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A new superconvergence property of Wilson nonconforming finite element

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Summary. In this paper the Wilson nonconforming finite element method is considered to solve a class of two-dimensional second-order elliptic boundary value problems. A new superconvergence property at the vertices and the midpoints of four edges of rectangular meshes is obtained.

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1. Introduction

The Wilson nonconforming finite element has been widely used in computational mechanics and structural engineering because of its good convergence behavior. In many practical cases, it seems better than the bilinear conforming element.However, it is shown in [2, 3, 4] that the convergence rate of Wilson rectangular element in the energy norm is of first order. As for the arbitrary quadrilateral meshes, a first-order convergence can also be retained provided a slight restriction on meshes is satisfied, see [6]. Furthermore, the first author has given an example [7] showing that the first-order convergence is optimal. Recently, Chen and Li [1] strictly proved this first-order optimality.

Meanwhile, computations have observed its superconvergence at the center of elements, thus the question of superconvergence was raised, see [7, 9]. Li justified in [5] this observation for the simplest model: $-\triangle u = f$. Subsequently, the result was extended to a class of second-order elliptic problems in [1], see Lemma 2.

On the other hand, following [8], it can be easily proved that the bilinear conforming element posseses the superconvergence at the center, as well as at the four vertices and the midpoints of four edges of rectangular meshes.

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In this paper we prove that besides the center of rectangles as shown in [1], Wilson element has also the superconvergence at these eight points like the bilinear element.

2. Wilson element

We consider the general second-order elliptic boundary value problem

$$\begin{cases} -\partial_x(a_1\partial_x u) - \partial_y(a_2\partial_y u) + a_3 u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where all funtions $a_i, i = 1,2,3$ and f are smooth enough (see Remark 1), and

$$0 < c_1 \le a_i \le c_2 < +\infty, i = 1,2, 0 \le a_3 \le c_3 < +\infty,$$

$\Omega \subset \mathbb{R}^2$ is a rectangular domain.

Let $\mathcal{T}_h$ be a rectangular partition of Ω, satisfying the regularity assumption [2], $z_0 = (x_0, y_0)$ is the center of $K \in \mathcal{T}_h$, $2h_x$ and $2h_y$ are the lengths of two edges of K in x and y direction respectively. $h = \max_K(h_x, h_y)$.

Definition. $\mathcal{T}_h$ is called a uniform partition when all h_x are equal and so are all h_y.

The variational problem of (2.1) is to find $u \in H_0^1(\Omega)$ such that

$$A(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where

$$A(u,v) = \iint_\Omega (a_1\partial_x u\partial_x v + a_2\partial_y u\partial_y v + a_3 uv)\, dx\, dy,$$

$$(f,v) = \iint_\Omega f v\, dx\, dy.$$

The Wilson element solution $w^* \in W_h$ of (2.2) satisfies

$$A_h(w^*,v) = (f,v) \quad \forall v \in W_h, \tag{2.3}$$

where

$$A_h(u,v) = \sum_{K\in\mathcal{T}_h} \iint_K (a_1\partial_x u\partial_x v + a_2\partial_y u\partial_y v + a_3 uv) dx\, dy,$$

and the finite element space $W_h = \{w_h, w_h|_K \in P_2(K)$ is determined by the function values at the four vertices of K and the mean values of its two second derivatives $\partial_{xx} w_h$ and $\partial_{yy} w_h$ on K, $w_h = 0$ at vertices belonging to $\partial\Omega\}$.

The bilinear element solution $u^* \in Q_h$ satisfies

$$A_h(u^*,v) = (f,v) \quad \forall v \in Q_h, \tag{2.4}$$

where $Q_h = \{u_h, u_h|_K \in Q_1(K)$ is determined by its function values at four vertices of K, $u_h|_{\partial\Omega} = 0\}$.

In the following, we assume that c(with or without subscript) is a generic constant which may take different values at different places and is independent of the mesh size h and the solution u.

3. Some lemmas

The following lemmas are known or can be easily derived.

Lemma 1 [2].

$$|u - w^*|_{1,h} \leq ch\|u\|_{2,\Omega}, \tag{3.1}$$

$$|u - w^*|_{0,\Omega} \leq ch^2\|u\|_{2,\Omega}, \tag{3.2}$$

where the semi-norm

$$|.|_{1,h} = (\sum_K |.|_{1,K}^2)^{\frac{1}{2}}.$$

Lemma 2 [1, Th.2].

$$|\nabla(u - w^*)(z_0)| \leq ch^2|\ln h|\|u\|_{3,\infty}. \tag{3.3}$$

Lemma 3 [8].

$$|\nabla(u - u^*)(z_0)| \leq ch^2|\ln h|\|u\|_{3,\infty}. \tag{3.4}$$

If the partition is uniform, z^* is a vertex or a midpoint of edges of K, then

$$|\overline{\nabla}(u - u^*)(z^*)| \leq ch^2|\ln h|\|u\|_{3,\infty}, \tag{3.5}$$

where $\overline{\nabla}$ refers to taking average over all neighbouring elements of z^*.

This lemma is a superconvergence result of the bilinear element, providing $u \in W^{3,\infty}(\Omega)\bigcap H_0^1(\Omega)$.

Let $w^* = (w^*)^{\mathrm{I}} + v^*$, where $(w^*)^{\mathrm{I}}$ is the bilinear interpolation of w^* at the vertices of elements, clearly, $w^* \in C^0(\overline{\Omega})$.Therefore, $(w^*)^{\mathrm{I}}$ and v^* are respectively the conforming and nonconforming part of Wilson element approximation w^*.

By definition,

$$(v^*)_K = (\partial_{xx}w^*)_K\frac{h_x^2}{2}[\frac{(x-x_0)^2}{h_x^2} - 1] + (\partial_{yy}w^*)_K\frac{h_y^2}{2}[\frac{(y-y_0)^2}{h_y^2} - 1]. \tag{3.6}$$

Lemma 4 [1, Cor.1].

$$\|w^*\|_{2,\infty,h} \leq c\|u\|_{2,\infty}.$$

Lemma 4 implies

$$|(\partial_{xx}w^*)_K| \leq c\|u\|_{2,\infty},$$

$$|(\partial_{yy}w^*)_K| \leq c\|u\|_{2,\infty}.$$

From Lemma 4 and (3.6), we have

Lemma 5.

$$|v^*|_{1,\infty,h} \le ch\|u\|_{2,\infty}, \tag{3.7}$$

$$|v^*|_{0,\infty} \le ch^2\|u\|_{2,\infty}. \tag{3.8}$$

Lemma 6 [1, Lemma 3 and Cor.1].

$$\|(w^*)^{\mathrm{I}} - u^*\|_{1,\infty} \le ch^2|\ln h|\|u\|_{2,\infty}. \tag{3.9}$$

Lemma 7.

$$|u - w^*|_{1,\infty} \le ch\|u\|_{2,\infty}, \tag{3.10}$$

$$|u - w^*|_{0,\infty} \le ch^2|\ln h|\|u\|_{2,\infty}. \tag{3.11}$$

Proof. From [8], we can see

$$|u - u^*|_{1,\infty} \le ch\|u\|_{2,\infty}, \tag{3.12}$$

$$|u - u^*|_{0,\infty} \le ch^2|\ln h|\|u\|_{2,\infty}. \tag{3.13}$$

Combination of Lemma 5, Lemma 6, (3.12) and (3.13) completes the proof. □

4. Superconvergence estimates

Theorem 1. *Suppose u, w^* are the solutions of (2.2), (2.3) respectively, $u \in W^{3,\infty}(\Omega)\bigcap H_0^1(\Omega)$, and the rectangular partition $\mathscr{T}_h$ is uniform, K_1 and K_2 are two adjacent elements, (x_i, y_i) is the center of $K_i, i = 1, 2$.*

Case 1. If $x_1 = x_2$, i.e., the elements are up-down adjacent, then

$$|(\partial_{yy} w^*)_{K_1} - (\partial_{yy} w^*)_{K_2})| \le ch\|u\|_{3,\infty}. \tag{4.1}$$

Case 2. If $y_1 = y_2$, i.e., they are right-left adjacent, then

$$|(\partial_{xx} w^*)_{K_1} - (\partial_{xx} w^*)_{K_2}| \le ch\|u\|_{3,\infty}. \tag{4.2}$$

Proof. Case 1. Suppose $K_1 = \square P_1P_2P_3P_4$ and $K_2 = \square P_4P_3P_6P_5$ are up-down adjacent as shown below. All points needed in the proof are shown in the Fig. 1. $2h_{ix}, 2h_{iy}$ are the lengths of edges of K_i in x and y direction respectively. Since the partition is uniform, $h_{1y} = h_{2y} = h_y$, and $h_{1x} = h_{2x} = h_x$.

The finite element equation (2.3) can be written as

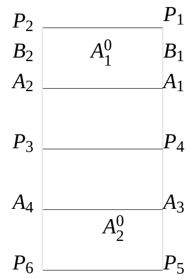

Fig. 1.

$$A_h(w^*, v_h) = A_h(u, v_h) + [(f, v_h) - A_h(u, v_h)] \quad \forall v_h \in W_h \tag{4.3}$$

Let

$$v_h = \begin{cases} \frac{h_y^2}{2}[\frac{(y-y_1)^2}{h_y^2} - 1], & (x,y) \in K_1, \\ -\frac{h_y^2}{2}[\frac{(y-y_2)^2}{h_y^2} - 1], & (x,y) \in K_2, \\ 0, & (x,y) \in \Omega - K_1 - K_2. \end{cases} \tag{4.4}$$

From (4.3), we have

$$\begin{aligned} \int_{K_1\cup K_2} a_2 \partial_y w^* \partial_y v_h dxdy &= \int_{K_1\cup K_2} a_2 \partial_y u \partial_y v_h dxdy \\ &+ \int_{K_1\cup K_2} a_3(u - w^*) v_h dxdy + E, \end{aligned} \tag{4.5}$$

where $E = (f, v_h) - A_h(u, v_h)$ is known as the consistency error of a nonconforming element.

In view of (3.6) and (4.4), the left side of (4.5)

$$\begin{aligned} \int_{K_1\cup K_2} a_2 \partial_y w^* \partial_y v_h dxdy &= \int_{K_1\cup K_2} a_2 \partial_y (w^*)^{\mathrm{I}} \partial_y v_h dxdy \\ &+ \int_{K_1} a_2 (y - y_1)^2 dxdy (w^*_{yy})_{K_1} \\ &- \int_{K_2} a_2 (y - y_2)^2 dxdy (w^*_{yy})_{K_2}. \end{aligned} \tag{4.6}$$

Since $(y - y_i)^2 \geq 0$, there exist $A_1^0 \in K_1$ and $A_2^0 \in K_2$ such that

$$\int_{K_1} a_2 (y - y_1)^2 dxdy = a_2(A_1^0) \frac{4}{3} h_x h_y^3,$$

$$\int_{K_2} a_2 (y - y_2)^2 dxdy = a_2(A_2^0) \frac{4}{3} h_x h_y^3.$$

Therefore, the last two terms on the right side of (4.6)

$$
\begin{aligned}
&\int_{K_1} a_2(y-y_1)^2 dxdy (w^*_{yy})_{K_1} - \int_{K_2} a_2(y-y_2)^2 dxdy (w^*_{yy})_{K_2} \\
&= a_2(A_1^0)((w^*_{yy})_{K_1} - (w^*_{yy})_{K_2})\frac{4}{3} h_x h_y^3 \\
&+ (a_2(A_1^0) - a_2(A_2^0))\frac{4}{3} h_x h_y^3 (w^*_{yy})_{K_2}.
\end{aligned}
\tag{4.7}
$$

Inserting (4.6), (4.7) into (4.5), we obtain

$$
\begin{aligned}
&a_2(A_1^0)((w^*_{yy})_{K_1} - (w^*_{yy})_{K_2})\frac{4}{3} h_x h_y^3 \\
&= (a_2(A_2^0) - a_2(A_1^0))\frac{4}{3} h_x h_y^3 (w^*_{yy})_{K_2} \\
&+ \int_{K_1\cup K_2} a_2 \partial_y (u-w^*)^{\mathrm{I}} \partial_y v_h dxdxy + \int_{K_1\cup K_2} a_2 \partial_y (u-u^{\mathrm{I}}) \partial_y v_h dxdy \\
&+ \int_{K_1\cup K_2} a_3 (u-w^*) v_h dxdy + E \\
&= J_1 + J_2 + J_3 + J_4 + E\,.
\end{aligned}
\tag{4.8}
$$

Using Lemma 4, the first term J_1 on the right side of (4.8) can be bounded from above

$$
|J_1| \le ch^5 \|u\|_{2,\infty}. \tag{4.9}
$$

The second term J_2 is estimated as follows. It is easily verified that

$$
\int_{K_i} \partial_y (u-w^*)^{\mathrm{I}} \partial_y v_h dxdy = 0. \tag{4.10}
$$

Let S be the midpoint of $P_3P_4 = K_1 \bigcap K_2$. Using (4.10), the standard interpolation error estimate, Lemma 7 and Lemma 4, we get

$$
\begin{aligned}
|J_2| &= \Big|\int_{K_1\cup K_2} (a_2 - a_2(S)) \partial_y (u-w^*)^{\mathrm{I}} \partial_y v_h dxdy\Big| \\
&\le ch^4 |(u-w^*)^{\mathrm{I}}|_{1,\infty} \\
&\le ch^4 (|u-w^*|_{1,\infty} + |(u-w^*) - (u-w^*)^{\mathrm{I}}|_{1,\infty}) \\
&\le ch^4 (|u-w^*|_{1,\infty} + h|u-w^*|_{2,\infty}) \\
&\le ch^5 \|u\|_{2,\infty}.
\end{aligned}
\tag{4.11}
$$

Now we consider the third term J_3. From Taylor expansion formula, it's easily seen that

$$
u - u^{\mathrm{I}} = \frac{1}{2}\partial_{xx} u [(x-x_i)^2 - h_x^2] + \frac{1}{2}\partial_{yy} u [(y-y_i)^2 - h_y^2] + R_2(u) \quad in \quad K_i\,,
$$

and

$$
R_2(u) = 0 \quad \forall u \in P_2(K).
$$

Therefore, Bramble-Hilbert lemma yields

$$|R_2(u)|_{m,\infty,K} \le ch^{3-m}|u|_{3,\infty,K}, \quad m = 0, 1.$$

Then we have

$$\begin{aligned}\partial_y(u - u^{\mathrm{I}}) = \; & \frac{1}{2}\partial_{xxy}u[(x - x_i)^2 - h_x^2] + \frac{1}{2}\partial_{yyy}u[(y - y_i)^2 - h_y^2] \\ & + \partial_{yy}u(y - y_i) + \partial_y R_2,\end{aligned}$$

and

$$|\partial_y R_2| \le ch^2|u|_{3,\infty}.$$

Hence

$$|J_3| \le ch^5\|u\|_{3,\infty} + |\int_{K_1} a_2\partial_{yy}u(y - y_1)^2 dxdy - \int_{K_2} a_2\partial_{yy}u(y - y_2)^2 dxdy|.$$

Let r_0 denote the last term of the above inequality, using the same technique dealing with the last two terms on the right side of (4.6), we get

$$|r_0| \le ch^5\|u\|_{3,\infty},$$

hence

$$|J_3| \le ch^5\|u\|_{3,\infty}. \tag{4.12}$$

The estimate of J_4 is easy. In fact, Lemma 1 gives

$$\begin{aligned}|J_4| & \le |\int_{K_1\cup K_2} a_3(u - w^*)v_h dxdy| \\ & \le ch^3|u - w^*|_{0,\Omega} \\ & \le ch^5\|u\|_{2,\infty}.\end{aligned} \tag{4.13}$$

Finally, application of (4.9), (4.11), (4.12), (4.13) to (4.8) implies

$$|(w_{yy}^*)_{K_1} - (w_{yy}^*)_{K_2}| \le ch\|u\|_{3,\infty} + ch^{-4}|E|. \tag{4.14}$$

Now we estimate E.

Applying Gauss integration formula, the equation (2.1) and the definition of v_h, we have

$$\begin{aligned}|E| & = |\int_{\partial K_1}[a_1\partial_x u v_h\cos(n, x) + a_2\partial_y u v_h\cos(n, y)]ds \\ & + \int_{\partial K_2}[a_1\partial_x u v_h\cos(n, x) + a_2\partial_y u v_h\cos(n, y)]ds| \\ & = |\int_{P_4P_1}(a_1\partial_x u v_h)dy - \int_{P_3P_2}(a_1\partial_x u v_h)dy \\ & + \int_{P_5P_4}(a_1\partial_x u v_h)dy - \int_{P_6P_3}(a_1\partial_x u v_h)dy|.\end{aligned} \tag{4.15}$$

Let $q(x,y) = a_1\partial_x u$, so that $q^{\mathrm{I}} \in Q_h$, then the mean-value theorem gives $B_1 \in P_4P_1, B_2 \in P_3P_2$, such that

$$
\begin{aligned}
&|\int_{P_4P_1}(q - q^{\mathrm{I}})v_h dy - \int_{P_3P_2}(q - q^{\mathrm{I}})v_h dy| \\
&= |[(q - q^{\mathrm{I}})(B_1) - (q - q^{\mathrm{I}})(B_2)]\int_{P_4P_1} v_h ds| \\
&\le ch^4|q - q^{\mathrm{I}}|_{1,\infty} \\
&\le ch^5|q|_{2,\infty} \\
&\le ch^5\|u\|_{3,\infty}.
\end{aligned}
\tag{4.16}
$$

Similarly,

$$
|\int_{P_5P_4}(q - q^{\mathrm{I}})v_h dy - \int_{P_6P_3}(q - q^{\mathrm{I}})v_h dy| \le ch^5\|u\|_{3,\infty}. \tag{4.17}
$$

On the other hand, Simpson rule gives

$$
\begin{aligned}
\int_{P_4P_1} q^{\mathrm{I}} v_h ds &= \frac{2}{3}q^{\mathrm{I}}v_h(A_1)2h_y \\
&= -\frac{2}{3}h_y^3 q^{\mathrm{I}}(A_1), \\
\int_{P_3P_2} q^{\mathrm{I}} v_h ds &= -\frac{2}{3}h_y^3 q^{\mathrm{I}}(A_2), \\
\int_{P_5P_4} q^{\mathrm{I}} v_h ds &= \frac{2}{3}h_y^3 q^{\mathrm{I}}(A_3), \\
\int_{P_6P_3} q^{\mathrm{I}} v_h ds &= \frac{2}{3}h_y^3 q^{\mathrm{I}}(A_4),
\end{aligned}
$$

where A_1, A_2, A_3, A_4 are the midpoints of $P_4P_1, P_3P_2, P_5P_4, P_6P_3$ respectively.

The last four equalities together with (4.15), (4.16), (4.17) imply

$$
|E| \le ch^5\|u\|_{3,\infty} + \frac{2}{3}|q^{\mathrm{I}}(A_1) - q^{\mathrm{I}}(A_2) - q^{\mathrm{I}}(A_3) + q^{\mathrm{I}}(A_4)|h_y^3. \tag{4.18}
$$

Since $q^{\mathrm{I}} \in Q_h$, we have

$$
\begin{aligned}
&|q^{\mathrm{I}}(A_1) - q^{\mathrm{I}}(A_2) - q^{\mathrm{I}}(A_3) + q^{\mathrm{I}}(A_4)| \\
&= \frac{1}{2}|q(P_1) - q(P_2) - q(P_5) + q(P_6)| \\
&\le ch^2|q|_{2,\infty} \\
&\le ch^2\|u\|_{3,\infty}.
\end{aligned}
\tag{4.19}
$$

Therefore

$$
|E| \le ch^5\|u\|_{3,\infty}. \tag{4.20}
$$

Combining (4.14)and (4.20) yields the inequality (4.1).
Case 2 can be proved in the same way. □

Lemma 8. *Let v^* be the nonconforming part of w^*, $\mathcal{T}_h$ is a uniform rectangular partition, z^* is a vertex or a midpoint of edges of K, then*

$$|\overline{\nabla} v^*(z^*)| \leq ch^2 \|u\|_{3,\infty}. \tag{4.21}$$

The proof of Lemma 8 can be derived directly by using Theorem 1.

Since

$$\nabla(u - w^*)(z^*) = \nabla(u - u^*)(z^*) + \nabla(u^* - (w^*)^{\mathrm{I}})(z^*) + \nabla v^*(z^*), \tag{4.22}$$

applying Lemma 3, Lemma 6 and Lemma 8, we get the following superconvergence result.

Theorem 2. *If the rectangular partition $\mathcal{T}_h$ is uniform, then there holds the superconvergence estimate*

$$|\overline{\nabla}(u - w^*)(z^*)| \leq ch^2 |\ln h| \|u\|_{3,\infty}. \tag{4.23}$$

Remark 1. Following [1], [8] and the proof of Theorem 1, the regularity of the coefficients a_i of the equation (2.1) may be stated as follows:

$$a_1, a_2 \in W^{2,\infty}(\Omega), \quad a_3 \in W^{1,\infty}(\Omega).$$

Remark 2. The uniform partition condition can be weaken to C-uniform partition, under which Theorem 1 and 2 still hold. C-uniform rectangular partition means that for two adjacent elements K_1 and K_2 , if $y_1 = y_2$, then $h_{1x} - h_{2x} = O(h^2)$, and if $x_1 = x_2$, then $h_{1y} - h_{2y} = O(h^2)$.

Remark 3. So far it has been shown that the asymptotic convergence rate of Wilson nonconforming element either in the energy norm, in the maximum norm or at the nine special superconvergence points of each element is not superior to the bilinear conforming element. The question is still open why the numerical performance of Wilson element is better than the bilinear one in many engineering computations.

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