Cycloids and Paths

An MST 501 Project Presentation
by
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Under the direction of
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A **cycloid** is the path traced by a point on a circle as the circle rolls along a flat line in two dimensions.

My Paper:

- Introduction and history of cycloids.
- Roberval's derivation of the area under a cycloid.
- Showing that a pendulum constrained by two inverted cycloids will swing in a path of a congruent, inverted cycloid.

My Curriculum Project

- **Lesson 1:** Intro to Cycloids/Deriving the Parametric Equation of a Cycloid.
- **Lesson 2:** Roberval's Derivation of the Area Under a Cycloid.
- **Lesson 3:** Using Integration to Find the Arc Length of a Cycloid and Area Under a Cycloid.
- **Lesson 4:** Showing that a Pendulum Constrained by two Inverted Cycloids Swings in the Path of a Congruent, Inverted Cycloid.

Why Cycloids?

- The basic idea is easy to comprehend and engaging.
- Utilizes concepts from algebra, geometry, trigonometry and calculus.
- Has a rich mathematical history that in many ways parallels the development of calculus.

A Brief History of the Cycloid

- **Charles de Bovelles** (1475-1566): First to study the curve.
- **Galileo Galilei** (1564-1642): Named the curve and popularized it.
- **Marin Mersenne** (1588-1648): First precise mathematical definition.
- **Gilles de Roberval** (1602-1675): Used Cavalieri's Principle to find the area under the curve.

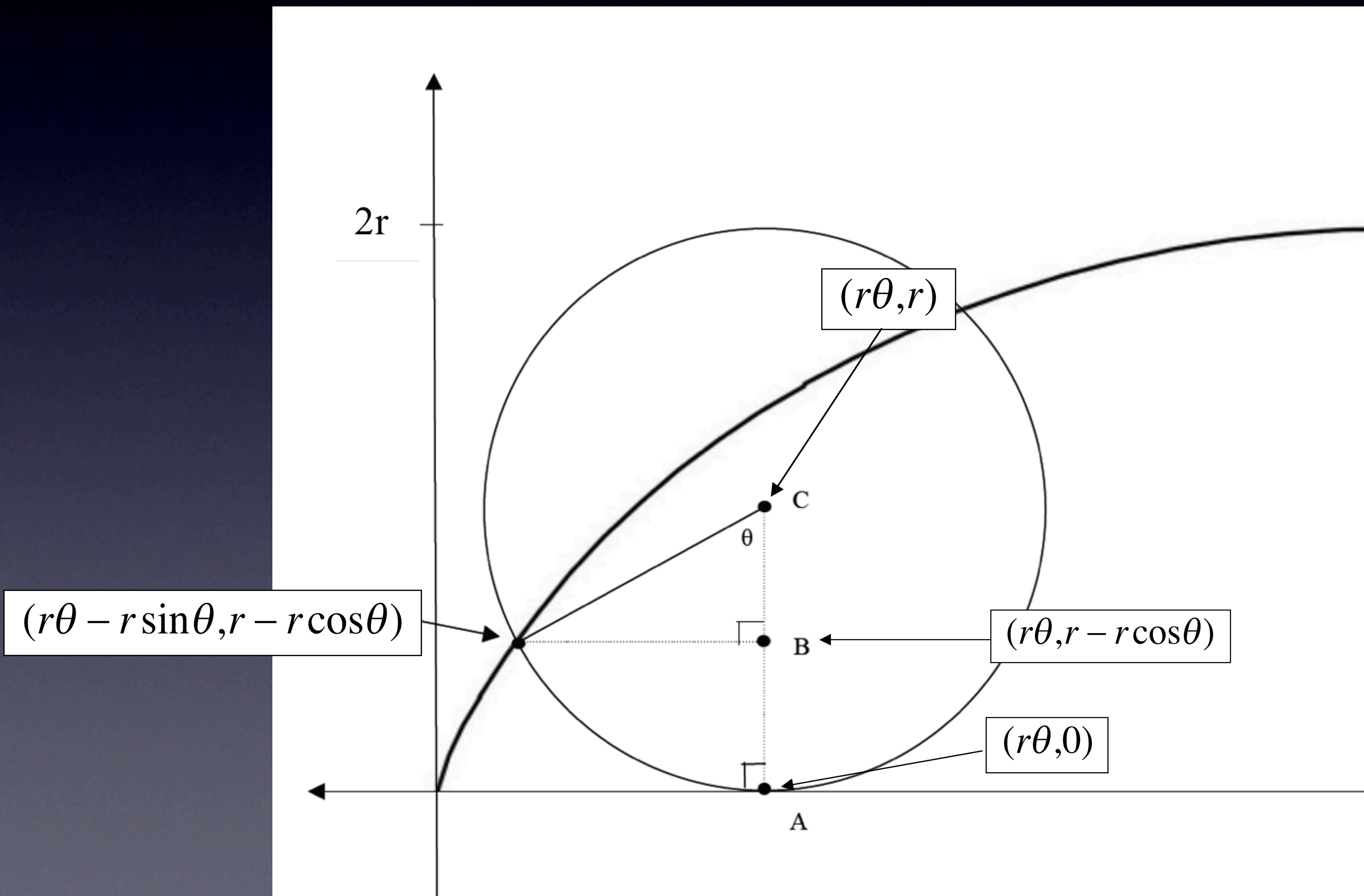
History of Cycloids (continued)

- **Blaise Pascal** (1623-1662): Used "indivisibles" to find the area under and arc length of the cycloid.
- **Christiaan Huygens** (1629-1695): Studied cycloid-constrained pendulum and developed "tautochrone" property.
- **Gottfried Leibniz** (1646-1716): Developed the first explicit formula for a cycloid.
- **Jacob** (1654-1705) and **Johann** (1667-1748) **Bernoulli**: Discovered "Brachistachrone" property of cycloids.

Did we forget anyone?

- **Rene Descartes** (1596-1650), **Pierre de Fermat**, (1601-1665), **Christopher Wren**, (1632-1723) and **Isaac Newton** (1642-1727) all studied and contributed to our knowledge of cycloids.
- So many famous mathematicians and scientists have been drawn to the study of the cycloid, the curve has been called, "**The Helen of Geometers**."

Deriving a Parametric Equation for a Cycloid



Parametric Equation for a Cycloid:

For a generating circle of radius r, with θ being the amount of rotation of the circe in radians, the cycloid curve is given by the parametric equation:

$$x = r(\theta - \sin\theta)$$

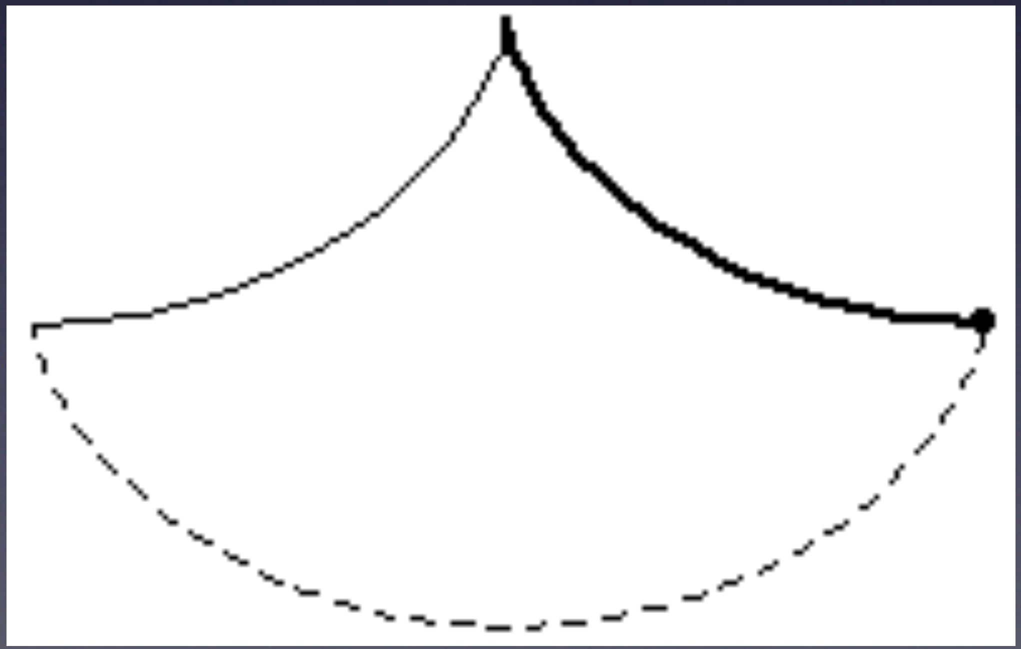
$$y = r(1 - \cos\theta)$$

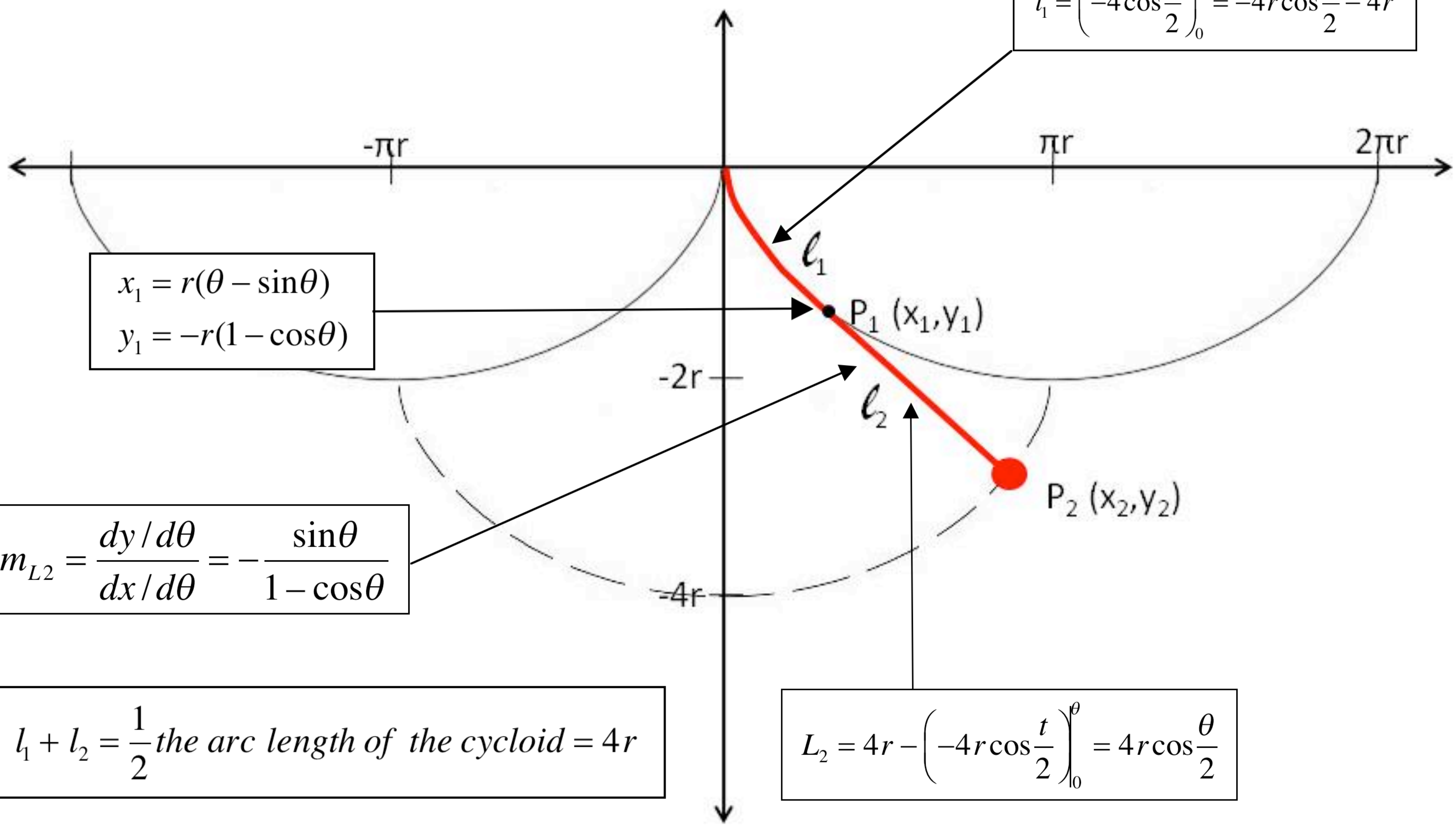
Two Important Properties of Cycloids

- The area under a cycloid curve is 3 times that of its generating circle, or $3\pi r^2$.

- The arc length of a cycloid is 8 times the radius of its generating circle.

Showing that a pendulum constrained by two inverted cycloids will swing in the path of a congruent, inverted cycloid.





$$l_1 = \left(-4\cos\frac{t}{2}\right)_0^{\theta} = -4r\cos\frac{\theta}{2} - 4r$$

$$x_1 = r(\theta - \sin\theta)$$
$$y_1 = -r(1-\cos\theta)$$

$$m_{L2} = \frac{dy/d\theta}{dx/d\theta} = -\frac{\sin\theta}{1-\cos\theta}$$

$$l_1 + l_2 = \frac{1}{2}\,the\ arc\ length\ of\ the\ cycloid = 4r$$

$$L_2 = 4r - \left(-4r\cos\frac{t}{2}\right)\Big|_0^{\theta} = 4r\cos\frac{\theta}{2}$$

Given an endpoint (x,y), the slope, m, and length, ℓ, of a line segment, find the coordinates of the other endpoint.

$$slope = \frac{y_2 - y_1}{x_2 - x_1}$$

+

$$l = \sqrt{(x_2 - x_1) + (y_2 - y_1)}$$

$$x_2 = \pm\sqrt{\frac{l^2}{m^2 + 1}} + x_1$$

$$y_2 = m\left(\pm\sqrt{\frac{l^2}{m^2 + 1}}\right) + y_1$$

Finding x_2

$$x_2 = \pm\sqrt{\frac{l^2}{m^2+1}} + x_1$$

$$x_2 = \pm\frac{4r\cos\frac{\theta}{2}}{\sqrt{\left(\frac{\sin\theta}{1-\cos\theta}\right)^2+1}} + r\theta - r\sin\theta$$

...after some algebra and trig identities...

$$x_2 = \pm\frac{4r\sqrt{1-\cos^2\theta}}{2} + r\theta - r\sin\theta$$

$$x_2 = r(\sin\theta + \theta) \quad or \quad x_2 = r(\theta - 3\sin\theta)$$

Finding y_2

$$y_2 = m\left(\pm\sqrt{\frac{l^2}{m^2+1}}\right) + y_1$$

$$y_2 = \left(-\frac{\sin\theta}{1-\cos\theta}\right)\left(\pm 2r\sin\theta\right) - r(1-\cos\theta)$$

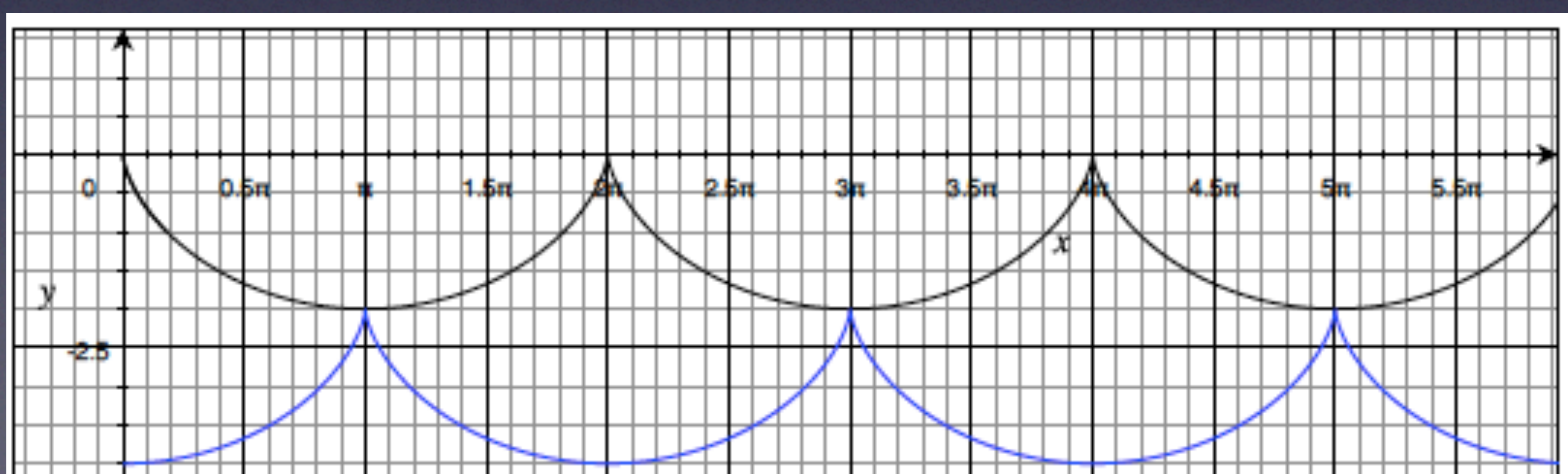
$$y_2 = \pm\left(\frac{2r(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}\right) - r(1-\cos\theta)$$

$$y_2 = 3r + 3r\cos\theta \quad or \quad y_2 = -3r - r\cos\theta$$

Which Equations Work?

$$x_2 = r(\sin\theta + \theta) \quad or \quad x_2 = r(\theta - 3\sin\theta)$$

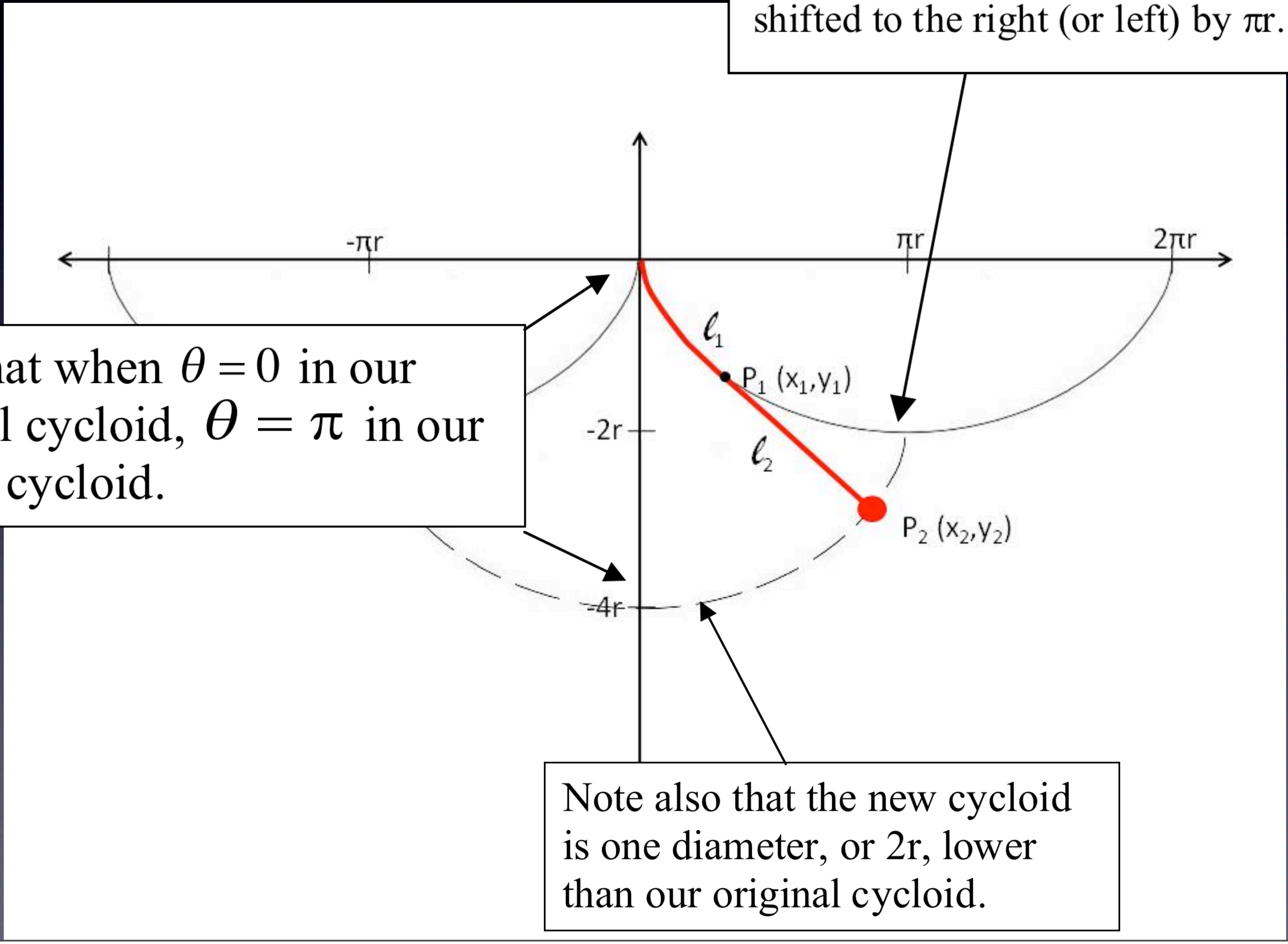
$$y_2 = 3r + 3r\cos\theta \quad or \quad y_2 = -3r - r\cos\theta$$



Showing that the new parametric is a shifted cycloid.

Note that the second cycloid is shifted to the right (or left) by πr.

Note that when $\theta = 0$ in our original cycloid, $\theta = \pi$ in our shifted cycloid.



Note also that the new cycloid is one diameter, or 2r, lower than our original cycloid.

Showing that the new parametric is a shifted cycloid (continued).

Beginning with our original inverted cycloid:

$$x = r(\theta - \sin\theta)$$

$$y = -r(1 - \cos\theta)$$

Replacing θ with (θ+π), and subtracting πr gives us:

$$x = r((\theta + \pi) - \sin(\theta + \pi)) - \pi r$$

Replacing θ with (θ+π), and subtracting 2r gives us:

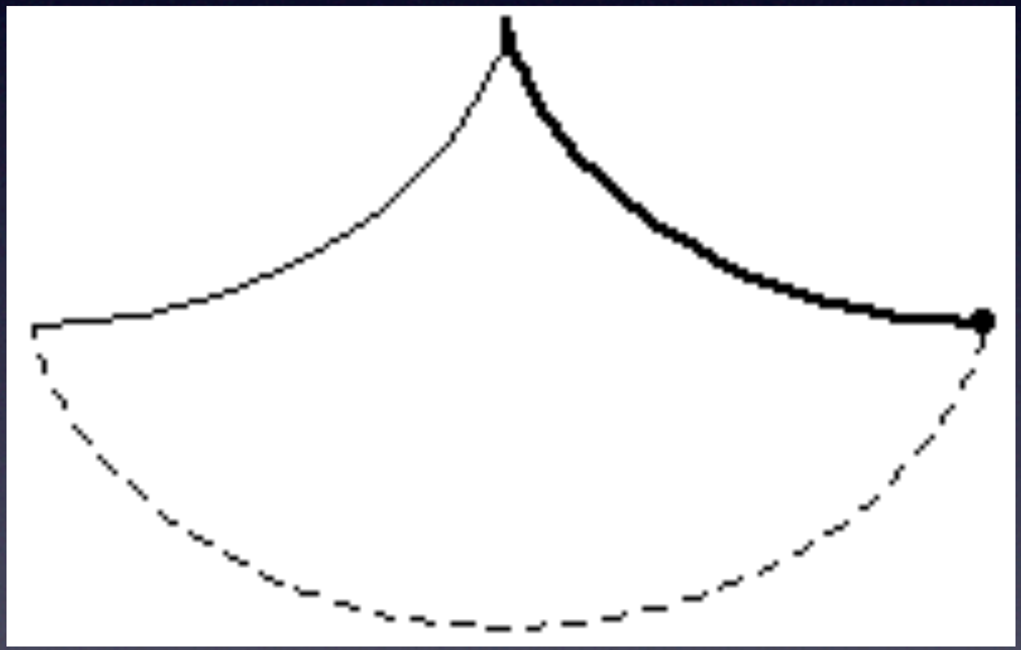
$$y = -r(1 - \cos(\theta + \pi)) - 2r$$

Which simplify to:

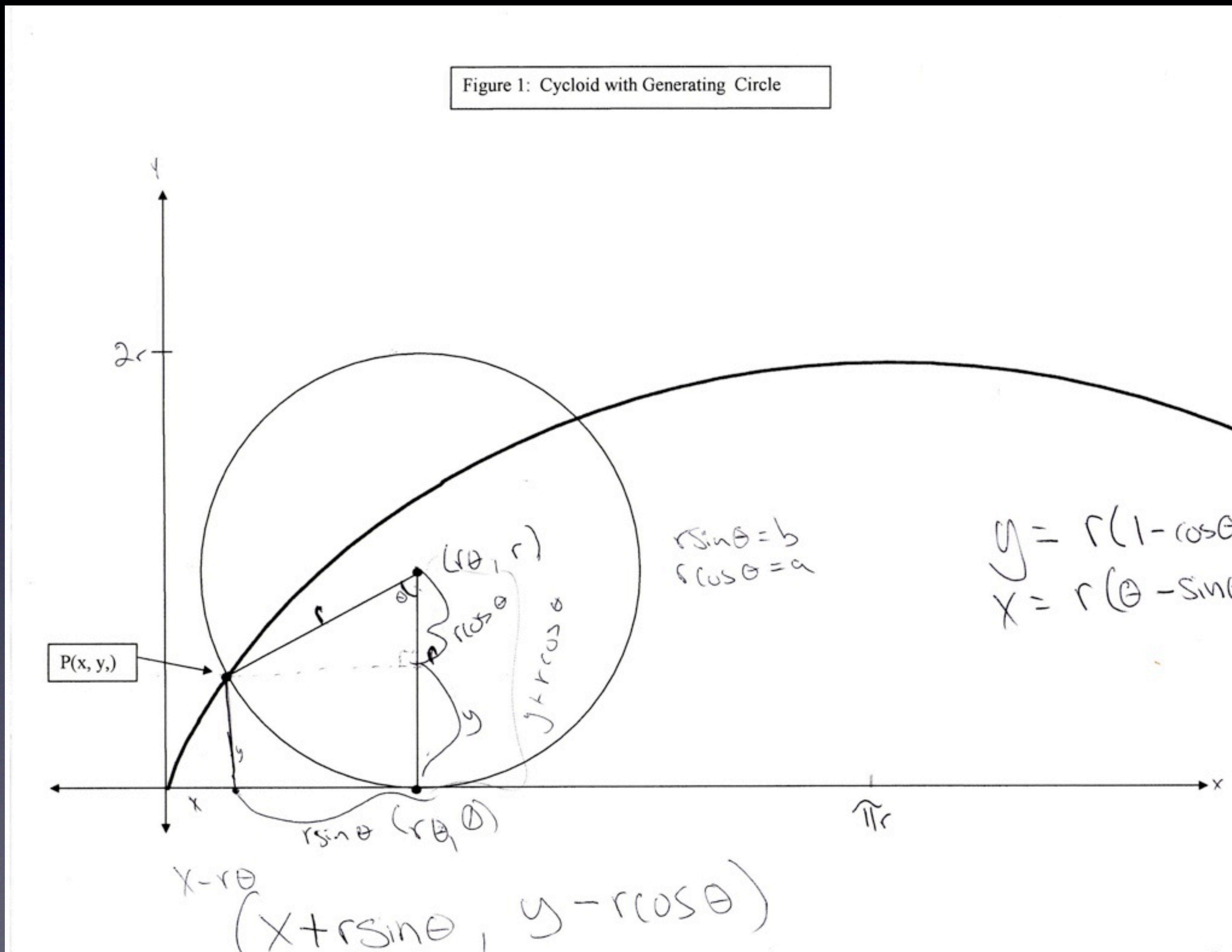
$$x = r(\sin\theta + \theta)$$ and $$y = -3r - r\cos(\theta)$$

Which we recognize as our parametric for x_2 , y_2.

So, we have shown that a pendulum constrained by inverted cycloids will indeed swing in the path of a congruent, inverted cycloid.



Lesson 1: Intro and Deriving the Parametric



Lesson 2: Roberval's Area Under the Cycloid

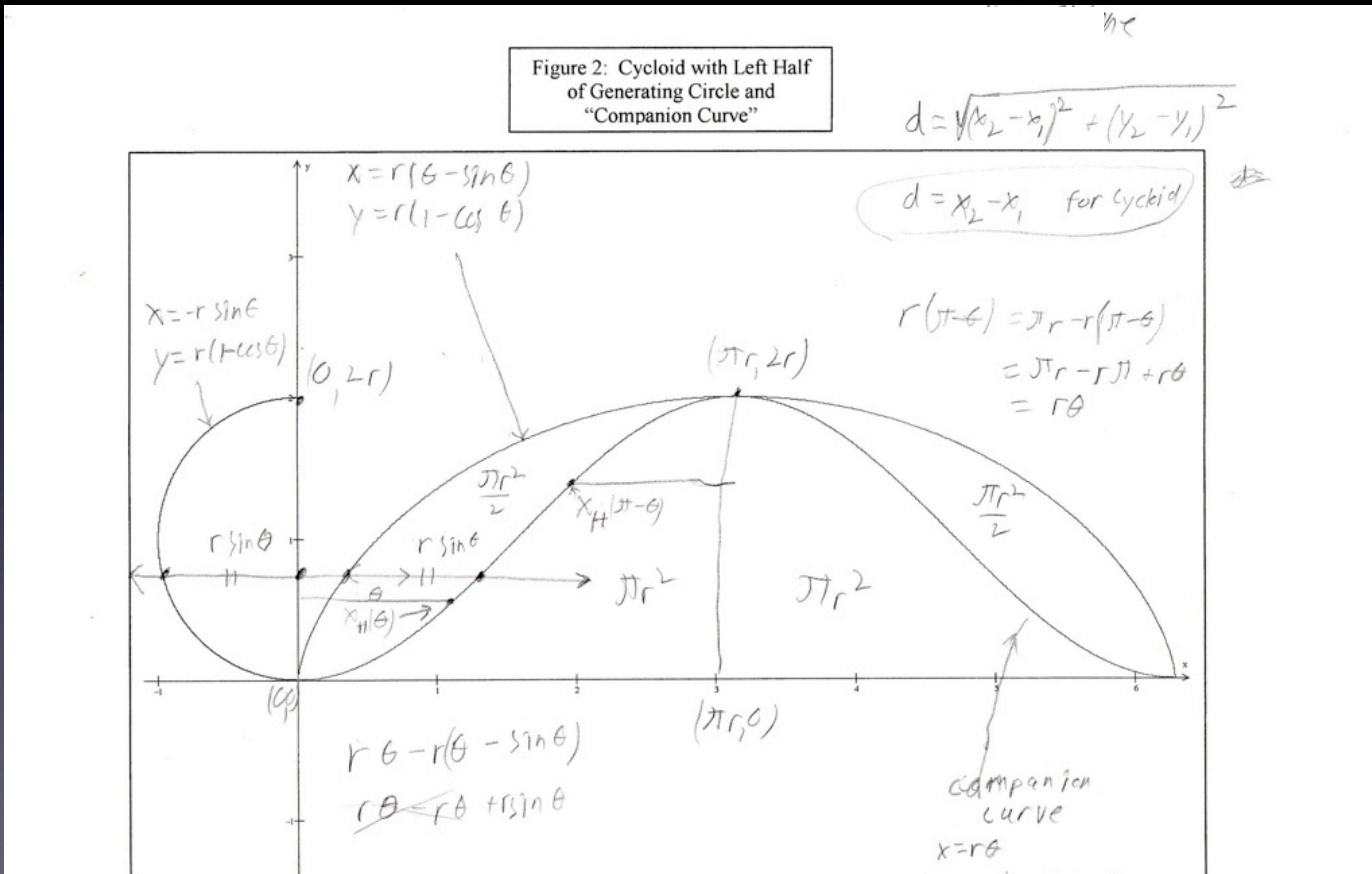


Figure 2: Cycloid with Left Half of Generating Circle and "Companion Curve"

Lesson 3: Using Integrals to Find the Arc Length and Area Under the Cycloid

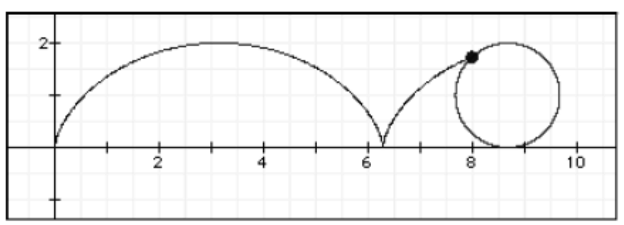
Worksheet 3.1 (p.1 of 3)

Name____________________ Date________________

In-Class Assignment: Area Under and Arc Length of a Cycloid

Part 1: Review of Cycloids

Recall that a cycloid is the curve made by a point on a circle as the circle rolls along a flat surface.



A cycloid has the parametric equation
x= r(θ-sinθ) and y= r(1-cosθ) where r is the radius of the generating circle and θ is the amount of rotation of the circle in radians.

Part 1: Area Under the Cycloid

Recall the formula for the area under a parametric curve:

If **x** = f(t) and y= g(t)

then

$$A = \int_{t1}^{t2} y dx = \int_{t1}^{t2} g(t) \cdot f'(t)$$

Step 1: Find f'(t)

Step 2: Substitute g(t) and f'(t) into the formula above.

Worksheet 3.1 (p.3 of 3)

Part 2: Arc Length of a Cycloid

Recall the arc length formula:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Step 1: find $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and set the limits of integration.

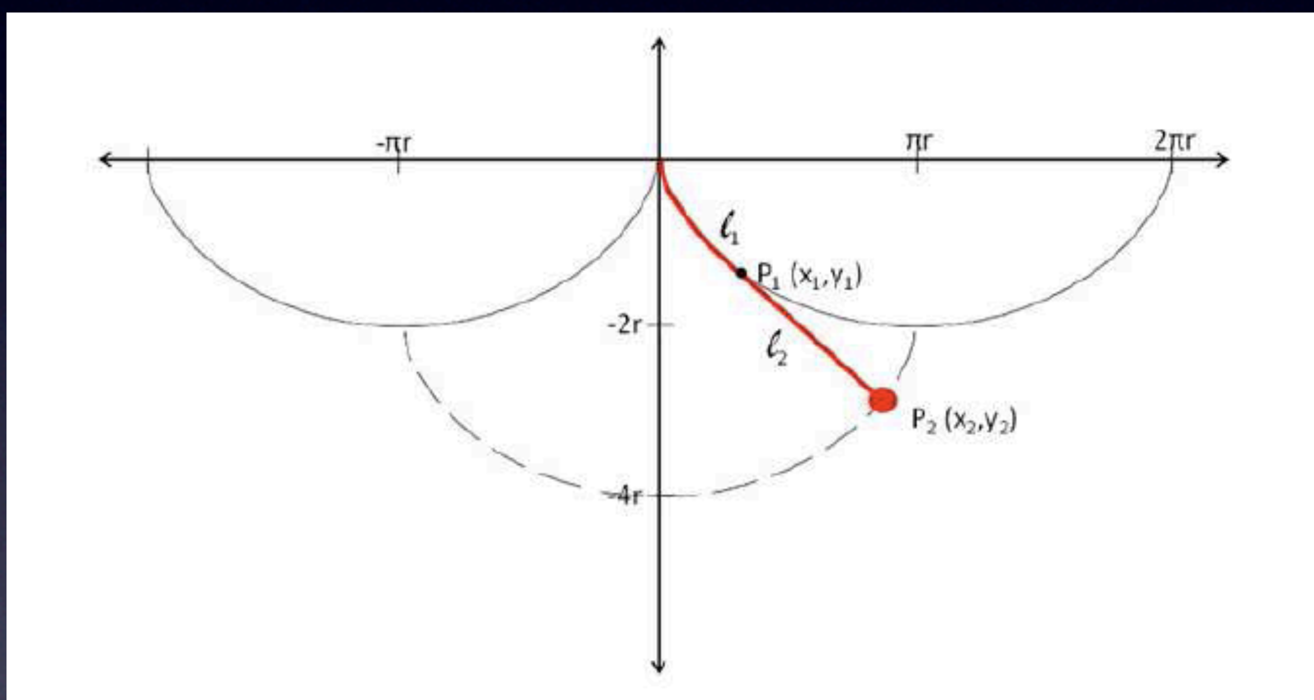
Step 2: Substitute and expand, factoring out the r^2.

Step 3: Substitute, using the trig identity $1 - \cos t = 2\sin^2\left(\frac{t}{2}\right)$

Step 4: Eliminate the radical, and integrate.

Step 5: Evaluate over the given limits.

Lesson 4: Showing that a Pendulum Constrained by two Inverted Cycloids follows the Path of a Congruent, Inverted Cycloid



Activity 4: Showing that a Pendulum Constrained by Two Inverted Cycloids Swings in the Shape of a Congruent, Inverted Cycloid

Because of the complexity of the algebra involved in the actual derivation (see section 3 of Part 1), I've eliminated some of the more difficult steps. This activity should probably be done as follows:

Intro: Teacher presents Diagram 3.1 and goes over the basic problem.

Students work in pairs or small groups or solo to solve the following problem.

Activity 1: Given the coordinates of one endpoint of a line segment, the length of that segment and the slope of that segment, find a formula to find the other endpoint.

Let (x_1, y_1) be the known endpoint. Let L be the length of the segment, and m be its slope.

(Hint: combine the slope formula and the distance formula. Your "formula" will have two parts, one to find the **x** coordinate of the missing endpoint, and one to find the y coordinate).

Once everyone has the formulas, we begin activity two.

Activity 2: Looking at the diagram, what expressions can we plug into our formulas for the following variables. Be ready to give a short explanation for your answer.

x_1=

y_1=

m=

L=

Thank You!