Distortion Colourings Of Cubic Graphs

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Abstract

In this MTH 501 project, we present an expansion of the 2013 paper, "Delay Colourings Of Cubic Graphs" by Agelos Georgakopoulos [3]. The paper introduces a novel graph-theoretic object, called a distortion, and proves that every cubic bipartite multigraph admits a distortion colouring.

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Contents

1 Introduction

In this MTH 501 project, we offer an expanded presentation of the results found in the 2013 paper, "Delay Colourings Of Cubic Graphs" by Agelos Georgakopoulos [3]. Our aim in this project is to present the main mathematical results, filling in occasional supplementary details of the paper that are left to the reader, and to offer additional commentary which will serve to illustrate results and elucidate proof methods wherever possible.

1.1 Context

Graph colorings form an important topic in applications of graph theory; many real-world problems modeled by graphs involve partitioning the vertices or edges of a given graph into disjoint sets such that the vertices or edges within these sets are non-adjacent or non-incident. For example, such constraints commonly arise in problems concerning scheduling, either to avoid conflicts or to minimize the total time of composite tasks when various subtasks might be able to be performed simultaneously.

The present paper is focused on distortion colourings of the edges of multigraphs, which is an mathematical abstraction rooted in the design of optical networks [7]. In that conceptualization, a color assigned to an edge may appear distorted in some way, visually or temporally, from the vantage point of one endpoint than from the other endpoint.

More specifically, in the paper under consideration, [3], the author defines a novel graph-theoretic object, called a distortion, which associates a restricted set of ordered pairs of colours with a given graph. The goal is to extend the notion of proper edge colourings to this more general context and to show that these novel proper edge colourings exist under specific constraints.

1.2 Statement of Main Result

The main result of the paper proves that every cubic bipartite multigraph admits a distortion colouring. In particular, we have the following.

Theorem 1. *Let G be a bipartite multigraph with maximum degree d, partition classes A and B, and a distortion (definitions in the next section) on $d+1$ colours. If $d = 3$, then G is properly $(d+1)$-distortion-colourable.*

1.3 Organization of Paper

For ease of reading, the content of this project is organized into five chapters. Chapter 1 contains a very brief introduction to the topic of distortion colourings, offering a bit of historical and mathematical context, and stating the main result. Chapter 2 presents a summary of the necessary definitions concerning graphs, colourings, matchings, and distortions. Chapter 3 collects a number of important preliminary results to facilitate the proof of the main result, including the main construction that will establish the 4-colourability of the graphs under consideration. In Chapter 4, the main proof shows that the construction achieves the desired goal. Finally, in Chapter 5 we conclude with a brief discussion of extent, related work and potential directions for further research.

2 Background and Definitions

This chapter houses some basic definitions from graph theory to provide the context for the exposition. For more detail on these concepts, we refer the reader to any standard textbook on graph theory, such as West [6]. More importantly, it also introduces and rigorously defines the novel objects associated with distortions.

2.1 Multigraphs

The main object of concern for this paper will be finite, loopless multigraphs, defined immediately below.

A (finite, loopless) **multigraph** G consists of a (finite) **vertex set** $V(G)$, a (finite) **edge set** $E(G)$, and an **incidence relation** that associates, with each edge, an unordered set of two vertices, called its **endpoints**. When u and v are the endpoints of an edge, they are said to be **adjacent**, written $u \sim v$. The **degree** of a vertex v is the number of edges that have v as an endpoint. A multigraph in which every vertex has degree 3 is said to be **cubic**.

A multigraph G is said to be **bipartite** whenever the vertex set $V(G)$ can be partitioned into two nonempty subsets A, B such that every edge has one endpoint in each partition class.

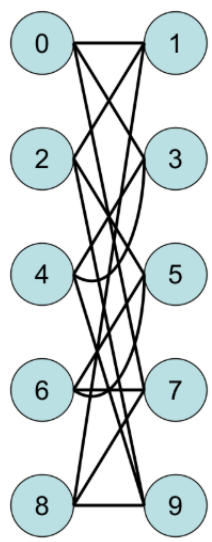

Figure 1. A bipartite cubic multigraph

2.2 Matchings and Colourings

A set M of edges in a multigraph G is called a **matching** if no two edges in M share an endpoint. A **perfect matching** of G is any matching M with the property that every vertex of G occurs as an endpoint of some edge in M.

An **edge-colouring** of a multigraph G is a labeling $f : E(G) \to S$, where S is a set of labels known as **colours**. When $|S| = r$, we refer to such a labeling f as an r-**colouring** and the set of edges assigned any given colour as a **colour class**.

An r-colouring is said to be **proper** whenever each colour class forms a matching. If a multigraph G requires at least r colors to be properly coloured, we refer to r as the **edge chromatic number**, $\chi'(G)$. In other words, the chromatic number is the minimum number of colours needed to label the edges so that edges incident with a vertex receive different colours.

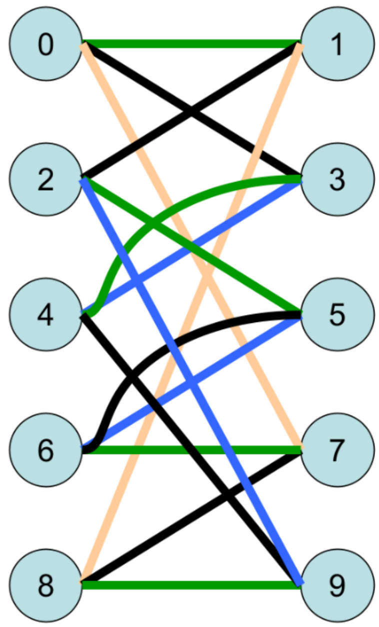

Figure 2. A proper 4-colouring of the edges of a multigraph

2.3 Distortion Colourings

In this section, let G denote a finite, loopless multigraph. Assume G is bipartite, with partition classes P_1 and P_2. The vocabulary regarding distortions is as follows.

Definition 1. *Let an* ***edge distortion*** d_{uv}*, be a set of ordered pairs (2-tuples) of colours that is:*

(1) a priori associated to edge uv, with

(2) the indices i in each tuple corresponding to partition classes P_i, where

(3) each colour is found exactly once in each index (over all tuples in the set).

For example, one edge distortion over the colours $\{\alpha, \beta, \gamma\}$ is

$$\{(\alpha, \alpha), (\beta, \gamma), (\gamma, \beta)\}.$$

If $u \in P_1$ and $v \in P_2$, the first element of a tuple corresponds to a potential colour for the u incidence, and the second element of a tuple corresponds to a potential colour for the v incidence. Furthermore, condition (3) satisfies the requirements for the values in any two indices of a tuple (over all tuples) to be a valid permutation of the underlying colours, by definition of permutation. Therefore, an edge distortion acts as a bijective function over the colours, mapping values in one index of a tuple to values in some other index of a tuple (over all tuples), e.g. $d_{uv}(\beta) = \gamma$. The inverse of the function is denoted using the same notation with the subscripts reversed: $d_{vu}(\gamma) = \beta$.

Definition 2. *For a given multigraph G, let a* ***distortion*** *be the set of all edge distortions for the edges of G.*

Definition 3. *For a given edge in a multigraph G, let a* ***distortion representative*** *be the tuple from an edge distortion used to colour the endpoints of that edge.*

Definition 4. *Let a* ***distortion colouring*** *be the labeling resulting from the assignment of distortion representatives to all edges in a graph. A distortion colouring is* ***proper*** *whenever no two edge endpoints incident to a given vertex are the same colour, over all vertices.*

3 Preliminary Results

This section collects a number of important preliminary results concerning bipartite cubic graphs and perfect matchings.

3.1 Bipartite Cubic Multigraphs

We first show that, if a bipartite multigraph has maximum degree 3, we can consider it to be a subgraph of a bipartite multigraph that is cubic.

Lemma 1. *Any bipartite multigraph with maximum degree 3 can be extended to a cubic bipartite multigraph.*

Proof. Let G be a non-cubic bipartite multigraph with maximum degree 3. Consider the following construction to create G':

1. G may contain some quantity of degree-0 vertices. If so, replace each degree-0 vertex with an instance of the complete bipartite graph on 6 vertices (commonly denoted $K_{3,3}$), making six degree-3 vertices. This will force G' to contain no degree-0 vertices.

2. G may contain degree-1 or degree-2 vertices in both partitions. If so, in one of the two partitions, for each non-degree-3 vertex, add and connect a new vertex to the non-degree-3 vertex with either one or two edges to make the non-degree-3 vertex degree-3. This will force G' to have any remaining degree-1 and degree-2 vertices in one partition.

3. G may contain more than three degree-1 vertices within a partition. If so, for each set S of these three degree-1 vertices, add three new vertices and six edges such that there are two edges incident to each degree-1 vertex and the new vertex paired with it, making those degree-1 vertices degree-3. This construction will force G' to contain fewer than three degree-1 vertices but will create new degree-2 vertices.

4. G may contain more than three degree-2 vertices within a partition. If so, for each set S of these three degree-2 vertices, add a new vertex with three edges such that each edge is incident to a vertex in S, making those degree-2 vertices degree-3. This will force G' to contain fewer than three degree-2 vertices.

5. G may contain at least one degree-1 vertex and at least one degree-2 vertex within a partition. If so, collect the degree-1 and degree-2 vertices in pairs until there are no more of one of the degrees; for each pair, add a new vertex and three edges: one from the new vertex to the degree-2

vertex, and two from the new vertex to the degree-1 vertex.

It remains to be shown that the resulting multigraph G' will be cubic.

Assume not. Then G' has either one or two degree-1 vertices, or one or two degree-2 vertices, all in one partition. Each degree-1 vertex can be connected to a new vertex via two edges to make degree-2 vertices. If there are two degree-2 vertices, a vertex can be added and connected to them to make a unique degree-2 vertex. A degree-2 vertex can be attached to a new vertex by a single edge to make a degree-1 vertex. Therefore, the four remaining cases can be treated as equal to the case of a unique degree-1 vertex without loss of generality.

As G' is bipartite, this implies the vertex partition with the unique non-degree-3 vertex is incident with some number of edges that is congruent to 1 (mod 3). Since each edge has exactly one endpoint in the other partition, the other vertex partition must also be incident to 1 (mod 3) edges, but all vertices in that partition are degree 3. This is a contradiction. Note that at every step of the construction, the number of edges incident to each partition consistently changes as follows:

$$0 \bmod 3 \hookrightarrow 0 \bmod 3$$
$$1 \bmod 3 \hookrightarrow 2 \bmod 3$$
$$2 \bmod 3 \hookrightarrow 1 \bmod 3$$

Since every step of the construction is fully reversible, the contradiction in G' implies an impossible starting graph. Thus, G' is cubic for any given G. □

3.2 Matchings in Bipartite Multigraphs

We next review two basic facts about matchings.

Lemma 2. *Suppose G is any bipartite, k-regular multigraph. The edge set of G is decomposable into k disjoint perfect matchings.*

Proof. Consider the following construction:

> For each $i \geq 1$, create the ith matching by choosing edges only incident to vertices of degree $k - i + 1$. As each edge is assigned to a matching, remove it from the graph.

Note that, by virtue of the multigraph being bipartite and regular, the two implied vertex partition classes must have the same cardinality via the Pigeonhole Principle. It follows that the iteration of the above construction produces the desired perfect matchings. □

Lemma 3. *Suppose G is any multigraph. The union of any two disjoint perfect matchings in G is a set of disjoint cycles that span all of the vertices.*

Proof. Every vertex appears in each perfect matching exactly once, by definition. Since the matchings are disjoint, every vertex in the union of two disjoint perfect matchings has degree 2. This is only possible in a forest of cycles. □

3.3 Agreement Among Distortions

Below is a highly useful combinatorial result which guarantees some agreement among the tuples of any two distortions.

Lemma 4. *Any two edge distortions (comprised of tuples with two indices) over the same set of colours have at least one tuple in common.*

Proof. Assume not. Given that each edge distortion is a permutation of colours, there are $n - (i - 1)$ choices for the ith tuple (given some arbitrary ordering) of the first edge distortion. Each tuple of the second edge distortion must avoid the first, so there are $n - (i - 2)$ choices for the ith tuple (given the same ordering as the first edge distortion). However, this forces 0 valid choices for the final tuple of the second edge distortion, which is impossible. Thus, two edge distortions (comprised of tuples with two indices) built from the same colour sets must have at least one tuple in common. □

3.4 The Main Construction

Suppose we are given a bipartite cubic multigraph G with a corresponding distortion. As the main result is an existence claim, we aim to provide a construction that always permits a proper 4-distortion-colouring. In this section, we present said main construction. Consider the following:

By Lemma 2, there exists a decomposition of the edges of G into three perfect disjoint matchings, M_1, M_2, M_3. By Lemma 3, we may choose any matching M_j, and let C_i be the cycles of the union of the other two matchings. Define an edge set $M_{j,i,A} = \{(a,b)|(a,b) \in M_j; a \in V(C_i \cap A)\}$, the set of edges in M_j with endpoints in the vertices of C_i that are in vertex partition A. Let the edge set $M_{j,i,B} = \{(a,b)|(a,b) \in M_j \setminus M_{j,i,A}; a \in V(C_i \cap B)\}$, the remaining edges in M_j with endpoints in the vertices of C_i that are in vertex partition B. Thus, $M_{j,i,A} \cap M_{j,i,B} = \{\}$ and $M_{j,i,A} \cup M_{j,i,B} = M_j$.

Furthermore, define uvy be a two-edge arc of C_i such that u and y are in vertex partition A. Let m_u and m_y be the edges in $M_{j,i,A}$ incident to u and y, respectively, and m_v the edge in $M_{j,i,B}$ incident to v. Finally, denote the four colours all edge distortions are defined over as α, β, γ, and δ.

We now prove the key property of the above construction in the following lemma.

Lemma 5. *For every C_i in $M_1 \cup M_2$, there is a 4-distortion-colouring f_A of $M_{3,i,A}$ such that for every 4-distortion-colouring f_B of $M_{3,i,B}$, there is a 4-distortion-colouring f_C of $E(C_i)$ such that $f_A \cup f_B \cup f_C$ is a proper 4-distortion-colouring of C_i.*

Proof. It will suffice to show the set of available distortion representatives for each edge of C_i during the process of distortion colouring is non-empty. Consider four cases: $d_{uv} = d_{yv}$ or $d_{uv} \neq d_{yv}$, and $u \neq y$ or $u = y$.

Case 1: If $d_{uv} = d_{yv}$ and $u \neq y$, that is, the edge distortions of uv and yv are identical and the arc uvy is not a 2-cycle, then assign distortion representatives to m_u and m_y (i.e. distortion-colour those edges) such that the underlying colours associated to the u and y endpoints are different. Without loss of generality, choose α for the association with u and β for y. Assign distortion representatives to the rest of M_j arbitrarily to complete f_A and f_B.

Distortion-colour $E(C_i)$ starting at edge vu and continue around the cycle ending with edge yv as follows. Since f_A and f_B are already distortion-coloured, if the colour associated to the m_v endpoint v by f_B is the same as the colour $d_{yv}(\beta)$, then there are three options for yv and two for vu. Choose either available distortion representative for vu. Note this reduces the options for yv to two; only the representative $(\beta, d_{yv}(\beta))$ and whichever representative conflicts with the uv representative are unavailable of the four tuples. Else, the colour associated to the m_v endpoint v by f_B is not the same as the colour of $d_{yv}(\beta)$. Since, in this case, $d_{yv}(\beta) = d_{uv}(\beta)$, colour uv with the distortion representative that would associate β to u. This ensures that yv still has at least two available options; only the representative $(\beta, d_{yv}(\beta))$ and whichever representative conflicts with the m_v representative by f_B are unavailable of the four tuples.

Since G is cubic and bipartite, every vertex in $V(C_i) \setminus \{u, v, y\}$ is incident to one edge in M_3, which has already been distortion-coloured by f_A or f_B. Having distortion-coloured vu, the next edge in the cycle (the third edge incident to u that is neither vu, the previous edge in the cycle, nor m_u, the edge in M_3), will have two available distortion representatives: namely, those not conflicting with the representatives for vu and m_v. Choose one of the two non-conflicting representatives. Continue this pattern until reaching yv, as for each new edge, there will be two choices (those not conflicting with the representatives for the previous edge in the cycle or the edge in M_3). At yv, the final edge of the cycle, of the two choices originally guaranteed by the strategic distortion-colouring of vu, only one will not conflict with the representative chosen for the previous edge in the cycle. Distortion-colour yv with this single non-conflicting representative. Then $E(C_i)$ has a 4-distortion-colouring.

Case 2: If $d_{uv} = d_{yv}$ and $u = y$, that is, the edge distortions of uv and yv are identical and the arc uvy is a 2-cycle, then assign any distortion representative to $m_u = m_y$. Without loss of generality, choose α for the association with $u = y$. Regardless of f_B, there are still at least two distortion representatives available for vu and yv: the tuples of d_{uv} with α not in the first index and whatever colour f_B associates with v in the second. Choose from these available representatives distinctly for each edge in arc uvy; then $E(C_i)$ has a 4-distortion-colouring.

Case 3: If $d_{uv} \neq d_{yv}$ and $u \neq y$, that is, the edge distortions of uv and yv are different and the arc uvy is not a 2-cycle, then the distortion-colouring of uvy is slightly more nuanced. Since $d_{uv} \neq d_{yv}$, there is at least one tuple in d_{uv} not in d_{yv} and at least one tuple in d_{yv} not in d_{uv}, the *non-shared* tuples. By Lemma 4, there is also at least one tuple in $d_{uv} \cap d_{yv}$, a *shared* tuple. Note that, combinatorially, non-shared tuples cannot exist in isolation; if they did, this would implies three shared tuples in each distortion, which would force the fourth tuple to be shared.

Distortion-colour m_u and m_y such that the underlying colours associated to the u and y endpoints are the same, say α, and such that the chosen representatives satisfy the relation $d_{uv}(\alpha) \neq d_{yv}(\alpha)$, (i.e. α is the first index in both members of a pair of non-shared tuples between d_{uv} and d_{yv}). Assign distortion representatives to the rest of M_j arbitrarily to complete f_A and f_B.

To distortion-colour $E(C_i)$, as in Case 1, the objective is to find distortion representatives such that a distortion-colouring of one the edges of uvy permits two options for the other edge, as then the logic for distortion-colouring the rest of the cycle is the same. There are three subcases. If the colour associated to the m_v endpoint v by f_B is α, then pick either edge vu or yv, and distortion-colour it with another non-shared tuple. This choice will force the other edge to have two valid options (at least one of which is a shared tuple), as desired.

If the colour associated to the m_v endpoint v by f_B is not α, but some colour χ which is the second index of another non-shared tuple pair, then one of the edges will have three valid options and the other will have two. This is because combinatorially, from the restrictions of this subcase, there are three non-shared tuple pairs, with (α, χ) being a non-shared tuple in one of the two distortions. Hence, that distortion will only have one invalid choice, instead of the two invalid choices expected (the tuple starting with α and the tuple ending with χ). To start the distortion-colouring process of the cycle, pick any representative from the edge distortion with two valid colours. This will invalidate a representative in the other edge distortion, bringing the number of options from three to two, as desired.

If the colour associated to the m_v endpoint v by f_B is not α, but some colour χ which is the second index of a shared tuple pair, then both edge distortions will have two valid options: some non-shared tuple associating α to v, and some other non-shared tuple (by the same logic as the previous subcase). Pick either edge and choose the representative that does not associate α to v. This ensures both options in the other edge distortion are available when closing the loop, as desired.

Thus, for all subcases, $E(C_i)$ has a 4-distortion-colouring.

Case 4: If $d_{uv} \neq d_{yv}$ and $u = y$, that is, the edge distortions of uv and yv are different and the arc uvy is a 2-cycle, then distortion-colour as in Case 3. If the colour associated to the m_v endpoint v by f_B is not α, then the logic is the same as the third subcase of Case 3. If it is α, then the logic is the same as the first subcase of Case 3. Either way, the other edge in the arc will have two options available, as desired, and it follows that $E(C_i)$ has a 4-distortion-colouring.

Since, for all cases, given the f_A distortion-colouring logic and an arbitrary f_B distortion-colouring, $E(C_i)$ has a 4-distortion-colouring, it immediately follows that there is a 4-distortion-colouring f_A of $M_{3,i,A}$ such that for every 4-distortion-colouring f_B of $M_{3,i,B}$, there is a 4-distortion-colouring f_C of $E(C_i)$ such that $f_A \cup f_B \cup f_C$ is a proper 4-distortion-colouring of C_i. □

4 Proof of Main Theorem

Recall the main statement to be proved: *Let G be a bipartite multigraph with maximum degree d, partition classes A and B, and a distortion on $d+1$ colours. If $d = 3$, then G is properly $d+1$-distortion-colourable.*

Proof: By Lemma 1, restrict G to cubic bipartite multigraphs. By Lemmas 2, 3, and 4, complete the construction shown in the proof of Lemma 5 for any single cycle as given by Lemma 3. By Lemma 3, all cycles are disjoint, so the construction in Lemma 5 can be trivially extended to all cycles (as given by Lemma 3) keeping the distortion colouring proper. This is because the edges distortion-coloured by f_A for any given i correspond to some subset of edges distortion-coloured by f_B for all other i, and f_B is specifically defined to be arbitrarily distortion-coloured for each cycle, there is no conflict in extending the distortion-colouring cycle by cycle. By Lemma 3, since the union of $M_{j,i,A}$ over i is M_j, this distortion colouring covers G. Therefore, for every bipartite multigraph G of maximum degree three, and given any distortions over G, there exists a proper distortion colouring on four colours. $\square$

5 Discussion and Related Research

Edge distortions, though defined here in generality, are found elsewhere in the literature, with several unanswered questions and partial results.

5.1 Delay Colourings

While this paper proves the $d = 3$ case, it is not presently known if bipartite multigraphs with maximum degree d are $(d + 1)$-distortion-colourable in general. However, if the edge-distortion is restricted to a specific type of permutation, $(\alpha\beta\gamma \ldots \omega)$ and its powers, further results are known.

Wilfong, et al [7] call this restriction a *delay colouring*, and, proving it true for $d = 4$, conjectured in 2001 that all bipartite multigraphs with maximum degree d are $d + 1$-delay-colourable. In 1952, Hall [4], not using the language of delay-colourability, proved a combinatorially equivalent to $(d + 1)$-delay-colourability for spindles via what is now known as the Fundamental Theorem of Juggling. In 2007, Alon and Asodi [1] showed other families of multigraphs are $(d + 1)$-delay-colourable, such as the d-regular multigraphs whose underlying simple graph is a simple cycle of even length where $d + 1$ is prime. Alon and Asodi further conjecture that multigraphs, not necessarily bipartite, are $(\chi'(G)+1)$-delay-colourable, attempting to associate the delay-colourability of G to the edge chromatic number of G.

5.2 Extending to Hypergraphs

There is also a different relaxation with a more standard application. Recall the language of the definition of an edge distortion: let an edge distortion be a set of ordered tuples of colours that is a priori associated to edge, with the indices i in each tuple corresponding to partition classes P_i, where each colour is found exactly once in each index (over all tuples in the set). This language was employed specifically to trivially generalize the definition to hyperedges (and hypergraphs). Indeed, consider the following conjecture from Wilfong et al [7].

Conjecture 1. *Let H be a tripartite 3-uniform hypergraph with partition classes A, B, and C, such that $|B| = |C| = |A| + 1$. Suppose that for every $x \in A$, the set of hyperedges containing x induces a perfect matching of $B \cup C$. Then A is matchable.*

Here, A being matchable means that there is a matching in H containing all vertices in A. The equivalence to the Main Proposition of this paper is as follows: represent each edge in the Main Proposition by a vertex in A, and let B and C be sets of size $d + 1 = |A| + 1$, to be thought of as the colour on the left endpoint and the right endpoint, respectively. This strengthens the well-known Brualdi-Stein conjecture, named by Brualdi [2] and Stein [5], who independently considered the question: in every $n \times n$ Latin square, does there exist a transversal of size $n - 1$?

The implication from the hyperedge construction is follows: construct a tripartite hypergraph H with A being the set of rows of a Latin square, B, the columns, C being the set $\{1, \ldots, n\}$, and each entry of the Latin square introducing a hyperedge contains the three corresponding vertices of H. Delete an arbitrary vertex in A with all edges containing it to obtain the hyperedge construction. As of this writing, the Brualdi-Stein conjecture is an open problem.

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