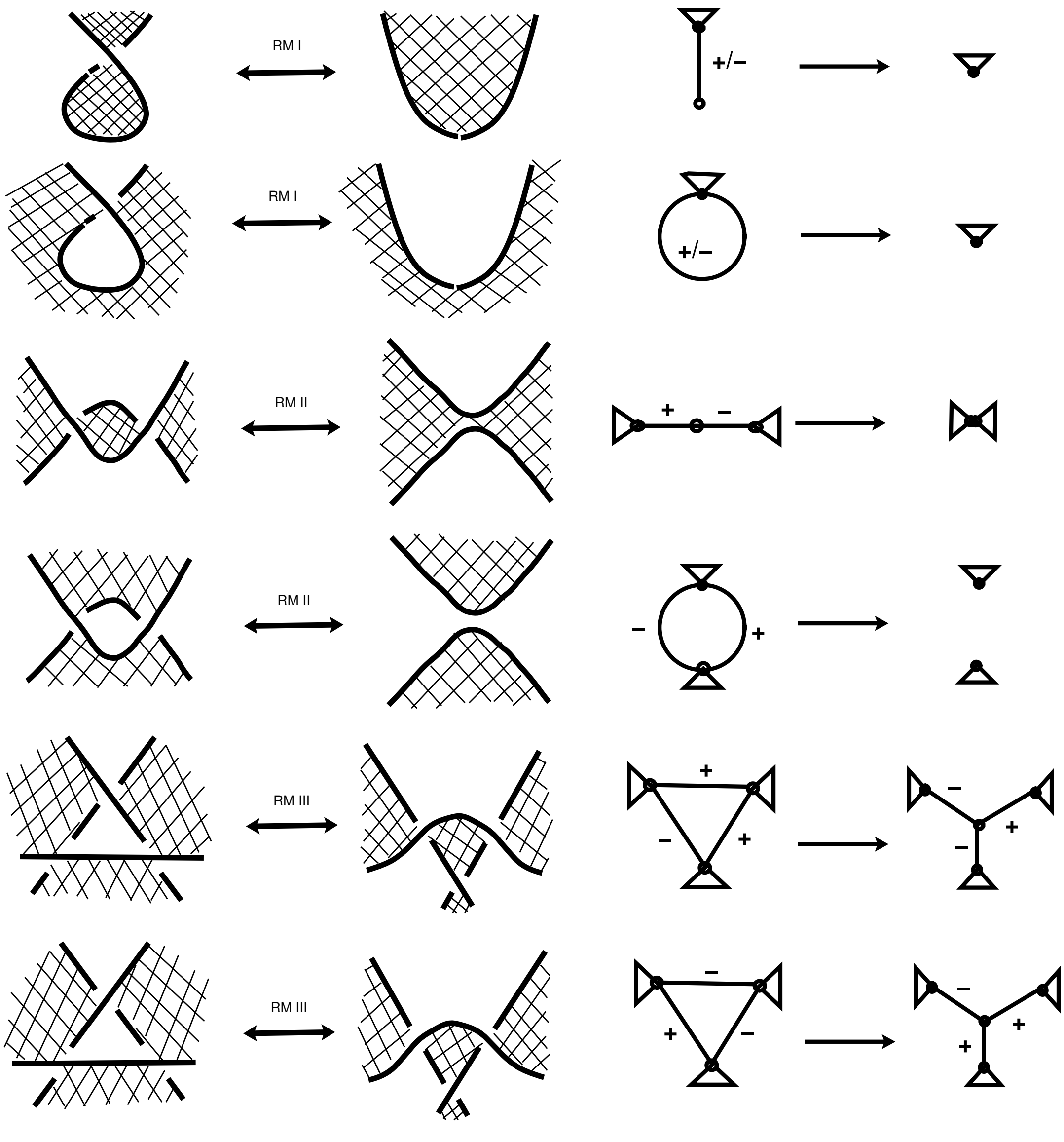

Figure 5: Graphical Reidemeister moves

III it needs to satisfy the following invariance equations for all $a, b, c \in X$:

$$\textbf{Type II:} \qquad \sum_{x\in X} W^+_{a,x}\, W^-_{b,x} \;=\; n\,\delta_{a,b} \tag{2.2}$$

$$\textbf{Type III:} \qquad \sum_{x\in X} W^+_{a,x}\, W^+_{b,x}\, W^-_{c,x} \;=\; \sqrt{n}\, W^+_{a,b}\, W^-_{a,c}\, W^-_{b,c}. \tag{2.3}$$

2.2.2 Lemma. *Given the Type II condition, the Type III condition is equivalent to*

$$\sum_{x\in X} W^+_{a,x}\, W^-_{b,x}\, W^-_{c,x} \;=\; \sqrt{n}\, W^+_{a,b}\, W^+_{a,c}\, W^-_{b,c} \qquad (a, b, c \in X). \tag{2.4}$$

Proof. To simplify our computations let us introduce the column vector $u_{b,c}$ whose $x-$entry is defined by $(u_{b,c})_x = W^+_{b,x}\, W^-_{c,x}$ for all $b, c \in X$. Using this column vector and the symmetry of W^+, W^-, we can now rewrite the Type III equation as follows:

$$\begin{aligned}
\sum_{x\in X} W^+_{a,x}\, W^+_{b,x}\, W^-_{c,x} &= \sqrt{n}\, W^+_{b,a}\, W^-_{c,a}\, W^-_{b,c} && (a, b, c \in X)\\
\sum_{x\in X} W^+_{a,x}\, u_{b,c} &= \sqrt{n}\, u_{b,c}\, W^-_{b,c} &&\\
(W^+\, u_{b,c})_a &= \sqrt{n}\, W^-_{b,c}\, (u_{b,c})_a &&\\
W^+\, u_{b,c} &= \sqrt{n}\, W^-_{b,c}\, u_{b,c} && (b, c \in X).
\end{aligned}$$

We can now manipulate this new equation using the Type II condition.

$$\begin{aligned}
W^+\, u_{b,c} &= \sqrt{n}\, W^-_{b,c}\, u_{b,c} && (b, c \in X)\\
W^+\, W^+_{b,c}\, u_{b,c} &= \sqrt{n}\, u_{b,c} && \textit{by } (2.1)\\
W^-\, W^+\, W^+_{b,c}\, u_{b,c} &= \sqrt{n}\, W^-\, u_{b,c} &&\\
n\, W^+_{b,c}\, u_{b,c} &= \sqrt{n}\, W^-\, u_{b,c} && \textit{by } (2.2)\\
\sqrt{n}\, W^+_{b,c}\, (u_{b,c})_a &= (W^-\, u_{b,c})_a && (a, b, c \in X)
\end{aligned}$$

$$\begin{aligned}\sqrt{n}\, W^+_{b,c}\, (u_{b,c})_a &= \sum_{x\in X} W^-_{a,x}\, u_{b,c}\\ \sqrt{n}\, W^+_{b,c}\, W^+_{b,a}\, W^-_{c,a} &= \sum_{x\in X} W^-_{a,x}\, W^+_{b,x}\, W^-_{c,x}.\end{aligned}$$

Since this is true for all $a, b, c \in X$, we can switch a and b in the equation and we will get the needed result. □

We remark that if we set $b = c$ in (2.3) we obtain the following:

$$\sum_{x\in X} W^+_{a,x} = \sqrt{n}\, W^-_{b,b}, \qquad (a, b \in X). \tag{2.5}$$

And if we let $a = c$ in (2.4) we obtain:

$$\sum_{x\in X} W^-_{b,x} = \sqrt{n}\, W^+_{a,a}, \qquad (a, b \in X). \tag{2.6}$$

Since the above equations hold for all a, b in X it follows that $W^{\pm}$ has constant row sums and constant diagonal entries. We define the **modulus** of the spin model to be the diagonal constant of W^+.

2.2.3 Example. The following is an example of a spin model with $X = \{1, 2, 3, 4\}$:

$$W^+ = \begin{pmatrix} -i & i & i & i \\ i & -i & i & i \\ i & i & -i & i \\ i & i & i & -i \end{pmatrix} \qquad W^- = \begin{pmatrix} i & -i & -i & -i \\ -i & i & -i & -i \\ -i & -i & i & -i \\ -i & -i & -i & i \end{pmatrix}$$

Notice that the modulus of the above spin model is $-i$ and the row sums of $W^{\pm}$ are $2i$ and $-2i$ respectively.

It is easy to see that our example satisfies the definition of a spin model. Moreover, a straightforward computation verifies that the introduced spin model satisfies the Type II and Type III equations. We use the Tait graph representation of a link diagram together with a spin model to define the following partial link invariant.

2.2.4 Definition. Let $S = (X, W^{\pm})$ be a spin model. Let L be any link diagram and $\mathcal{L}_L$ the Tait graph of L with vertices V. Let m denote the modulus of S. A **state** σ is a function from V to X, and we define the **partition function** to be

$$Z_L^S = \left(\frac{1}{\sqrt{n}}\right)^{|V|-1} \sum_{\substack{states \\ \sigma : V \to X}} \prod_{\substack{edges \\ v, v' \in \mathcal{L}_L}} W^{\pm}_{\sigma(v), \sigma(v')}. \tag{2.7}$$

In chapter 3, we will prove that the partition function remains invariant under the Reidemeister moves II and III and varies predictably under Reidemeister move I.

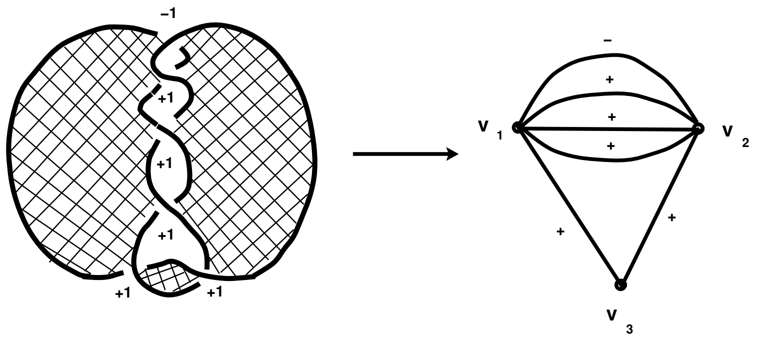

Figure 6: Figure-eight knot after RM II

2.2.5 Example. To illustrate the above definition we will compute the partition function for another link diagram of the figure-eight knot. Consider Figure 6 to see the link diagram as well as the corresponding Tait graph. Note that each edge is marked with a + or − depending on the crossings in the link diagram. To compute the partition function we will use the spin model of example 2.2.3. We are given

$V = \{v_1, v_2, v_3\}$ and $n = 4$. Then we can write

$$
\begin{aligned}
Z_L^S &= \left(\frac{1}{\sqrt{n}}\right)^{|V|-1} \sum_{\substack{states \\ \sigma: V \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_L}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{4}\right) \sum_{\substack{states \\ \sigma}} \Big([W^{+}_{\sigma(v_1),\sigma(v_2)}]^3 \;\; W^{-}_{\sigma(v_1),\sigma(v_2)} \\
&\qquad\qquad \times W^{+}_{\sigma(v_2),\sigma(v_3)} \;\; W^{+}_{\sigma(v_1),\sigma(v_3)} \Big)
\end{aligned}
$$

Grouping the states according to which vertices are assigned the same spins, we obtain:

$$
\begin{aligned}
Z_L^S = \left(\frac{1}{4}\right) \Big[& \sum_{\substack{\sigma(v_1)=\sigma(v_2) \\ \sigma(v_2)=\sigma(v_3)}} (-i)^3(i)(-i)(-i) \; + \sum_{\substack{\sigma(v_1)=\sigma(v_2) \\ \sigma(v_1)\neq\sigma(v_3) \\ \sigma(v_2)\neq\sigma(v_3)}} (-i)^3(i)(i)(i) \\
& + \sum_{\substack{\sigma(v_1)=\sigma(v_3) \\ \sigma(v_1)\neq\sigma(v_2) \\ \sigma(v_2)\neq\sigma(v_3)}} (i)^3(-i)(i)(-i) \; + \sum_{\substack{\sigma(v_2)=\sigma(v_3) \\ \sigma(v_1)\neq\sigma(v_3) \\ \sigma(v_1)\neq\sigma(v_2)}} (-i)^3(i)(-i)(i) \\
& + \sum_{\substack{\sigma(v_1)\neq\sigma(v_2) \\ \sigma(v_1)\neq\sigma(v_3) \\ \sigma(v_2)\neq\sigma(v_3)}} (i)^3(-i)(i)(i) \;\; \Big]
\end{aligned}
$$

By counting the number of states of each kind we find that:

$$
\begin{aligned}
Z_L^S &= \left(\frac{1}{4}\right) \Big[\; 4(-i)^3(i)(-i)(-i) \; + \; 12(-i)^3(i)(i)(i) + \; 12(i)^3(-i)(i)(-i) \\
&\qquad\qquad + 12(-i)^3(i)(-i)(i) \; + \; 24(i)^3(-i)(i)(i) \; \Big] \\
&= 4.
\end{aligned}
$$

Thus the partition function produces the value 4 for this projection of the figure-eight knot. Applying any sequence of Reidemeister moves II and III to this link diagram will not change the value of 4 given by the partition function, as we will see in Chapter 3. We will also see that Reidemeister moves of Type I only change this value by a power of the modulus $-i$.

2.3 Association Schemes and Bose-Mesner Algebras

2.3.1 Definition. Let $X = \{1, 2, ..., n\}$ and let $A_0, A_1, ..., A_D$ be symmetric zero-one matrices of size $n \times n$, each with rows and columns indexed by X. We say the pair $\mathcal{A} = (X, \{A_i\}_{0 \leq i \leq D})$ is a **symmetric association scheme** if the following properties hold:

$$A_0 = I, \tag{2.8}$$

$$A_0 + A_1 + \cdots + A_D = J, \tag{2.9}$$

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \qquad (0 \leq i, j \leq D), \tag{2.10}$$

where I is the identity matrix, J is the all 1's matrix, and the constants p_{ij}^h are non-negative integers. We refer to the matrices $A_0, A_1, ..., A_D$ as the **associate matrices** of $\mathcal{A}$.

Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices with entries in $\mathbb{C}$, where the rows and columns are indexed by X. By (2.8)-(2.10), $A_0, ..., A_D$ form a basis for a subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. M is known as the **Bose-Mesner algebra** of $\mathcal{A}$. Observe that M is commutative, since the associate matrices are symmetric. Furthermore, since each A_i is a zero-one matrix, M is closed under the entry-wise product, called the **Hadamard product**, and

$$A_i \circ A_j = \delta_{ij} A_i$$

for any i, j.

The following result provides a useful test for identifying Bose-Mesner algebras.

2.3.2 Theorem. *[5, p.57] An algebra of symmetric $n \times n$ matrices is the Bose-Mesner algebra of some symmetric association scheme iff it contains the identity* *matrix I, the all 1's matrix J, and is closed under the Hadamard product.*

A more complete introduction to association schemes can be found in [3], [8], or [22].

2.4 Distance-Regular Graphs

The most common examples of symmetric association schemes and Bose-Mesner algebras arise from distance-regular graphs.

Let Γ denote a connected graph with vertex set V, edge set E. For any vertices x, y let $\partial(x, y)$ = path length distance from x to y. Let $D = max\{\partial(x, y)|$ vertices $x, y\}$. Call D the **diameter** of Γ. The **sphere of radius** i about a vertex x is the set

$$\Gamma_i(x) \;=\; \{y \,|\, \partial(x, y) = i\}.$$

2.4.1 Definition. Γ is **distance regular** if for any 3 distances h, i, j there exists a constant p^h_{ij} such that for all vertices x, y with $\partial(x, y) = h$,

$$|\Gamma_i(x) \cap \Gamma_j(y)| \;=\; p^h_{ij}.$$

The constants p^h_{ij} are called the **intersection numbers** of Γ.

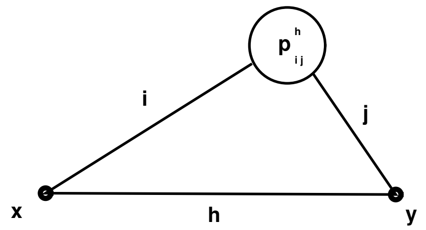

Figure 7: Intersection numbers for a distance-regular graph

The graphs formed by the edges and vertices of the five Platonic solids are examples of distance regular graphs.

Suppose we are given a distance regular graph Γ of diameter D. For each $i\,(0 \leq i \leq D)$, we define the ith **distance matrix** A_i to be the matrix in $\mathrm{Mat}_V(\mathbb{C})$ with $xy-$entry:

$$(A_i)_{xy} \;=\; \begin{cases} 1 & \text{if } \partial(x,y)=i, \\ 0 & \text{otherwise} \end{cases} \qquad (x,y \in V). \tag{2.11}$$

Observe that A_0 is the identity matrix, A_1 is the usual adjacency matrix, and A_2 the distance-2 matrix, and so on.

It is easily verified that the distance matrices $A_0, A_1, ..., A_D$ of a distance-regular graph will satisfy (2.8) and (2.9). We take a closer look at why these distance matrices satisfy (2.10). It is sufficient to show that for each $x, y \in V$

$$(A_iA_j)_{xy} = \left(\sum_{h=0}^{D} p_{ij}^h A_h\right)_{xy}.$$

To this end, pick $x, y \in V$ and let $\partial(x,y) = l$. Consider the xy entry of A_iA_j:

$$(A_iA_j)_{xy} \;=\; \sum_{z\in\Gamma}(A_i)_{xz}(A_j)_{zy}\ ,$$

but $(A_i)_{xz}(A_j)_{zy} = 0$ unless $\partial(x,z) = i$ and $\partial(z,y) = j$. Hence we have

$$(A_iA_j)_{xy} \;=\; p_{ij}^l\ .$$

Now consider the xy entry of $\sum_{h=0}^{D} p_{ij}^h A_h$. Since $\partial(x,y) = l$ and $(A_h)_{xy} = 0$ unless $h = l = \partial(x,y)$, we have that

$$\begin{aligned} \left(\sum_{h=0}^{D} p_{ij}^h A_h\right)_{xy} &= (p_{ij}^l A_l)_{xy} \\ &= p_{ij}^l. \end{aligned}$$

Hence (2.10) holds.

It follows now that the distance matrices $A_0, A_1, ..., A_D$ of any distance-regular graph form a symmetric association scheme and thus are a basis for the Bose-Mesner algebra $\mathcal{A}(\Gamma)$, called the **adjacency algebra** of Γ.

In chapter 4 we will see that the matrices W^+ and W^- from a spin model satisfying the Type II and Type III equations are contained in the Bose-Mesner algebra of an association scheme. Indeed, the matrices W^+ and W^- from the spin model we introduced in example 2.2.3 are contained in the adjacency algebra of the graph of the tetrahedron $\mathcal{A}(\Gamma_T)$, one of the platonic solids.

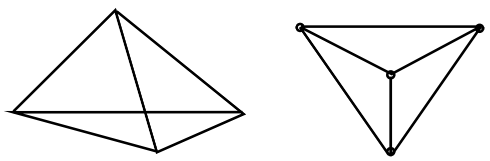

Figure 8: Tetrahedron and its distance-regular graph

To see this, note that the diameter of the tetrahedron is 1 and thus we have the distance matrices A_0 and A_1 forming a basis for $\mathcal{A}(\Gamma_T)$. In fact, $A_0 = I$ and $A_1 = J - I$. We can write

$$W^+ = -i \cdot A_0 + i \cdot A_1 \qquad \text{and} \qquad W^- = i \cdot A_1 - i \cdot A_0.$$

Therefore, W^+ and W^- are contained in $\mathcal{A}(\Gamma_T)$.

A more complete introduction to distance-regfular graphs can be found in [4], [5], or [8].

Chapter 3

Spin Models

3.0.2 Theorem. *Let S be a spin model and L_1, L_2 be connected link diagrams. Then $Z^S_{L_1} = Z^S_{L_2}$ if L_1 and L_2 differ only by Reidemeister II and III moves. Furthermore, if L_1 and L_2 differ only by a Reidemeister move of Type I, then $Z^S_{L_1} = m^p Z^S_{L_2}$, where p equals ± 1. The value of p depends only on the type of crossing involved.*

This chapter will give an explicit proof of the stated theorem by showing that the partition function Z^S_L is invariant under the Reidemeister moves II and III, and varies as described under Reidemeister move I.

3.1 Invariance under Reidemeister move II

3.1.1 Lemma. *Let S be a spin model and L_1, L_2 link diagrams with partition functions $Z^S_{L_1}$ and $Z^S_{L_2}$, respectively. If L_1 and L_2 differ only by Reidemeister move II then $Z^S_{L_1} = Z^S_{L_2}$.*

Proof. There are two possible cases for the shading of the diagrams. See Figure 9.

Case 1: Suppose Reidemeister move II is applied to L_1. For this case, the effect will be to separate the existing region into three regions and create two new crossings, one positive and the other negative. Notice that the middle region creates a new vertex in the Tait graph. The two outer regions may still be the same region, but

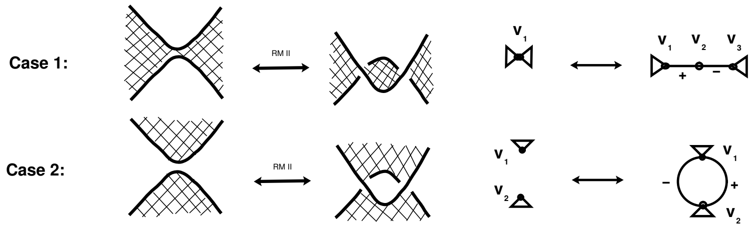

Figure 9: Two types of shadings for Reidemeister move II

we will give the Tait graph as if two new vertices were formed. We will consider the different possible states for those two vertices in the partition function. See Figure 9. Call the newly created link diagram L_2. Let V_1, V_2 denote the vertex sets of the Tait graphs $\mathcal{L}_{L_1}, \mathcal{L}_{L_2}$, respectively. Let v_1, v_2, v_3 be the vertices for the regions indicated in Figure 9. The partition function of L_2 will be as follows:

$$
\begin{aligned}
Z^S_{L_2} &= \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states \\ \sigma:V_2\to X}} \prod_{\substack{edges \\ v,v'\in\mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|+1} \sum_{\substack{states \\ \sigma:V_1\to X}} \prod_{\substack{edges \\ v,v'\in\mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \left[\sum_{\sigma(v_1)=\sigma(v_3)} W^{\pm}_{\sigma(v_1),x} W^{\mp}_{x,\sigma(v_1)} \right. \\
&\qquad \left. + \sum_{\sigma(v_1)\neq\sigma(v_3)} W^{\pm}_{\sigma(v_1),x} W^{\mp}_{x,\sigma(v_1)} \right] \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|+1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot (n\cdot\delta_{\sigma(v_1)=\sigma(v_3)} \\
&\qquad + n\cdot\delta_{\sigma(v_1)\neq\sigma(v_3)}) \qquad \text{by (2.2)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|+1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot n \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= Z^S_{L_1}
\end{aligned}
$$

Case 2: Applying Reidemeister move II to L_1 in this case creates two new crossings, one of which is positive and the other negative. Notice that the new region formed remains white thus no new vertex is formed in the Tait graph. Call the newly created link diagram L_2. Let V_1, V_2 denote the vertex sets of the Tait graphs $\mathcal{L}_{L_1}, \mathcal{L}_{L_2}$, respectively. Let v_1, v_2 be the vertices for the regions indicated in Figure 9. The partition function of L_2 will be as follows:

$$\begin{aligned}
Z^S_{L_2} &= \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states \\ \sigma:V_2\to X}} \prod_{\substack{edges \\ v,v'\in\mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{\substack{states \\ \sigma:V_1\to X}} \prod_{\substack{edges \\ v,v'\in\mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \cdot W^{\pm}_{\sigma(v_1),\sigma(v_2)} W^{\mp}_{\sigma(v_1),\sigma(v_2)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot 1 \qquad \text{by (2.1)} \\
&= Z^S_{L_1}
\end{aligned}$$

This completes the proof of the lemma. □

3.2 Invariance under Reidemeister move III

3.2.1 Lemma. *Let S be a spin model,and L_1, L_2 link diagrams with partition function $Z^S_{L_1}$ and $Z^S_{L_2}$, respectively. If L_1 and L_2 differ only by Reidemeister move III then $Z^S_{L_1} = Z^S_{L_2}$.*

Proof. The shading of regions does not play a role here since all possible shadings are already represented. See Figure 10. The star-triangle relation, as this move is called, is the most difficult of the three moves to deal with. Type III moves an arc past a crossing and thus removes three crossings between the three regions but creates a new region and three new crossings which connect the old regions with the new one. Let the new link diagram be L_2 and $Z^S_{L_2}$ be its partition function. There are two kinds of Type III moves depending on the crossing:

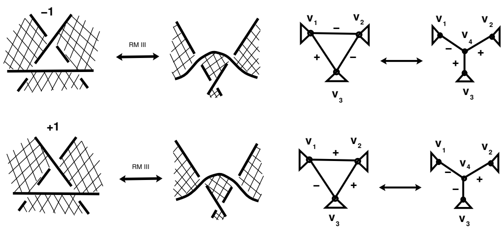

Figure 10: Two types of crossings for Reidemeister move III

Let V_1, V_2 denote the vertex sets of the Tait graphs $\mathcal{L}_{L_1}, \mathcal{L}_{L_2}$, respectively. Let v_1, v_2, v_3, v_4 be the vertices for the regions indicated in Figure 10. Invariance under the first kind is assured as follows:

$$
\begin{aligned}
Z^S_{L_2} &= \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states \\ \sigma: V_2 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{\substack{states \\ \sigma: V_1 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&\qquad \times \left(\frac{\sum_{x \in X} W^{+}_{\sigma(v_2),x} W^{+}_{\sigma(v_3),x} W^{-}_{\sigma(v_1),x}}{W^{+}_{\sigma(v_2),\sigma(v_3)} W^{-}_{\sigma(v_2),\sigma(v_1)} W^{-}_{\sigma(v_3),\sigma(v_1)}} \right) \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \\
&\qquad \times \left(\frac{\sqrt{n} \cdot W^{+}_{\sigma(v_2),\sigma(v_3)} W^{-}_{\sigma(v_2),\sigma(v_1)} W^{-}_{\sigma(v_3),\sigma(v_1)}}{W^{+}_{\sigma(v_2),\sigma(v_3)} W^{-}_{\sigma(v_2),\sigma(v_1)} W^{-}_{\sigma(v_3),\sigma(v_1)}} \right) \quad \text{by (2.3)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \sqrt{n} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= Z^S_{L_1}
\end{aligned}
$$

Let us now show that we have invariance under the second kind as well:

$$
\begin{aligned}
Z^S_{L_2} &= \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states \\ \sigma: V_2 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{\substack{states \\ \sigma: V_1 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&\qquad \times \left(\frac{\sum_{x \in X} W^{+}_{\sigma(v_2),x} W^{-}_{\sigma(v_3),x} W^{-}_{\sigma(v_1),x}}{W^{+}_{\sigma(v_2),\sigma(v_3)} W^{+}_{\sigma(v_2),\sigma(v_1)} W^{-}_{\sigma(v_3),\sigma(v_1)}} \right) \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \\
&\qquad \times \left(\frac{\sqrt{n} \cdot W^{+}_{\sigma(v_2),\sigma(v_3)} W^{+}_{\sigma(v_2),\sigma(v_1)} W^{-}_{\sigma(v_3),\sigma(v_1)}}{W^{+}_{\sigma(v_2),\sigma(v_3)} W^{+}_{\sigma(v_2),\sigma(v_1)} W^{-}_{\sigma(v_3),\sigma(v_1)}} \right) \quad \text{by (2.4)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \sqrt{n} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= Z^S_{L_1}
\end{aligned}
$$

□

3.2.2 Example. We give an example to illustrate how the Reidemeister moves II and III leave the partition function invariant. Recall that in section 2.2 we computed the partition function for a link diagram of the figure-eight knot. The computed value was 4. Let us now compute the partition function after applying Reidemeister moves II and III to the same link diagram. See Figure 11.

We will use the same spin model as before and are now given $V = \{v_1, v_2, v_3, v_4\}$

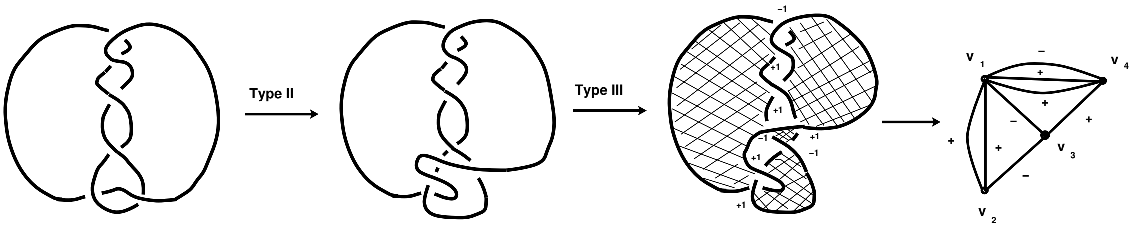

Figure 11: Figure-eight knot under Reidemeister moves II and III

and $n = 4$. Then we can write

$$\begin{aligned} Z_L^S &= \left(\frac{1}{\sqrt{n}}\right)^{|V|-1} \sum_{\substack{states \\ \sigma: V \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_L}} W^{\pm}_{\sigma(v),\sigma(v')} \\ &= \left(\frac{1}{8}\right) \sum_{\substack{states \\ \sigma}} \Big([\, W^+_{\sigma(v_1),\sigma(v_2)}\,]^2 \; W^-_{\sigma(v_2),\sigma(v_3)} \; W^+_{\sigma(v_3),\sigma(v_4)} \\ &\qquad \times \; [\, W^+_{\sigma(v_1),\sigma(v_4)}\,]^2 \; W^-_{\sigma(v_1),\sigma(v_4)} \; W^-_{\sigma(v_1),\sigma(v_3)} \Big) \end{aligned}$$

Grouping the states and counting the number of states of each kind as we did in Example (2.2.5) gives us $Z_L^S = 4$, which is precisely what we computed before.

3.3 Behavior of Z_L^S under Reidemeister move I

3.3.1 Lemma. *Let S be a spin model and L_1, L_2 link diagrams with partition functions $Z_{L_1}^S$ and $Z_{L_2}^S$, respectively. If L_1 and L_2 differ only by a Reidemeister move of Type I then $Z_{L_1}^S = m^p \, Z_{L_2}^S$, where p equals ± 1. The value of p depends only on the type of crossing involved.*

Proof. There are two possible shadings of the diagrams to consider:

Case 1: Suppose Reidemeister move I is applied to L_1 in such a way that it creates a positive crossing. This move will split the existing region in two, adding a region to the link diagram and thus a vertex to the Tait graph $\mathcal{L}_{L_1}$ with a positive labeled edge. Call the newly created link diagram L_2. Let V_1, V_2 denote the vertex sets

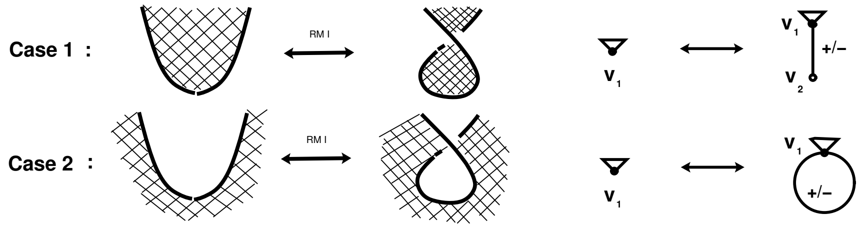

Figure 12: Two types of shadings for Reidemeister move I

of the Tait graphs $\mathcal{L}_{L_1}, \mathcal{L}_{L_2}$, respectively. Let v_1, v_2 be the vertices for the regions indicated in Figure 12. The partition function of L_2 will be as follows:

$$\begin{aligned}
Z^S_{L_2} &= \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states \\ \sigma: V_2 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{\substack{states \\ \sigma: V_1 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \sum_{x \in X} W^{+}_{\sigma(v_1),x} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \sqrt{n} \cdot W^{-}_{\sigma(v_2),\sigma(v_2)} \qquad \text{by (2.5)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot W^{-}_{\sigma(v_2),\sigma(v_2)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \left(\frac{1}{m}\right) \\
&= m^{-1}\, Z^S_{L_1}
\end{aligned}$$

Now suppose the Reidemeister move I creates a negative crossing. Again, we create a new region in the link diagram and therefore a new vertex in the Tait graph, but the formed edge is now negative. The partition function of L_2 will be as follows:

$$Z^S_{L_2} = \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states \\ \sigma: V_2 \to X}} \prod_{\substack{edges \\ v,v' \in \mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')}$$

$$
\begin{aligned}
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{\substack{states\\ \sigma:V_1\to X}} \prod_{\substack{edges\\ v,v'\in\mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \sum_{x\in X} W^{-}_{\sigma(v_1),x} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot \sqrt{n} \cdot W^{+}_{\sigma(v_2),\sigma(v_2)} \qquad \text{by (2.5)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot W^{+}_{\sigma(v_2),\sigma(v_2)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot m \\
&= m^1\ Z^S_{L_1}
\end{aligned}
$$

Case 2: Suppose that Reidemeister move I is applied to L_1. In this case, we create a new crossing, positive or negative connecting the same region. Thus we only create a new edge for an existing vertex (a loop) but not a new vertex. See Figure 12. The partition function of the new created link diagram, L_2 will then be as follows.

$$
\begin{aligned}
Z^S_{L_2} &= \left(\frac{1}{\sqrt{n}}\right)^{|V_2|-1} \sum_{\substack{states\\ \sigma:V_2\to X}} \prod_{\substack{edges\\ v,v'\in\mathcal{L}_{L_2}}} W^{\pm}_{\sigma(v),\sigma(v')} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{\substack{states\\ \sigma:V_1\to X}} \prod_{\substack{edges\\ v,v'\in\mathcal{L}_{L_1}}} W^{\pm}_{\sigma(v),\sigma(v')} \cdot W^{\pm}_{\sigma(v_1),\sigma(v_1)} \\
&= \left(\frac{1}{\sqrt{n}}\right)^{|V_1|-1} \sum_{states} \prod_{edges} W^{\pm}_{\sigma(v),\sigma(v')} \cdot m^{\pm 1} \\
&= m^{\pm 1} \cdot Z^S_{L_1}
\end{aligned}
$$

This completes the proof of the Lemma. □

Proof of Theorem 3.0.2.

Follows from (3.1.1), (3.2.1), and (3.3.1). $\square$

Chapter 4

The Bose-Mesner Algebra of a Spin Model

The big question that stands is: *Where can we find these spin models?*. It is quite a difficult task to find spin models which satisfy the invariance equations (2.2) - (2.3). Thanks to Jaeger [12] we now know to look at the Bose-Mesner algebras of association schemes to find spin models. In particular, we have the following theorem by Jaeger:

4.0.2 Theorem. *If a spin model $S = (X, W^+, W^-)$ satisfies the Type II and Type III equation, then it naturally gives rise to a Bose-Mesner algebra for an association scheme, such that the matrices W^+, W^- are elements of this algebra.*

The next two sections will cover the two main results needed to prove the theorem. Our proof will follow the one given by Nomura in [20]. In section 4.1 we will show that if a spin model satisfies the Type II equation then it gives rise to the Bose-Mesner algebra of an association scheme. Section 3 will show that if a spin model satisfies the Type III equation then the matrices W^+, W^- are contained in the Bose-Mesner algebra of an association scheme.

4.1 A Spin Model satisfying the Type II condition

Let $S = (X, W^+, W^-)$ denote a spin model with $|x| = n$. For each $b, c \in X$, define an n-dimensional column vector $u_{b,c}$ with x-entry

$$(u_{b,c})_x \;=\; W^+_{b,x}\, W^-_{c,x}. \tag{4.1}$$

Now define $\mathcal{N}$ to be the set of all symmetric $n \times n$ matrices A such that for all $b, c \in X$ $u_{b,c}$ is an eigenvector of A. Thus we have that for each $b, c \in X$

$$A\, u_{b,c} \;=\; \lambda^A_{b,c}\, u_{b,c}$$

where $\lambda^A_{b,c}$ denotes the eigenvalue of A on $u_{b,c}$.

Fist let us show that $\mathcal{N}$ is a subspace of $\mathrm{Mat}_n\mathbb{C}$. For $A, B \in \mathcal{N}$, we have:

$$\begin{aligned}(A+B)\, u_{b,c} &= A\, u_{b,c} + B\, u_{b,c} \\ &= \lambda^A_{b,c}\, u_{b,c} + \lambda^B_{b,c}\, u_{b,c} \\ &= (\lambda^A_{b,c} + \lambda^B_{b,c})\, u_{b,c}\end{aligned}$$

so that $\mathcal{N}$ is closed under addition. And for any $\alpha \in \mathbb{C}$ we have

$$\begin{aligned}(\alpha\, A)\, u_{b,c} &= \alpha\,(A\, u_{b,c}) \\ &= \alpha\,(\lambda^A_{b,c}\, u_{b,c}) \\ &= (\alpha\, \lambda^A_{b,c})\, u_{b,c}\end{aligned}$$

so that $\mathcal{N}$ is closed under scalar multiplication. The eigenvalues of $A + B$ and αA are given by $\lambda^A_{b,c} + \lambda^B_{b,c} = \lambda^{A+B}_{b,c}$ and $\alpha\, \lambda^A_{b,c} = \lambda^{\alpha A}_{b,c}$, respectively. Also, the identity matrix $I \in \mathcal{N}$ since $I u_{b,c} = 1 \cdot u_{b,c}$ and I is symmetric. Thus $\mathcal{N}$ is a subspace of the full matrix algebra $\mathrm{Mat}_n\mathbb{C}$.

For the remainder of this section, let us assume that the spin model
$S = (X, W^+, W^-)$ satisfies the Type II equation. Our goal is to show that $\mathcal{N}$ is the

Bose-Mesner algebra of an association scheme. Thus by Theorem 2.3.2 we need to show that $\mathcal{N}$ satisfies the following conditions:

(i) $\mathcal{N}$ is closed under the ordinary matrix product.

(ii) $\mathcal{N}$ is closed under the Hadamard (entry-wise) product.

(iii) $\mathcal{N}$ contains J, the all 1's matrix.

We now consider the first of these three conditions:

(i) $\mathcal{N}$ *is closed under the ordinary matrix product.*

Notice that as b, c range over the set X, we obtain n^2 column vectors $u_{b,c}$ of size n. In order to show that $\mathcal{N}$ is closed under the ordinary matrix product, we will need the vectors $u_{b,c}$ to span the n-dimensional space $\mathbb{C}^{\mathbb{X}}$. The proof of the following lemma will show that the vectors span using the Type II condition.

4.1.1 Lemma. *If a spin model S satisfies the Type II condition, then for any fixed $b \in X$, the n vectors $u_{b,x}$, $x \in X$ are linearly independent.*

Proof. Let the matrix $U = (u_{b,1}, \ldots, u_{b,n})$. Note that we can write $U = \triangle W^-$, where $\triangle$ denotes the diagonal matrix with diagonal entries $W^+_{b,1}, \ldots, W^+_{b,n}$. We know that W^+ has non-zero entries because it must satisfy that $W^+_{b,c} W^-_{b,c} = 1$ for all $b, c \in X$. Therefore the diagonal entries of $\triangle$ are all non-zero implying that it is invertible. We are given the Type II condition $W^+ W^- = n \cdot I$ and thus we have $\left(\frac{1}{n} W^+\right) W^- = I$. This implies that W^- is invertible; in particular, $\left(\frac{1}{n} W^+\right)$ is its inverse. Now we have that U is an invertible matrix, since it is the product of two invertible matrices. We want to show that $u_{b,1}, \ldots, u_{b,n}$ are linearly independent. Suppose not, then we have

$$0 = c_1 u_{b,1} + c_2 u_{b,2} + \ldots + c_n u_{b,n},$$

not all c_i's zero. Let C be a $n \times n$ matrix such that all entries of the first row are c_1, all entries of the second row c_2 and so on, in particular $C \neq 0$. Then we can write

$$\begin{aligned}
[0] &= \begin{pmatrix} \\ u_{b,1} & u_{b,2} & \dots & u_{b,n} \\ \\ \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_1 & \dots & c_1 \\ c_2 & c_2 & \dots & c_2 \\ \vdots & \vdots & \dots & \vdots \\ c_n & c_n & \dots & c_n \end{pmatrix} \\
[0] &= U \cdot C
\end{aligned}$$

We know that U is invertible so we can multiply the last equation by U^{-1} on the left to obtain $[0] = C$, a contradiction. Hence the column vectors of U, namely $u_{b,1}, \dots, u_{b,n}$, are linearly independent. $\square$

Now we have that the vectors $u_{b,c}$ $(b, c \in X)$ span $\mathbb{C}^{\mathbb{X}}$. We are now ready to show that $\mathcal{N}$ is closed under the ordinary matrix product.

4.1.2 Lemma. *If a spin model S satisfies the Type II equation, then for any $A, B \in \mathcal{N}$, we have $AB = BA \in \mathcal{N}$. The eigenvalue of AB is given by $\lambda_{b,c}^{AB} = \lambda_{b,c}^{A}\,\lambda_{b,c}^{B}$.*

Proof. By assumption, $A\,u_{b,c} = \lambda_{b,c}^{A}\,u_{b,c}$ and $B\,u_{b,c} = \lambda_{b,c}^{B}\,u_{b,c}$. Therefore we can write

$$\begin{aligned}
(AB)\,u_{b,c} &= A\,(B\,u_{b,c}) \\
&= A\,(\lambda_{b,c}^{B}\,u_{b,c}) \\
&= \lambda_{b,c}^{B}\,(A\,u_{b,c}) \\
&= \lambda_{b,c}^{B}\,(\lambda_{b,c}^{A}\,u_{b,c}).
\end{aligned}$$

Similarly we get that $(BA)\,u_{b,c} = \lambda_{b,c}^{A}\,(\lambda_{b,c}^{B}\,u_{b,c})$. So it follows that

$$(AB)\,u_{b,c} = (BA)\,u_{b,c} \qquad (\forall b, c \in X$$

We know that the vectors $u_{b,c}$ span $\mathbb{C}^{\mathbb{X}}$ and thus the above equality holds for all basis vectors, which implies that $AB = BA$. Since we know that A and B are symmetric and commute it is easy to verify that AB is also symmetric:

$$(AB)^T = B^T A^T = BA = AB.$$

Therefore $AB \in \mathcal{N}$. □

Next we show that:

(ii) $\mathcal{N}$ *is closed under the Hadamard product*

The Hadamard product is the entry-wise matrix product and for any two matrices A and B it is defined by $(A \circ B)_{b,c} = A_{b,c} B_{b,c}$. The following lemma will be crucial in showing that $\mathcal{N}$ is closed under the Hadamard product.

4.1.3 Lemma. *If a spin model S satisfies the Type II equation, then for all $n \times n$ matrices A and B, and for all $b, c \in X$,*

$$(A \circ B)_{b,c} = \frac{1}{n} \sum_{x \in X} (A\, u_{b,c}) \circ (B\, u_{b,c}).$$

Proof. Consider the $a-$entry of $(A \circ B)\, u_{b,c}$:

$$\begin{aligned} ((A \circ B)\, u_{b,c})_a &= \sum_{x \in X} A_{a,x} B_{a,x} W^+_{b,x} W^-_{c,x} \\ &= \sum_{x \in X} \sum_{y \in Y} \delta_{x,y}\, A_{a,x} B_{a,y} W^+_{b,x} W^-_{c,y} \end{aligned}$$

We introduced the extra sum and the Kronecker delta only because it lets us use the Type II condition as follows:

$$\text{Type II:} \qquad \delta_{x,y} = \frac{1}{n} \sum_{z \in X} W^+_{y,z} W^-_{x,z}$$

Thus we can replace the Kronecker delta in the above equation such that we get the following result:

$$\begin{aligned}
((A \circ B)\, u_{b,c})_a &= \sum_{x\in X}\sum_{y\in Y} \frac{1}{n} \sum_{z\in X} W^+_{y,z}\, W^-_{x,z}\, A_{a,x}\, B_{a,y}\, W^+_{b,x}\, W^-_{c,y} \\
&= \frac{1}{n} \sum_{z\in X}\sum_{x\in Y} A_{a,x}\, W^+_{b,x}\, W^-_{z,x} \sum_{y\in Y} B_{a,y}\, W^+_{z,y}\, W^-_{c,y} \\
&= \frac{1}{n} \sum_{z\in X}\sum_{x\in Y} A_{a,x}\, (u_{b,z})_x \sum_{y\in Y} B_{a,y}\, (u_{z,c})_y \\
&= \frac{1}{n} \sum_{z\in X} (A\, u_{b,z})_a\, (B\, u_{z,c})_a \\
&= \frac{1}{n} \sum_{z\in X} ((A\, u_{b,z}) \circ (B\, u_{z,c}))_a
\end{aligned}$$

Note that we use the symmetry of W^+ and W^-, as well as the definition of matrix-column vector entry-wise multiplication in the above reasoning. This proves the lemma. □

Now consider the following lemma, it will provide us with the needed closure under $\circ$:

4.1.4 Lemma. *If a spin model S satisfies the Type II equation, then for all $A, B \in \mathcal{N}$, we have $A \circ B \in \mathcal{N}$. The eigenvalue of $A \circ B$ is given by* $\lambda^{A\circ B}_{b,c} = \frac{1}{n} \sum_{z\in X} \lambda^A_{b,x}\, \lambda^B_{x,c}$

Proof. For $x \in X$, we have

$$\begin{aligned}
(A\, u_{b,z}) \circ (B\, u_{z,c}) &= (\lambda^A_{b,x}\, u_{b,x}) \circ (\lambda^B_{x,c}\, u_{x,c}) \\
&= \lambda^A_{b,x}\, \lambda^B_{x,c}\, (u_{b,x} \circ u_{x,c}).
\end{aligned}$$

We know that the $y-$entry of $u_{b,x} \circ u_{x,c}$ is as follows:

$$\begin{aligned}
(u_{b,x} \circ u_{x,c})_y &= W^+_{b,y}\, W^-_{x,y} \cdot W^+_{x,y}\, W^-_{c,y} \\
&= W^+_{b,y}\, W^-_{c,y} && \text{by (2.1)} \\
&= (u_{b,c})_y && \text{by (4.1)}
\end{aligned}$$

Thus

$$u_{b,x} \circ u_{x,c} \;=\; u_{b,c}.$$

Then lemma 4.1.3 implies

$$\begin{aligned}(A \circ B)\, u_{b,c} \quad &= \quad \frac{1}{n} \sum_{x \in X} (A\, u_{b,z}) \circ (B\, u_{z,c}) \\ &= \quad \frac{1}{n} \sum_{x \in X} \lambda^{A}_{b,x}\, \lambda^{B}_{x,c}\, u_{b,c}\end{aligned}$$

Since A and B are symmetric, $A \circ B$ is symmetric. Thus we have shown that $(A \circ B) \in \mathcal{N}$. □

Now we are only left to show that J, the all 1's matrix, is contained in $\mathcal{N}$.

(iii) $\mathcal{N}$ *contains* J

Let J be the $n \times n$ matrix with all entries equal to 1.

4.1.5 Lemma. $J \in \mathcal{N}$

Proof. The consider the $a-$entry of $J\, u_{b,c}$:

$$\begin{aligned}`(J\, u_{b,c})_a \quad &= \quad \sum_{x \in X} J_{a,x}\, (u_{b,c})_x \\ &= \quad \sum_{x \in X} 1 \cdot W^{+}_{b,x}\, W^{-}_{c,x} \\ &= \quad n \cdot \delta_{b,c} \qquad\qquad \text{by Type II condition}\end{aligned}$$

Recall that $\delta_{b,c} = 1$ if $b = c$ and it equals zero otherwise. So $J\, u_{b,c} \;=\; \lambda^{J}_{b,c}\, u_{b,c}$ holds with $\lambda^{J}_{b,c} \;=\; n \cdot \delta_{b,c}$. That if $b = c$ then J has eigenvalue 1 and otherwise 0. Clearly J is a symmetric matrix and thus $J \in \mathcal{N}$. □

4.1.6 Theorem. *If a spin model S satisfies the Type II condition, then* $\mathcal{N}$ *is the Bose-Mesner algebra of some association scheme.*

Proof. Lemmas 4.1.2, 4.1.4, and 4.1.5 show that $\mathcal{N}$ is a commutative algebra closed under the Hadamard product and containing the all 1's matrix J and identity I. By Theorem 2.3.2 we know that this implies that $\mathcal{N}$ is the Bose-Mesner algebra of some association scheme. □

4.2 A Spin Model satisfying the Type III condition

In this section we will show that if a spin model satisfies the Type III equation then W^+ actually has $u_{b,c}$ as an eigenvector and thus is contained in $\mathcal{N}$. The Type III equation is what caused Nomura to believe that the matrices of a spin model actually sit inside the Bose-Mesner algebra of an association scheme. Recall from linear algebra that a set of pair-wise commuting matrices can be simultaneously diagonalized. See, for example [7, p.291]. In particular, the matrices $A_0, A_1, \dots, A_D$ of an association scheme share a common basis of eigenvectors. When Nomura saw that the Type III equation implies that W^+ has eigenvectors $u_{b,c}$, he realized that these eigenvectors were the key to constructing the desired Bose-Mesner algebra containing W^+. Thus he was able to prove the following lemma.

4.2.1 Lemma. *If S satisfies the Type III condition, then $W^+ \in \mathcal{N}$.*

Proof. The Type III condition states that

$$\sum_{x\in X} W^+_{a,x}\, W^+_{b,x}\, W^-_{c,x} \;=\; \sqrt{n}\, W^+_{a,b}\, W^-_{a,c}\, W^-_{b,c} \qquad (a,b,c \in X).$$

Since W^+, W^- are symmetric, we can rewrite the Type III equation as follows and then using the definition of $u_{b,c}$ to make the needed substitution.

$$\sum_{x\in X} W^+_{a,x}\, W^+_{b,x}\, W^-_{c,x} \;=\; \sqrt{n}\, W^+_{b,a}\, W^-_{c,a}\, W^-_{b,c}$$

$$\begin{aligned}\sum_{x \in X} W^+_{a,x}\, u_{b,c} &= \sqrt{n}\, u_{b,c}\, W^-_{b,c} \\ (W^+\, u_{b,c})_a &= \sqrt{n}\, W^-_{b,c}\, (u_{b,c})_a \qquad (a, b, c \in X).\end{aligned}$$

This implies that $u_{b,c}$ is an eigenvector of W^+ with eigenvalue $\sqrt{n}\, W^-_{b,c}$. Therefore $W^+ \in \mathcal{N}$. $\square$

Moreover, we can say that $W^- \in \mathcal{N}$ since the Type III condition is equivalent to

$$\sum_{x \in X} W^+_{a,x}\, W^-_{b,x}\, W^-_{c,x} = \sqrt{n}\, W^+_{a,b}\, W^+_{a,c}\, W^-_{b,c} \qquad (a, b, c \in X),$$

as shown in section 2.2. Similarly to the proof given above we can show that $u_{b,c}$ is an eigenvector of W^-, thus showing that $W^- \in \mathcal{N}$.

Proof of Theorem 4.0.2.

Theorem 4.1.6 and lemma 4.2.1 let us conclude that if a spin model S satisfies the Type II and Type III conditions then S gives rise to a Bose-Mesner algebra, such that W^+, W^- are contained in this algebra. $\square$

Chapter 5

Conclusion

In this paper, we have studied spin models and their use in distinguishing links given their link diagrams. In particular, we introduced the partition function Z_L^S that is constructed using these spin models. We have shown how the partition function stays invariant under the Reidemeister moves II and III as well as how it behaves under Reidemeister move I. Furthermore, we considered the question of how to find spin models by exploring their connection to the Bose-Mesner algebra of an association scheme.

There are many extensions of these results and directions for further research. For example, in this paper we have only considered unoriented links under which the partition function is only partially invariant. It is interesting to note that the partition function can be normalized to produce a true link invariant of oriented links. For more information on this topic we refer the reader to [10], [14], [15].

Moreover, the concept of a spin model can be generalized to allow the matrices W^+, W^- to be non-symmetric without losing any of the invariant properties of the partition function Z_L^S. It is necessary, however, to adjust some of the axioms and invariance equations. In a similar vein, the concept of an association scheme generalizes to the non-symmetric case, and non-symmetric spin models are related to these non-symmetric association schemes in manner entirely analogous to our Theorem 4.0.2. This line of research is explained in [14].

Probably the most active research done in this area concerns the question of which association schemes give rise to spin models. In this paper we have shown that all spin models satisfying the Type II and III equations give rise to association schemes, but it is not true that every association scheme contains the matrices of a spin model. The classification of spin models is still an open problem. For partial results in this direction, see [1], [2], [6], [13], [18], [19], and [21].

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