On the Connectivity of Connected Bipartite Graphs With Two Orbits

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Abstract

In this 501 project, we review the article "Connectivity of Connected Bipartite Graphs with Two Orbits" by Meng [1]. In that paper it is claimed that any connected bipartite graph with two vertex orbits has connectivity equal to its minimum degree. This paper exhibits a counterexample to that claim, as well as a modified version of their argument to prove a weaker statement. Specifically, we show that the conclusion holds for any connected graph with two vertex orbits, where the orbits coincide with the cells of the bipartition. In other words, all half-vertex transitive graphs have connectivity equal to their minimum degree.

1 Introduction

One of the first topics covered in any introductory graph theory course is the connectivity of a graph. The concept, on its face, is very simple; the connectivity of a graph is the minimum number of vertices that must be removed in order to disconnect it. Despite the simplicity of the the definition, in practice it can be quite difficult to determine the connectivity of an arbitrary graph, to the extent that algorithms have been developed to assist with the process [4]. Of particular interest are those graphs with either very high or very low connectivity, as edge case examples tend to be the most useful in applications. Therefore, the business of determining families of highly connected graphs is an active area of study [5][6]. In this paper, we consider the work of Meng [1], in which it is claimed that that any connected bipartite graph with two vertex orbits has connectivity equal to its minimum degree. Here we exhibit a counterexample to that claim, as well as a modified version of their argument to prove a weaker statement. Specifically, we show that the conclusion holds for any connected graph with two vertex orbits, where the orbits coincide with the cells of the bipartition. In other words, all half-vertex transitive graphs have connectivity equal to their minimum degree.

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2 Definitions

Let $X = (V, E)$ be a connected graph with vertex set V and edge set E. Let Y be a non-empty subset of V. The *neighborhood* of Y, denoted $N(Y)$ is defined as:

$$N(Y) = \{v \in V \backslash Y : \exists y \in Y \text{ s.t. } xy \in E\} \tag{1}$$

In other words, the neighborhood of Y consists of all vertices in $V \backslash Y$ that are adjacent to a vertex in Y. A *cut set* U of X is any set of vertices for which $V \backslash U$ induces a subgraph of X that is either not connected or is isomorphic to K_1. The *connectivity*, κ, of X is the minimum cardinality of all cut sets of X. A subset F of V is said to be a *fragment* if $N(F)$ is a minimal cut set of X. A fragment of minimum cardinality is called an *atom* of X.

The *degree* of a vertex $v \in V$ is defined to be the number of edges that have v as an endpoint. The *minimum degree*, δ, of X is the minimum degree of all vertices of X. Since $N(v)$ is a cutset for any vertex v, an immediate upper bound on the connectivity of a connected graph is that:

$$\kappa \leq \delta \tag{2}$$

A graph is said to be *bipartite* if there exists a partition of V into two parts P_1, P_2 such that $vu \notin E(X)$ for any $v, u \in P_i$, $(i = 1, 2)$.

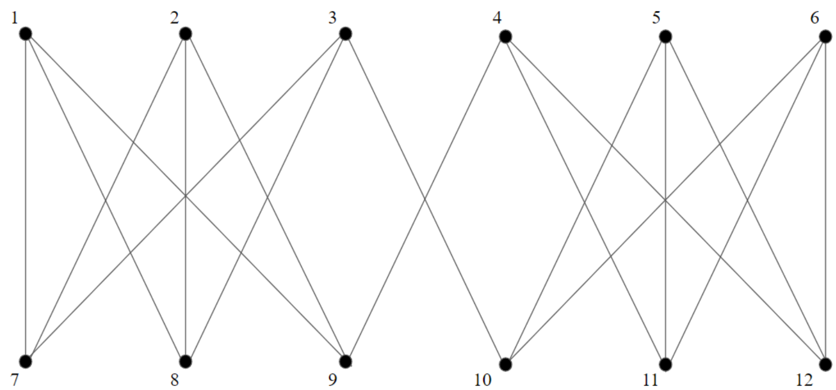
Now let $Aut(X)$ denote the automorphism group of X. Graph X is said to be *vertex transitive* if, for any pair of vertices u, v, there exists some $g \in Aut(X)$ such that $g(u) = v$. An *orbit* of $Aut(X)$, (or equivalently an orbit of X) is a set $\{x^g : g \in Aut(X)\}$ for some $x \in V(X)$. The distinct orbits of a graph form a partition of the vertex set of the graph. A graph is said to be *half-vertex transitive* if it is bipartite with partition P_1, P_2 and has two orbits, O_1, O_2 such that (without loss of generality) $P_1 = O_1$ and $P_2 = O_2$.

Another important concept is an *imprimitive block*. Let T be a set and G a permutation group that acts on it. A proper subset A of T of size greater than 1 is said to be an imprimitive block (or equivalently a block of imprimitivity) of G on T if every $\sigma \in G$ has either $\sigma(A) = A$ or $\sigma(A) \cap A = \emptyset$. In the context of graph theory, this means that, given a graph $X(V, E)$, a subset $Y \subset V$ is a block of imprimitivity of X if it is an imprimitive block of $Aut(X)$ on $V(X)$.

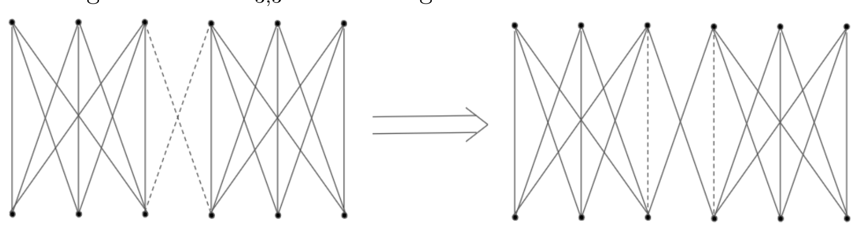
3 Previous Claim

As mentioned above, the bound (1) stems from the property that removing every vertex adjacent to a minimum degree vertex in a graph is sufficient to disconnect the graph or isolate that vertex. What is of interest to us are graphs that meet this upper bound, in other words graphs that have connectivity equal to their minimum degree. One example of a family of such graphs are the complete graphs, each of which can only be disconnected by removing all but one of their vertices. This claim of completeness is a very strong condition; of interest is finding weaker conditions that also result in these highly connected graphs. One well established condition is that any edge transitive graph has connectivity equal to its minimum degree [2]. Edge transitive graphs are either vertex transitive or are half-vertex transitive [3], so the family of edge transitive, half-vertex transitive graphs have connectivity equal their degree. The question that follows is whether this family can be extended to a larger family of graphs that also has the property of high connectivity.

To this end, Meng [1] claims that any connected bipartite graph with two orbits has connectivity equal to its degree. Unfortunately, this claim is not true. An assumption is made in their argument that any bipartite graph with two orbits is half-vertex transitive. This need not be the case, as the graph G below illustrates:



This graph is bipartite, with bipartition {1,2,3,4,5,6} and {7,8,9,10,11,12}, and has two orbits, {1,2,5,6,7,8,11,12} and {3,4,9,10}, but its orbits do not coincide with its bipartition, so it is not half-vertex transitive. Additionally, G has minimum degree 3, but both {3,9} and {4,10} are cut sets of size two, so $\kappa(G) < \delta(G)$. In fact, this graph is but one of an entire family of bipartite graphs with two orbits which have connectivity less than their degree. Graph G can be constructed by taking two $K_{3,3}$ graphs and performing a two-switch on a one edge from one $K_{3,3}$ and one edge from the other.



This same process can be repeated for any $K_{n,n}$, the result being an n-regular, connected, bipartite graph with two orbits and a cut set of size 2. Therefore for any positive integer k, the graph H constructed in this way using two copies of $K_{k+2,k+2}$ has $\delta(H) - \kappa(H) = k$. In other words, not only can the equality between the connectivity and minimum degree be broken, but their difference can be made arbitrarily large.

Fortunately, while the claim of the paper is incorrect, the weaker claim that half-vertex transitive graphs have connectivity equal to their degree is true, and is proved by the argument of the paper. I will therefore attempt to recover the this weaker claim by following the argument laid out in the paper.

The argument relies on three previously proven theorems. These are:

Theorem 1 [2] *Let $X = (V, E)$ be connected graph which is not a complete graph.*

(i) $\kappa(X) = \delta(X)$ if and only if every atom of X has cardinality 1;

(ii) if $\kappa(X) < \delta(X)$, then each atom has cardinality at most $[(|V| - \kappa(X))/2]$ and induces a connected subgraph of X.

Theorem 2 [7] *If X=(V,E) is a connected graph which is not a complete graph, then distinct atoms of X are disjoint. Thus if $\kappa(X) < \delta(X)$, the atoms of X are imprimitive blocks of X.*

Theorem 3 [3] *Let X=(V,E) be a connected graph. If W is a minimum vertex cut set and A an atom of X, then $A \cap W = \emptyset$ or $A \subseteq W$.*

4 Recovering The Proof

Let $X = (V, E)$ be a connected half-vertex transitive graph. Since each of the vertices in each half of the bipartition is the image under an automorphism of every other vertex in that half of the bipartition, X is semi-regular. In this section X_0 and X_1 will denote the two halves of the biparition, or equivalently X_0 and X_1 will denote the two orbits of $Aut(X)$. Let m denote the valency of vertices in X_0, and let n denote the valency of vertices in X_1. Without loss of generality assume that $m \leq n$. Therefore $\delta(X) = m$. Let A be an atom of X, and let $A_0 = X_0 \cap A$, $A_1 = X_1 \cap A$. Since X_0 and X_1 partition X, $A = A_0 \cup A_1$.

Lemma 1 *Let X=(V,E) be a connected half-vertex transitive graph, and A be an atom of X. If κ(X) < δ(X), then $A_i = A \cap X_i$ (i=0,1) have size greater than 1.*

Proof: Since $X = (V, E)$ has two orbits, it must have at least two vertices. Graph X is bipartite, so the only complete graph that X could be is K_2, but K_2 has $\delta(K_2) = \kappa(K_2)$, so X is not complete. Therefore, since X is connected, Theorem 1 indicates that $Y = X[A]$ is a connected subgraph of X with at least two vertices. Since Y is connected and bipartite, each half of the bipartition must have at least one element, so $|A_i| \geq 1$ for $i \in 0, 1$. Now suppose that $|A_i| = 1$ for one of the two possible values of i.

Case 1: $|A_0| = 1$. Then, $|A_1| \leq m$ since Y is connected, so every vertex in A_1 must be adjacent to the single vertex in A_0.

Subcase 1.1: Let $|A_1| = m$. Then $|N(A)| \geq n-1$, as each vertex in A_1 must be adjacent to n vertices, only one of which is in A_0. If $|N(A)| = n-1$, then every vertex in $N(A)$ is adjacent to all m vertices of A_1, so $N(N(A)) = A_1$, implying that $N(A) \cup A$ is a component of X. Since X is connected, this means that $N(A) \cup A = X$, which would mean that A is not an atom of X, a contradiction. Therefore $|N(A)| > n-1$, or equivalently,

$$\kappa(X) = |N(A)| \geq n \geq m = \delta(X),$$

again a contradiction.

Subcase 1.2: Let $|A_1| = p < m$. Then $N(A_0)\backslash A_1$, the set of neighbors of the single vertex in A_0 that are not in A_1, has $|N(A_0)\backslash A_1| = m - p$. Let $q = |N(A_1)\backslash A_0|$. Each vertex in A_1 is adjacent to n vertices in X_0, one of which is in A_0, so $q \geq n-1$. Each vertex in A_0 is adjacent to m vertices in X_1, one of which is in A_1, so $q \geq m-1$. Then:

$$|N(A)| = |N(A_0)\backslash A_1| + |N(A_1)\backslash A_0| = m - p + q \geq m + n - p - 1.$$

Since $|N(A)| = \kappa(X) < \delta(X) = m$, these two inequality chains combine to define the inequality $m > m + n - p - 1$, which implies that $0 > n - p - 1$, or equivalently, $p + 1 > n$. I defined $p < m$, so $m \geq p + 1 > n$, a contradiction, as $m \leq n$.

Case 2: $|A_1| = 1$. Then $|A_0| \leq n$ since Y is connected, so every vertex in A_0 must be adjacent to the lone vertex in A_1.

Subcase 2.1: Assume $|A_0| = n$. Then we have $|N(A)| \geq m-1$, as each vertex in A_0 is adjacent to m vertices of X_1, only one of which is in A_1. Since $|N(A)| = \kappa(X) < m$, $|N(A)| = m-1$. Then each vertex in A_0 is adjacent to the same m vertices of X_1, call this set M. Each vertex of M is adjacent the n vertices of A_0, but the valency of elements of M is n, so each element of M is adjacent to only elements of A_0. Since $N(A) \subset M$, this means that $N(A) \cup A$ is a component of X, but X is connected so $N(A) \cup A = X$. Thus $N(A)$ is not a cut set of X, so A is not an atom of X, a contradiction.

Subcase 2.2: Now assume $|A_0| = p < n$. Then $|N(A_1)\backslash A_0| = n - p$, as the single vertex in A_1 has n neighbors, p of which are in A_1. Let $q = |N(A_0)\backslash A_1|$. Each vertex in A_0 is adjacent to m vertices in X_1, one of which is in A_1, so $q \geq m-1$. Then $|N(A)| = |N(A_1)\backslash A_0| + |N(A_0)\backslash A_1|$, so

$$m > \kappa(X) = |N(A)| = n - p + q \geq n + m - p - 1.$$

So $m > n + m - p - 1$, or equivalently, $p + 1 > n$. We defined $p < n$, so $p < n < p + 1$, a contradiction, as p and n are integers. Thus, neither $|A_0|$ nor $|A_1|$ can have cardinality 1. □

Lemma 2 *Let $X = (V, E)$ be a connected half-vertex transitive graph, and A be an atom of X, with Y=X[A]. If $\kappa(X) < \delta(X)$, then $Aut(Y)$ acts transitively on $A_i = A \cap X_i (i = 0, 1)$.*

Proof: By Lemma 1, $A_0 = A \cap X_0$ and $A_1 = A \cap X_1$ have at least two vertices. Therefore there exist a pair of vertices v, $u \in A_0$, and since X is half-vertex transitive, there exists an automorphism $\phi : X \to X$ such that $\phi(u) = v$.

Therefore $v \in A$ and $v \in \phi(A)$, so $A \cap \phi(A) \neq \emptyset$. Therefore, by Theorem 2, $\phi(A) = A$. No element of A_0 can be mapped to A_1 or vice-versa, as A_1 and A_0 are contained in different orbits of X. Therefore $\phi(A_0) = A_0$ and $\phi(A_1) = A_1$. Thus the restriction of ϕ to Y is an automorphism on Y. The set of all such restrictions is transitive on A_0 and A_1, as $Aut(X)$ is transitive on X_0 and X_1, so the automorphism group of Y acts transitively on A_0 and A_1. □

Lemma 3 *Let X=(V,E) be a connected half-vertex transitive graph with two orbits. If* $\kappa(X) = \delta(x)$*, then:*

(i) Every vertex of X lies in an atom, and

(ii) Every atom A satisfies $|A| \leq \kappa(x)$.

Proof: (i) By Lemma 1, the subgraph Y of X induced by any atom A contains at least two vertices of both X_0 and X_1. Therefore at least one element of X_0 and X_1 are contained in an atom. The automorphism group $Aut(X)$ is transitive on X_0 and X_1, so every element of X_0 is the image of an element of A_0 and every element of X_1 is the image of an element of A_1. For any automorphism $\Phi : X \to X$, $N(\Phi(A)) = \Phi(N(A))$, so $\Phi(A)$ is also an atom of X. Therefore the images of vertices of A_0 and A_1 under automorphisms are contained in atoms, so every element of X lies in an atom.
(ii) Let $F = N(A)$. By (i), every vertex in X lies in an atom of X, so any vertex $v \in F$ likewise lies in some atom A' of X. Then, by Theorem 3, since $A' \cap F \neq \emptyset$, $A' \subseteq F$, so $|A| = |A'| \leq |F| = \kappa(X)$. □

Theorem 4 *If X=(V,E) is a half-vertex transitive graph, then* $\kappa(X) = \delta(X)$.

Proof: Assume for sake of contradiction that $\kappa(X) < \delta(X)$. By Lemma 3, every vertex of X lies in an atom, and atoms are blocks of imprimtivity, so they are disjoint. Thus $V(X)$ is the disjoint union of distinct atoms of X. Therefore if A is an atom of X, there is a set of permutations in $\{\sigma_1, ..., \sigma_k\} \subseteq Aut(X)$, such that

$$V(X) = \bigcup_{i=1}^{k} \sigma_i(A),$$

and $\sigma_i(A) \cap \sigma_j(A) = \emptyset$ for $i \neq j$. By Theorem 1, the subgraph $Y = X[A]$ induced by A, is a connected subgraph of X with more than one vertex, and by Lemma 1, $A_0 = X_0 \cap A$ and $A_1 = X_1 \cap A$ both have cardinality greater than 1. Since $Aut(X)$ has orbits X_0 and X_1, $\sigma_i(A_0) \subseteq X_0$ for any $i \in (1, ..., k)$, and likewise $\sigma(A_1)$ is contained in X_1. Since $\sigma_i(A) \cap \sigma_j(A) = \emptyset$, for $i \neq j$, $\sigma_i(A_0) \cap \sigma_j(A_0) = \emptyset = \sigma_i(A_1) \cap \sigma_j(A_1)$ for $i \neq j$. Thus $X_0 = \bigcup_{i=1}^{k} \sigma_i(A_0)$, and $X_1 = \bigcup_{i=1}^{k} \sigma_i(A_1)$, which in turn implies that $|X_i|/|A_i| = k$ for $i = 0$ and $i = 1$. By the Handshaking Lemma, $|X_0|/|X_1| = n/m$, so $|A_0|/|A_1| = nk/mk = n/m$ as well.

By Lemma 2, $Aut(Y)$ acts transitively on A_1 and A_2, so $Y_0 = X[A_0]$ and $Y_1 = X[A_1]$ are regular, implying that $Y = X[A]$ is semi-regular. Consider $\delta(Y)$.

Since $|A_0|/|A_1| = n/m$, A_0 has at least as many vertices as A_1 and the same number of incident edges, so by the Handshaking Lemma, $\delta_{A_0} \leq \delta_{A_1}$. Therefore $\delta(Y) = \delta_{A_0}$. Certainly, $|A_0|\delta_{A_0} = |A_1|\delta_{A_1}$, so equivalently $\delta_{A_0}|A_0|/|A_1| = \delta_{A_1}$. Let $d = \delta(Y) = \delta_{A_0}$. Since $|A_0|/|A_1| = n/m$, then $\delta_{A_1} = dn/m$. Therefore every vertex in A_0 has $m-d$ neighbors in $N(A)$ and every vertex in A_1 has $n - dn/m$ neighbors in $N(A)$. Summing these values gives $|N(A)| \geq m - d + n - dn/m = m + n - (m+n)d/m$. By assumption, $m > |N(A)|$, so

$$m > m + n - (m+n)d/m.$$

It follows that

$$(m+n)d/m > n.$$

Multiplication through by $m/(n+m)$ gives

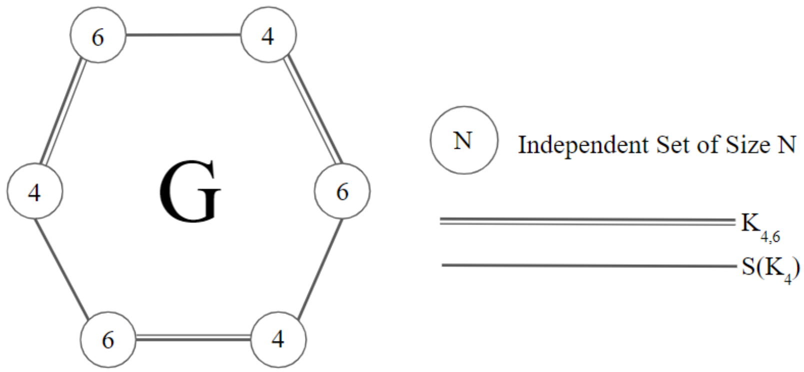
$$d > mn/(m+n).$$

The size of A_0 must be at least as large as the minimum degree of A_1 in Y, and likewise the size of A_1 must be at least as large as the minimum degree of A_0 in Y. Therefore $|A| = |A_0| + |A_1| \geq \delta_{A_1} + \delta_{A_0} = dn/m + d = d(m+n)/m$, which, with the aid of our inequality regarding d, can be converted into $|A| > (mn/(m+n))(m+n)/m = n \geq m > \kappa(X)$. Lemma 3 states that $|A| \leq \kappa(X)$, a contradiction. Thus $\kappa(X) \geq \delta(X)$ and $\kappa(X) \leq \delta(X)$, so $\kappa(X) = \delta(X)$. $\square$

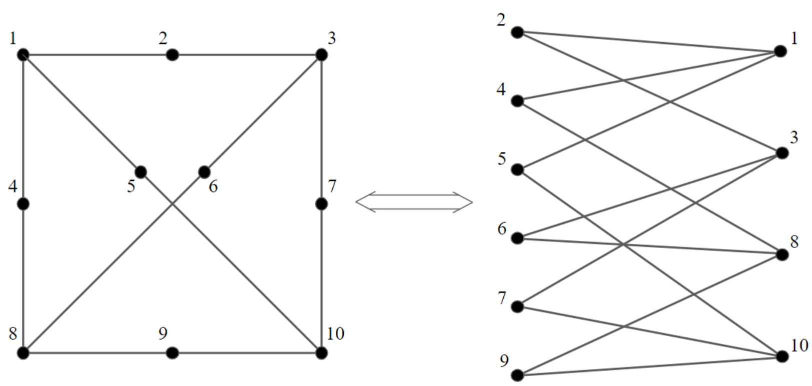
5 Final Considerations and an Example

Connected graphs that are edge transitive must be either vertex transitive or half-vertex transitive. It has previously been shown [2], that edge transitive graphs have connectivity equal to their minimum degree. With the revised conclusion of our argument, this claim can be extended to all half-vertex transitive graphs. After a quick survey of the literature, we were unable to find an example of a half-vertex transitive graph that is not edge transitive, nor a proof that all half-vertex transitive graphs must be edge transitive. So for the sake of completeness, we offer the following construction.

Let G be a graph with 30 vertices and an edge set indicated in the figure below. Let $K_{4,6}$ denote the complete bipartite graph with independent sets of size 4 and 6. Let K_4 be the complete graph on 4 vertices. Then $S(K_4)$ denotes the graph formed by subdividing the edges of K_4.



Graph G consists of three independent sets of size 6 and three independent sets of size 4 such that each independent set of size 6 has edges between it and two of the independent sets of size 4. In the future, these independent sets will be referred to as 6-sets and 4-sets respectively. The subgraph induced by any adjacent pair of these sets is either isomorphic to $K_{4,6}$ or $S(K_4)$, alternating so that each independent set is contained in one induced $K_{4,6}$ and one induced $S(K_4)$. The structure of the induced $S(K_4)$ subgraph is shown below.

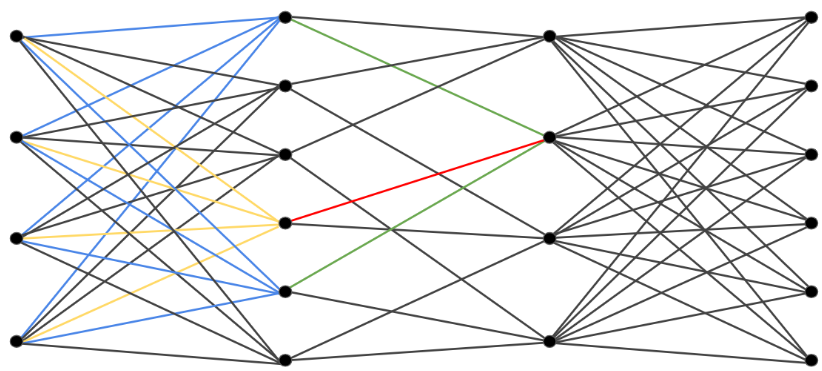


Theorem 5 *Graph G is half-vertex transitive, but not edge transitive.*

Proof: First we show that G is half-vertex transitive. There are no edges between any two of the 6-sets or 4-sets, so G is bipartite with all 6-sets in one cell of the bipartion, and all 4-sets in the other. It remains to show that G has two orbits on these color classes. Each vertex in a 6-set has degree 6 and each vertex in a 4-set has degree 9, so G is not vertex transitive. By examining the figure above, it is clear that G has a 3-fold rotational symmetry, so any 6-set (or 4-set) is the image of any other 6-set (or 4-set) under an automorphism. Now I must simply show that $Aut(G)$ is transitive on each of the 4 and 6-sets. Since K_4 is vertex and edge transitive, it follows that $S(K_4)$ is half-vertex transitive with its edge and vertex induced vertices forming the two cells of its bipartition. Then since each $S(K_4)$ shares its vertices with two copies of $K_{4,6}$, there is a subgroup of $Aut(G)$ for each induced $S(K_4)$ that acts as $Aut(S(K_4))$ on that subgraph and is the identity everywhere else. Then, since every 4-set and 6-set is contained in an induced $S(K_4)$, $Aut(G)$ is transitive on each of the 4-sets and 6-sets of G, so G is half-vertex transitive.

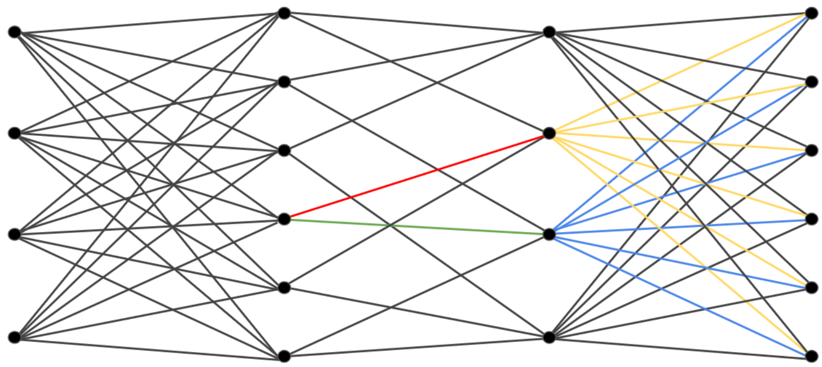
Now I show that G is not edge transitive. Let e_0 be an edge in G such that e_0 is contained in an induced $S(K_4)$. I count the number of 4-cycles containing e_0. Since $S(K_4)$ contains no 4-cycles, any such 4-cycle must intersect one of the two copies of $K_{4,6}$ that share vertices with the induced $S(K_4)$. There are therefore two cases, one for each of the two copies of $K_{4,6}$ used.

Case 1: Assume that the 4-cycle uses edges contained in the induced $K_{4,6}$ that shares 6 vertices with $S(K_4)$. Call this induced $K_{4,6}$ N_1.



Let e_0 be the red edge in the diagram above. The second edge in the cycle (green) must have one vertex in N_1 and the other vertex in e_0, so there are two possibilities. The third edge (blue) can be any edge in N_1 that shares a vertex with the second, so there are 4 possibilities. The last edge (yellow) must return to the vertex of e in N_1, so there is only one possible choice. Thus there are 8 4-cycles of this type.

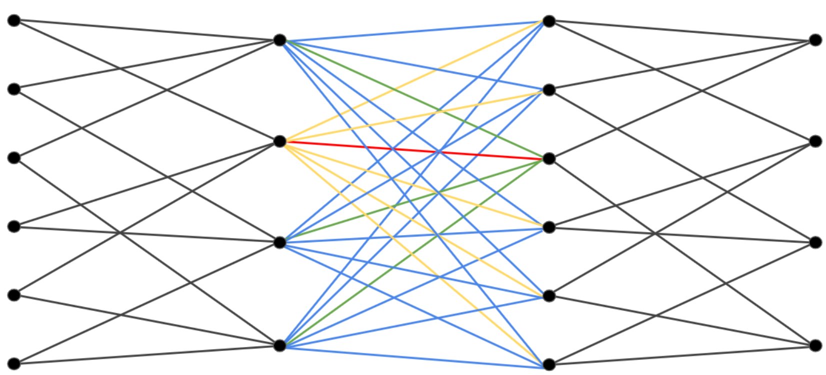
Case 2: Assume that the 4-cycle uses edges contained in the induced $K_{4,6}$ that shares 4 vertices with $S(K_4)$. Call this induced $K_{4,6}$ N_2.



Let e_0 be the red edge in the diagram above. The second edge in the cycle (green) must have one vertex in N_2 and the other vetex in e_0, so there is one possibility. The third edge (blue) can be any edge in N_2 incident to the green edge, so there are 6 possibilities. The final edge (yellow) must return to e_0, so there is only one choice. Thus there are 6 4-cycles of this type, for a total of 14 4-cycles containing e_0.

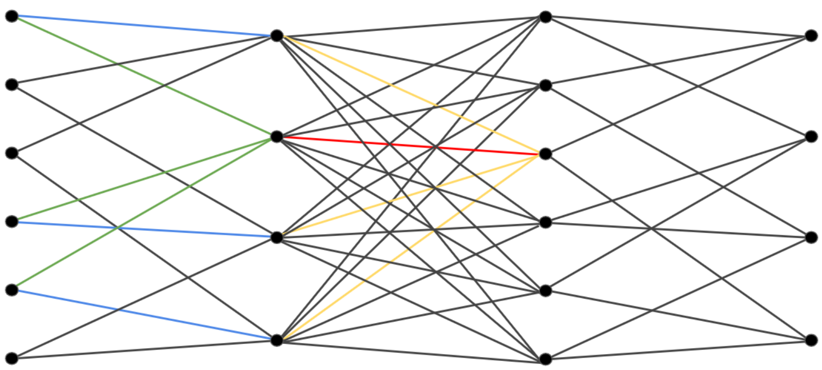
Now let e_1 be an edge in G such that e_1 is contained in an induced $K_{4,6}$. I count the number of 4-cycles containing e_1. There are three cases, one where the 4-cycle is contained in the $K_{4,6}$ and two where the 4-cycle uses edges from the neighboring copies of $S(K_4)$.

Case 1: Assume first that the 4-cycle is entirely contained in the induced $K_{4,6}$.



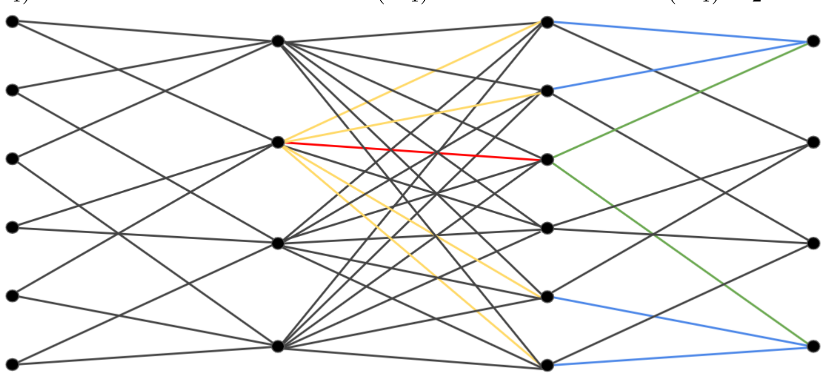
Let e_1 be the red edge in the diagram above. There are 3 choices for the edge (green) incident with the rightmost vertex of e_1, and 5 choices for the edge (yellow) incident with the leftmost vertex of e_1. The final edge (blue) must connect the end vertices of these two edges, so there is one choice available. Therefore there are 15 4-cycles of this type.

Case 2: Assume now that the 4-cycle uses edges contained in the induced $S(K_4)$ that shares 4 vertices with $S(K_4)$. Call this induced $S(K_4)$ N_1.



Let e_1 be the red edge in the diagram above. There are three edges incident with e_1 in N_1 (green), each of which must be followed up by the single incident (blue) edge with a vertex in the induced $K_{4,6}$. The final edge (yellow) must return to the first vertex of the cycle, so there are 3 4-cycles of this type.

Case 3: Finally, assume that the 4-cycle uses edges contained in the induced $S(K_4)$ that shares 6 vertices with $S(K_4)$. call this induced $S(K_4)$ N_2.



Let e_1 be the red edge in the induced subgraph above. There are two (green) edges incident with e_1 in N_2, each of which has two incident edges (blue) that have a vertex in the induced $K_{4,6}$ containing e_1. The final edge (yellow) must return to the vertex in e_1 that is also in N_1, so there are four 4-cycles of this type, for a total of 22 4-cycles containing e_1. Since e_0 and e_1 are contained in a different number of 4-cycles e_1 cannot be the image of e_0 under any automorphism of G. Thus G is not edge transitive. □

References

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