Non-overlapping Domain Decomposition and Heterogeneous Modeling Used in Solving Free Boundary Problems

by

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Abstract

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Non-overlapping domain decomposition techniques for free boundary problems are extensively considered in this dissertation. We use a model problem to test the effectiveness of various kinds of DDM schemes.

The problem of fluid flow past a truncated concave shaped profile between walls is solved using conformal mapping techniques. An open wake is formed behind the profile. The problem formulated in a hodograph plane is decomposed into two non-overlapping domains. We use different modeling techniques to describe the problems. First, a heterogeneous model is used, i.e., we use different functions in different sub-domains to describe the problem. In one of these domains, a Baiocchi type transformation is used to obtain a fixed domain formulation for the part of the transformed problem containing an unknown boundary. The second method is a heterogeneous modeled problem where the Baiocchi type transformation is extended into the second domain. Next, a parallel version of the latter model is considered. Numerical results show all the methods have good agreement with a published solution. Furthermore, the parallel version of the DDM method is extended to solve other free boundary problems.

Finally, the convergence issue from a mathematical point of view is considered. The existence and convergence properties of the free boundary problems considered

in this dissertation, (including the problem of flow past a concave shaped profile, the rectangular dam problem and the problem of flow through a porous dam with a toe drain), are proved.

Contents

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Chapter 1

Introduction

Parallel Programs have been extensively used recently in solving large scale scientific and engineering problems with the development of multiple-CPU structure(Cray T3E and SGI Origin 2000). Domain decomposition methods(DDM) and multigrid methods also become hot topics since they are highly related to the parallel programs. The general idea of domain decomposition methods is to split the domain of the partial differential equation(PDE) into two(or more) subdomains and then to obtain solutions of related PDE problems on each subdomain. The solution on all the subdomains are then combined to obtain the solution on the whole domain. Domain decomposition is advantageous in that it allows the efficient use of parallel processing in an obvious manner.

There are two types of domain decomposition methods. Let us use two-subdomain splitting to clarify this. The first type is the overlapping domain decomposition method, where there is a common intersection area of these two subdomains. For this type of decomposition, we can solve the PDE on each subdomain with Dirichlet boundary condition.

The second type is the non-overlapping domain decomposition method, where the intersection of these two subdomains is a common boundary interface. For this type of decomposition, we can solve one subdomain PDE with the Dirichlet condition on the common boundary while solving the second subdomain PDE with Neumann condition on the common boundary.

In this dissertation, we will use the non-overlapping domain decomposition method to solve free boundary problems arising from fluid flow problems(such as flow past a concave profile and flow through porous media). Flow through porous media has great importance in many facets of engineering practice. In particular such flows with free surfaces make up a significant part of the seepage phenomena that occur in nature. Some examples of these are seepage through earth dams; seepage from open channels such as rivers, ponds, irrigation systems and recharge basins.

The flow past concave(or convex) profile arises from many practical engineering problems, such as the flow past bridge piers and channel constrictions. This will fall into the category of potential flow with a free streamline.

Dormiani et al.[11] considered the flow past symmetric convex profiles with open wakes. A fixed domain approach and a Baiocchi type transformation in conjunction with a modified Schwarz alternating scheme are used to solve this problem. The flow is such that an open wake is formed behind the profile. Overlapping domain decomposition methods are used. Then, Bruch et al.[7] used a non-overlapping domain decomposition approach to solve the same problem and the numerical results were in good agreement with Davis[10] and Dormiani et al.[11]. However, it is difficult to prove the existence and uniqueness of this problem and the convergence of the numerical scheme from a mathematical view point, since a different type of function is defined in each subdomain and the interface condition involving these two different functions is complicated. This became an open problem.

Bruch et al.[8] considered the flow past symmetric concave profiles with open wakes. The non-overlapping domain decomposition method is used together with heterogeneous modeling, that is, a different dependent variable is used in each subdomain. With this method, we can show the numerical results can approximate a published solution within 0.6%. However it is still difficult to prove the convergence and existence.

Jiang et al.[13] proposed a new non-overlapping DDM scheme with the same function extended across the interface between the two subdomains to solve the flow past symmetric concave profiles with open wakes. In this way, the mathematical formulation becomes very simple and the proof of existence and uniqueness becomes possible. Furthermore, the convergence of the numerical scheme can also be proved due to the simplicity of the interface conditions expressed by the same function from both sides.

One thing to note is that we can not use this method to handle the flow past symmetric convex profiles with open wakes.

To speed up the parallel computation of the above problem, we devised another version of DDM scheme which can execute the computation on both domains at the same time when more than one CPU is available such as a Cray T3E. The convergence speed will be twice as fast as before.

We also used this new parallel DDM scheme for problems of flow through porous media. The numerical results showed that this new method is also advantageous on these problems compared with the traditional DDM.

Finally, we proved the existence and uniqueness of the solution of flow past a concave profile with open wake and show the convergence of the numerical scheme to the true solution.

For the flow through porous media problems, we also considered the mathematical proof of the convergence of the numerical solution to the true solution which is known to exist and be unique[3].

The remainder of this dissertation is organized as follows. In Chapter 2, the model problem of flow past a concave profile with open wake is solved using the non-overlapping method and heterogeneous modeling along with the original dependent variable in one subdomain and a Baiocchi type transformation variable in the other subdomain, which shows excellent performance. In Chapter 3 non-overlapping DDM and heterogeneous modeling are again used to handle the same problem. This time the Baiocchi type transformation is extended into the second subdomain. The performance is still as good as before. In Chapter 4, a revised parallel DDM scheme and heterogeneous modeling for the latter problem is used and it is shown that the new scheme is almost twice as fast as the old one. In Chapter 5, the new DDM scheme is applied to other fluids problems and there is also speed up. In Chapter 6 and Chapter 7, the uniqueness, existence of the true solution and convergence of the numerical scheme toward the true solution for the above mentioned fluids problems are given.

Chapter 2

Flow Past a Concave Profile With Open Wake in a Channel

2.1 Introduction

The physical problem to be investigated in this chapter is flow past a concave shaped profile which is situated in a channel. This type of flow falls into the category of potential flow with a free streamline. Figure 2.1 shows such a case where the location of the free streamline is unknown *a priori*. This two-dimensional, incompressible and inviscid flow is an approximate model of the basic flows that occur in many practical engineering problems.The objective herein is to provide basic potential-flow solutions to the problem and in particular to determine the location of the free streamline.

The physical problem will be formulated in a hodograph plane using conformal mapping techniques. See Bruch and Dormiani[5] and the references therein for work done using this approach. The basic technique that will be used to solve this problem is the fixed domain method in conjunction with the Baiocchi transformation. This approach has had considerable success in solving a wide variety of free and moving boundary problems.

Although the fixed domain approach and a Baiocchi type transformation are not applicable over the entire solution domain, they will be used in conjunction with a non-overlapping domain decomposition and a modified alternating iteration scheme. The numerical results that are obtained herein for flow past a profile between walls will be compared with those of Lesnic et al.[15].

2.2 Formulation of the Problem

This study is concerned with Helmholtz motions, defined as follows:

(a) the motion takes place in free space, i.e., gravity is neglected.

(b) the motion is steady, i.e., $p + \frac{1}{2}\rho u^2 =$ constant, where p, ρ and u are the pressure, the density and the speed of the fluid, respectively.

The flow field includes a pair of free streamlines on which the pressure and velocity are constants, p_c and q_c, respectively. The channel height, $2h$, the velocity on the boundary of the cavity, q_c, and the profile shape are assumed to be known. However, the upstream velocity in the channel, q_∞, and the free streamline locations are to be found(see Figure 2.1).

Because of symmetry the flow region under consideration, $\mathcal{R}$, is bounded between the axis of symmetry, AB, half of the profile, BC, the free streamline, CD, and the wall of the channel, $D'A'$. In this region the stream function, ψ, identically satisfies the continuity equation and the irrotationality condition which gives Laplace's equation as the governing differential equation. The boundary $ABCD$ is the $\psi = 0$ streamline and $\psi = q_\infty h$ on the wall $D'A'$. The downstream jet half-width, d_c, is found from the conservation of mass relation

$$q_c d_c = q_\infty h \tag{2.1}$$

and q_c may be set to unity without any loss of generality. Therefore, the mathematical formulation of the problem in the physical plane becomes: find $\psi(x, y)$ such that

$$\triangle\psi(x,y) = 0 \quad \text{in } \mathcal{R} \tag{2.2a}$$

$$\psi(x,y) = 0 \quad \text{on ABCD} \tag{2.2b}$$

$$\psi = q_\infty h \quad \text{on D'A'} \tag{2.2c}$$

$$\lim_{x\to\infty} \psi(x,y) = y q_\infty \quad \text{on AA'} \tag{2.2d}$$

Figure 2.1: The Physical Problem

$$\lim_{x\to\infty}\psi(x,y) = [y-(h-d_c)]q_c \quad \text{on DD'} \tag{2.2e}$$

$$|\nabla\psi| = q_c \quad \text{on CD(free surface).} \tag{2.2f}$$

Note that velocity at each point is

$$\vec{q} = (q_1, q_2) = qe^{i\theta} = q(\cos\theta + i\sin\theta).$$

We can choose q_1 and q_2 as our variables instead of x and y. We perform the transformation $T_1 : (x,y) \to (q_1, q_2)$, and we can easily prove

$$\frac{\partial^2\psi}{\partial q_1^2} + \frac{\partial^2\psi}{\partial q_2^2} = 0 \tag{2.3}$$

in the region $R_1 = T_1(\mathcal{R})$ on the $q_1 - q_2$ plane(the hodograph plane) (see Figure 2.2).

Since

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -q_2dx + q_1dy = -q\sin\theta dx + q\cos\theta dy, \tag{2.4a}$$

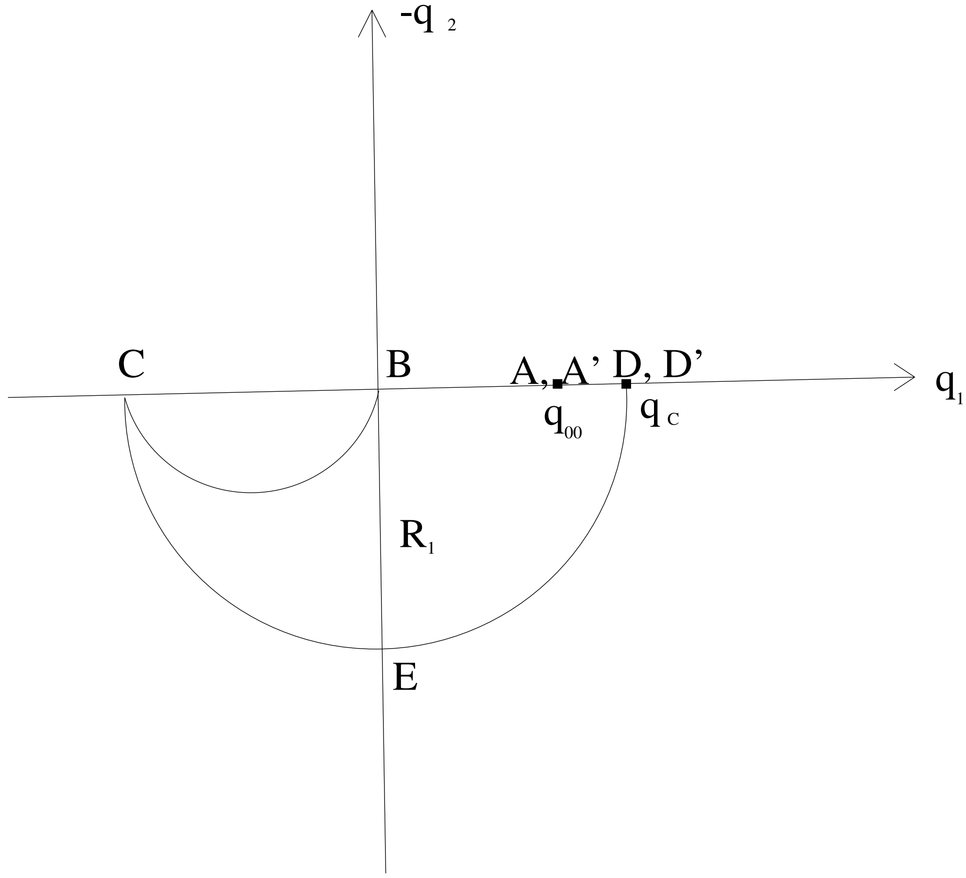

Figure 2.2: The Transformed Problem in q_1-q_2 Plane

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = q_1 dx + q_2 dy = q \cos\theta dx + q \sin\theta dy, \tag{2.4b}$$

we obtain

$$dx + idy = \frac{e^{i\theta}}{q}(d\phi + id\psi). \tag{2.5}$$

It is convenient to introduce the variable σ by

$$q = q_c e^{-\sigma} \text{or} \quad \sigma = -ln\frac{q}{q_c}. \tag{2.6}$$

Now the variables are (θ, σ) instead of (q_1, q_2).

The coordinate transformation $(x, y) \to (\theta, \sigma)$ maps the problem in the physical plane onto the hodograph plane, where θ is the polar angle of velocity and $\sigma =$

$-ln(q/q_c)$. Values of the harmonic function ψ are unchanged on the boundaries of the region under the conformal mapping.

Since the profile is concave, the coordinates X and Y of the curve in the (θ, σ) hodograph plane representing the surface of the profile can be expressed in terms of θ, where θ is the angle between tangent to the curve and x-axis. Let $X(\theta)$ and $Y(\theta)$ be this parameterization. Then $\widetilde{R}(\theta)$, the algebraic radius of curvature, is

$$\widetilde{R}(\theta) = -\sqrt{X'(\theta)^2 + Y'(\theta)^2} \tag{2.7}$$

and bounded.

Now (2.5) can be written as

$$dx + idy = \frac{e^{\sigma + i\theta}}{q_c}(d\phi + id\psi). \tag{2.8}$$

The left hand side of this equation is the total differential so the right hand side must be too,

$$\begin{aligned} \tfrac{e^{\sigma+i\theta}}{q_c}(d\phi + id\psi) &= \tfrac{e^{\sigma+i\theta}}{q_c}(\tfrac{\partial\phi}{\partial\theta}d\theta + \tfrac{\partial\phi}{\partial\sigma}d\sigma + i\tfrac{\partial\psi}{\partial\theta}d\theta + i\tfrac{\partial\psi}{\partial\sigma}d\sigma) \\ &= \tfrac{e^{\sigma+i\theta}}{q_c}[(\tfrac{\partial\phi}{\partial\theta} + i\tfrac{\partial\psi}{\partial\theta})d\theta + (\tfrac{\partial\phi}{\partial\sigma} + i\tfrac{\partial\psi}{\partial\sigma})d\sigma]. \end{aligned} \tag{2.9}$$

Now we can write the condition of total differentials for the right hand side of (2.9) namely

$$\frac{\partial}{\partial\sigma}[\frac{e^{\sigma+i\theta}}{q_c}(\frac{\partial\phi}{\partial\theta} + i\frac{\partial\psi}{\partial\theta})] = \frac{\partial}{\partial\theta}[\frac{e^{\sigma+i\theta}}{q_c}(\frac{\partial\phi}{\partial\sigma} + i\frac{\partial\psi}{\partial\sigma})].$$

After differentiating and simplifying we obtain

$$\frac{\partial\phi}{\partial\sigma} = \frac{\partial\psi}{\partial\theta}, -\frac{\partial\psi}{\partial\sigma} = \frac{\partial\phi}{\partial\theta}. \tag{2.10}$$

From which we deduce, eliminating ϕ,

$$\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial\sigma^2} = \triangle\psi = 0 \qquad \text{on } R = T(\mathcal{R}), \tag{2.11}$$

where T represents the transformation $(x, y) \to (\theta, \sigma)$ from the physical to the hodograph plane (see Figure 2.3). Note

$$\Gamma = T(P) \tag{2.12}$$

is the transformation of the profile, and

$$\sigma = l(\theta) \tag{2.13}$$

is the curve in the hodograph plane corresponding to the profile in the physical plane.

The boundary conditions for ψ in the hodograph plane are

$$\begin{aligned} \psi(0,\sigma) &= 0 \qquad \text{on } \sigma \geq \sigma_\infty \\ \psi(0,\sigma) &= hq_\infty \qquad \text{on } 0 < \sigma \leq \sigma_\infty \\ \psi(\theta,0) &= 0 \quad 0 \leq \theta \leq \theta_1 \end{aligned} \tag{2.14}$$

$$\psi(\theta,\sigma) = 0 \qquad \text{on } \Gamma. \tag{2.15}$$

On Γ, the transformation of the profile boundary to the hodograph plane, the radius of curvature is

$$\begin{aligned} \widetilde{R}(\theta) &= -\sqrt{X'(\theta)^2 + Y'(\theta)^2} \\ &= -[(\tfrac{dX}{d\theta})^2 + (\tfrac{dY}{d\theta})^2]^{\frac{1}{2}}, \end{aligned} \tag{2.16}$$

where θ is the angle between the tangent to the curve and the x-axis. and we can write (2.8) along the boundary of profile where $\psi = 0$ and $d\psi = 0$. Therefore

$$\begin{aligned} dX + idY &= \tfrac{e^{\sigma+i\theta}}{q_c}(d\phi + id\psi) = \tfrac{e^{\sigma+i\theta}}{q_c}(d\phi) \\ &= \tfrac{e^{\sigma+i\theta}}{q_c}(\tfrac{\partial\phi}{\partial\theta}d\theta + \tfrac{\partial\phi}{\partial\sigma}d\sigma). \end{aligned} \tag{2.17}$$

From (2.10) we have $\frac{\partial\phi}{\partial\sigma} = \frac{\partial\psi}{\partial\theta}$ and $\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial\sigma}$, therefore (2.17) becomes

$$dX + idY = \frac{e^{\sigma+i\theta}}{q_c}(-\frac{\partial\psi}{\partial\sigma}d\theta + \frac{\partial\psi}{\partial\theta}d\sigma). \tag{2.18}$$

Also note that on the profile $d\psi = 0$,

$$d\psi = \frac{\partial\psi}{\partial\theta}d\theta + \frac{\partial\psi}{\partial\sigma}d\sigma = 0,$$

hence

$$\frac{\partial\psi}{\partial\sigma}l'(\theta) = -\frac{\partial\psi}{\partial\theta}. \tag{2.19}$$

Therefore (2.18) becomes

$$\begin{aligned} dX + idY &= \tfrac{e^{\sigma+i\theta}}{q_c}(-\tfrac{\partial\psi}{\partial\sigma}d\theta - \tfrac{\partial\psi}{\partial\sigma}l'(\theta)d\sigma) \\ &= \tfrac{e^{\sigma+i\theta}}{q_c}(-\tfrac{\partial\psi}{\partial\sigma})[1 + l'(\theta)^2]d\theta, \end{aligned}$$

then

$$\frac{dX}{d\theta} + i\frac{dY}{d\theta} = -\frac{e^{\sigma+i\theta}}{q_c}(\frac{\partial\psi}{\partial\sigma})(1 + l'(\theta)^2).$$

Consider (2.16) and the fact that $\frac{\partial\psi}{\partial\sigma} < 0$ on Γ(since $\psi = 0$ on Γ and $\psi > 0$ below Γ), we obtain

$$\frac{\partial\psi}{\partial\sigma} = \frac{q_c e^{-\sigma}\widetilde{R}(\theta)}{1 + l'(\theta)^2} \quad \text{on} \quad \Gamma. \tag{2.20}$$

From (2.19)

$$\frac{\partial\psi}{\partial\theta} = -l'(\theta)\frac{\partial\psi}{\partial\sigma} = -\frac{q_c e^{-\sigma}\widetilde{R}(\theta)l'(\theta)}{1 + l'(\theta)^2} \quad \text{on} \quad \Gamma. \tag{2.21}$$

Finally, on the wake boundary we have

$$\psi(\theta, 0) = 0 \quad 0 \leq \theta \leq \theta_1, \tag{2.22}$$

where θ_1 is the polar angle of common tangent to the wake and profile at the connecting point. Therefore, the governing equation and boundary conditions take the following form in the hodograph plane:

$$\triangle\psi = 0 \qquad \text{on } R = T(\mathcal{R}) \tag{2.23}$$

$$\psi(0, \sigma) = 0 \qquad \text{on } \sigma \geq \sigma_\infty$$

$$\psi(0, \sigma) = hq_\infty \qquad \text{on } 0 < \sigma \leq \sigma_\infty \tag{2.23a}$$

$$\psi(\theta, \sigma) = 0 \qquad \text{on } \Gamma \tag{2.23b}$$

$$\frac{\partial\psi}{\partial\sigma} = \frac{q_c e^{-\sigma}\widetilde{R}(\theta)}{1 + l'(\theta)^2} \quad \text{on } \Gamma \tag{2.23c}$$

$$\frac{\partial\psi}{\partial\theta} = -\frac{q_c e^{-\sigma}\widetilde{R}(\theta)l'(\theta)}{1 + l'(\theta)^2} \quad \text{on } \Gamma \tag{2.23d}$$

$$\psi(\theta, 0) = 0 \quad 0 \leq \theta \leq \theta_1. \tag{2.23e}$$

These equations are identical to (2.11), (2.14), (2.15), (2.20), (2.21) and (2.22), respectively, and R is the image of region $\mathcal{R}$ under the transformation and Γ is the representation of the profile in the hodograph plane (see Figure 2.3). On Γ, $\sigma = l(\theta)$. Note that the location of Γ and the point $(0, \sigma_\infty)$ are unknown *a priori*.

The region R of the problem in the hodograph plane is next divided into two non-overlapping regions R_ψ and R_{u_1}(see Figure 2.3) such that

$$R = R_\psi \cup R_{u_1} \cup \Gamma_1, \text{ where } R_\psi = \{(\theta, \sigma) | 0 < \theta < \theta_0, \quad \sigma > 0\},$$

$$R_{u_1} = \{(\theta, 0) | \theta_0 < \theta < \theta_1, \sigma < l(\theta)\}, \quad \Gamma_1 = \{(\theta, \sigma) | \theta = \theta_0, \quad \sigma > 0\},$$

in which θ_0 is the value of θ at the stagnation point and θ_1 is the value at the detachment point of the cavity boundary from the profile.

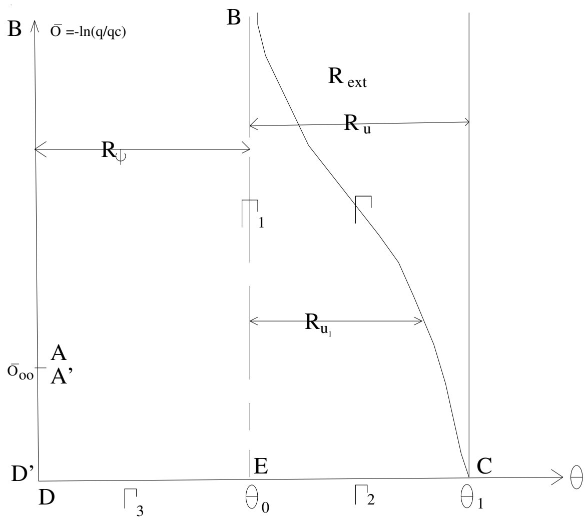

Figure 2.3: The Transformed Problem in θ-σ Plane

Define an integrated stream function by using the Baiocchi type transformation

$$u_1(\theta, \sigma) = \frac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi(\theta, \tau) d\tau \tag{2.24}$$

on the region R_{u_1}. Note that $u_1 > 0$ in R_{u_1} since $\psi > 0$ there.

Next, the dependent variable u_1 is continuously extended across the boundary Γ, on which $u_1 = 0$, into

$$R_{ext} = \{(\theta, \sigma) | \theta_0 < \theta < \theta_1, l(\theta) < \sigma < \infty\},$$

such that u_1 is zero in R_{ext}. Let $R_u = R_{u_1} \cup R_{ext} \cup \Gamma$ and u_1 is defined in (2.24) for the region R_{u_1}. Therefore, $\forall \phi \in C_0^\infty(R_u)$, $u_1(\theta, \infty) = 0$, $u_{1\sigma}(\theta, \infty) = 0$ in the region

R_{ext}, we obtain

$$\begin{aligned}\iint_{R_u} \nabla u_1 \cdot \nabla \phi d\theta d\sigma &= \iint_{R_u} (u_{1\theta}\phi_\theta + u_{1\sigma}\phi_\sigma) d\theta d\sigma \\ &= \iint_{R_{u_1}} (u_{1\theta\theta} + u_{1\sigma\sigma})\phi d\theta d\sigma.\end{aligned}$$

Since

$$\begin{aligned}u_{1\theta} &= \tfrac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi_\theta(\theta,\tau) d\tau + \tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi(\theta, l(\theta)) l'(\theta) \\ &= \tfrac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi_\theta(\theta,\tau) d\tau\end{aligned}$$

and

$$u_{1\sigma} = -\frac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi(\theta,\tau) d\tau - \frac{e^{-\sigma}}{q_c} e^\sigma \psi(\theta,\sigma),$$

then

$$\begin{aligned}u_{1\theta\theta} &= \tfrac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi_{\theta\theta}(\theta,\tau) d\tau + \tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi_\theta(\theta, l(\theta)) l'(\theta) \\ &= -\tfrac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi_{\tau\tau}(\theta,\tau) d\tau + \tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi_\theta(\theta, l(\theta)) l'(\theta) \\ &= -\tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi_\sigma(\theta, l(\theta)) + \tfrac{e^{-\sigma}}{q_c} e^\sigma \psi_\sigma(\theta,\sigma) \\ &\quad + \tfrac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi_\tau(\theta,\tau) d\tau + \tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi_\theta(\theta, l(\theta)) l'(\theta) \\ &= -\tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi_\sigma(\theta, l(\theta)) + \tfrac{1}{q_c} \psi_\sigma(\theta,\sigma) + \tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi_\theta(\theta, l(\theta)) l'(\theta) \\ &\quad + \tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} \psi(\theta, l(\theta)) - \tfrac{1}{q_c} \psi(\theta,\sigma) - \tfrac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi(\theta,\tau) d\tau\end{aligned} \tag{2.25}$$

and

$$u_{1\sigma\sigma} = \frac{e^{-\sigma}}{q_c} \int_\sigma^{l(\theta)} e^\tau \psi(\theta,\tau) d\tau + \frac{e^{-\sigma}}{q_c} e^\sigma \psi(\theta,\sigma) - \frac{1}{q_c} \psi_\sigma(\theta,\sigma), \tag{2.26}$$

where subscripts σ and θ denote differentiation with respect to that variable. Therefore,

$$\begin{aligned}u_{1\theta\theta} + u_{1\sigma\sigma} &= -\tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} [\psi_\sigma(\theta, l(\theta)) - \psi_\theta(\theta, l(\theta)) l'(\theta)] \\ &= -\tfrac{e^{-\sigma}}{q_c} e^{l(\theta)} [\psi_\sigma(\theta, l(\theta)) + \psi_\sigma(\theta, l(\theta)) l'(\theta)^2] \\ &= -e^{-\sigma} \widetilde{R}(\theta).\end{aligned}$$

Hence,

$$\iint_{R_u} \nabla u_1 \cdot \nabla \phi d\theta d\sigma = \iint_{R_u} [e^{-\sigma} \widetilde{R}(\theta)] \phi d\theta d\sigma. \tag{2.27}$$

Therefore, we obtain

$$\triangle u_1 = \frac{\partial^2 u_1}{\partial \theta^2} + \frac{\partial^2 u_1}{\partial \sigma^2} = -\widetilde{R}(\theta) e^{-\sigma} \chi_{R_{u_1}} \quad \text{in } R_u, \tag{2.28}$$

where $\chi_{R_{u_1}}$ is the characteristic function defined by $\chi_{R_{u_1}} = 1$ in R_{u_1} and $\chi_{R_{u_1}} = 0$ otherwise.

Since $u_1 \geq 0, [-\triangle u_1(\theta,\sigma) - \widetilde{R}(\theta)e^{-\sigma}] \geq 0$, then

$$u_1[-\triangle u_1(\theta,\sigma) - \widetilde{R}(\theta)e^{-\sigma}] = 0. \tag{2.29}$$

In the right region R_{u_1}, only u_1 is defined, but in the left region R_ψ, ψ is defined, and these unknowns have some interface conditions on Γ_1 that connect them, therefore our original problem can be stated using ψ and u_1 in the two non-overlapping regions as follows:

$$\triangle\psi = 0 \qquad \text{in } R_\psi \tag{2.30a}$$

$$\psi(\theta,0) = 0 \qquad 0 \leq \theta \leq \theta_0 \tag{2.30b}$$

$$\psi(0,\sigma) = hq_\infty \quad 0 \leq \sigma \leq \sigma_\infty = -ln\frac{q_\infty}{q_c} \tag{2.30c}$$

$$\psi(0,\sigma) = 0 \quad \sigma > \sigma_\infty$$

$$\psi(\theta,\infty) = 0 \quad 0 \leq \theta \leq \theta_0$$

$$\psi(\theta_0,\sigma) = -q_c(u_1 + u_{1\sigma}) \quad \text{on } \Gamma_1 \tag{2.30d}$$

and

$$\triangle u_1 = -\widetilde{R}(\theta)e^{-\sigma}\chi_{R_{u_1}} \qquad \text{in } R_u \tag{2.31a}$$

$$u_1(\theta_1,\sigma) = 0 \qquad \sigma \geq 0 \tag{2.31b}$$

$$u_1 + u_{1\sigma} = 0 \quad \text{on } \Gamma_2 \tag{2.31c}$$

$$u_1(\theta,\infty) = 0 \quad \theta_0 \leq \theta \leq \theta_1$$

$$u_{1\theta}(\theta_0,\sigma) = \frac{e^{-\sigma}}{q_c}\int_\sigma^\infty e^\tau\psi_\theta(\theta_0,\tau)d\tau \quad \text{on } \Gamma_1, \tag{2.31d}$$

where $\Gamma_2 = \{(\theta,\sigma)|\theta_0 \leq \theta \leq \theta_1, \sigma = 0\}$.

Co-ordinate transformation expressions describing the co-ordinates of the physical plane in terms of the co-ordinates of the hodograph plane, are needed for calculating the co-ordinates of the wake boundary $x = x(\theta,\sigma)$ and $y = y(\theta,\sigma)$.

Since $\sigma = 0$ thus $d\sigma = 0$ on the wake boundary, from (2.18) we obtain

$$dx = -\frac{1}{q_c}\frac{\partial\psi}{\partial\sigma}\cos\theta d\theta \text{ and } \quad dy = -\frac{1}{q_c}\frac{\partial\psi}{\partial\sigma}\sin\theta d\theta \text{ on } \Gamma_2 \cup \Gamma_3, \tag{2.32}$$

where $\Gamma_3 = \{(\theta,\sigma)|0 \leq \theta \leq \theta_0, \sigma = 0\}$.

2.3 Numerical Procedure and Results

The problem posed in the hodograph plane by equations (2.30) and (2.31) is formulated in a region which is unbounded in the positive σ- direction. For numerical computations, the region will be truncated. Toward this end, a σ_u is chosen which is sufficiently large so that for all practical purposes the values of ψ and u_1 for $\sigma > \sigma_u$ are approximately zero.

Note that since the function defining σ is logarithmic, $\sigma = -\ln(\frac{q}{q_c})$, and u_1 is weighted by an exponential function, the truncation has little or no effect on the numerical results. Hence, σ_u provides an upper bound for $R_{u_1} \cup R_\psi \cup \Gamma$. The solution algorithm is a finite difference successive over- relaxation scheme for both u_1 and ψ with projection for the u_1-problem only. A grid of mesh points is superimposed on the bounded region, where each node is specified by row i and column j. Therefore, the field equation for ψ, equation (2.30a), can be written as the following difference equation:

$$\begin{aligned} \psi_{i,j}^{(n+\frac{1}{2})} &= \tfrac{\alpha\beta(\triangle\theta)^2(\triangle\sigma)^2}{2[\alpha(\triangle\theta)^2+\beta(\triangle\sigma)^2]}[\tfrac{2}{(1+\alpha)(\triangle\theta)^2}\psi_{i,j-1}^{(n+1)} + \tfrac{2}{\alpha(1+\alpha)(\triangle\theta)^2}\psi_{i,j+1}^{(n)} \\ &\quad + \tfrac{2}{(1+\beta)(\triangle\sigma)^2}\psi_{i-1,j}^{(n+1)} + \tfrac{2}{\beta(1+\beta)(\triangle\sigma)^2}\psi_{i+1,j}^{(n)}] \end{aligned} \tag{2.33a}$$

and

$$\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)} + \omega(\psi_{i,j}^{(n+\frac{1}{2})} - \psi_{i,j}^{(n)}), \tag{2.33b}$$

where $\triangle\theta$ and $\triangle\sigma$ are the spacings in θ and σ- direction, respectively, α and β provide for unequal divisions for mesh points, ω is the over-relaxation parameter and $\psi_{i,j}^{(n)}$ is the value of ψ at node i, j for the nth iteration. Similarly, for u_1 in the region R_{u_1} (see equation(2.31a)):

$$\begin{aligned} u_{1(i,j)}^{(n+\frac{1}{2})} &= \tfrac{\alpha\beta(\triangle\theta)^2(\triangle\sigma)^2}{2[\alpha(\triangle\theta)^2+\beta(\triangle\sigma)^2]}[\tfrac{2}{(1+\alpha)(\triangle\theta)^2}u_{1(i,j-1)}^{(n+1)} + \tfrac{2}{\alpha(1+\alpha)(\triangle\theta)^2}u_{1(i,j+1)}^{(n)} \\ &\quad + \tfrac{2}{(1+\beta)(\triangle\sigma)^2}u_{1(i-1,j)}^{(n+1)} + \tfrac{2}{\beta(1+\beta)(\triangle\sigma)^2}u_{1(i+1,j)}^{(n)} + \widetilde{R}(\theta_j)e^{-\sigma_i}] \end{aligned} \tag{2.34a}$$

and

$$u_{1(i,j)}^{(n+1)} = max\{0, u_{1(i,j)}^{(n)} + \omega(u_{1(i,j)}^{(n+\frac{1}{2})} - u_{1(i,j)}^{(n)})\}, \tag{2.34b}$$

where $u_{1(i,j)}^{(n)}$ is the value of u_1 at node i,j for the nth iteration. These iterations are stopped when

$$\max_{i,j} |\psi_{i,j}^{(n+1)} - \psi_{i,j}^{(n)}| < \epsilon \quad \text{and} \quad \max_{i,j} |u_{1(i,j)}^{(n+1)} - u_{1(i,j)}^{(n)}| < \epsilon, \tag{2.35}$$

where ϵ is some fixed positive constant.

Note that on the boundary $\Gamma_4 = \{(\theta,\sigma)|0 \leq \theta \leq \theta_0, \sigma = \sigma_u\}$, which corresponds to the stagnation point in the physical plane , $\psi \approx 0$, for the region R_ψ. On the boundary $\Gamma_5 = \{(\theta,\sigma)|\theta_0 \leq \theta \leq \theta_1, \sigma = \sigma_u\}$, $\quad u_1 = 0$.

The values of ψ at the mesh points on Γ_1 are calculated using equation (2.30d), in which $u_{1\sigma}$ is approximated by a central difference expression; therefore for $\theta_0 = n_1\triangle\theta$, where n_1 is the number of spacings in the θ-direction in R_ψ,

$$\psi_{i,n_1}^{(n+1)} = q_c[u_{1i,0}^{(n)} + \frac{u_{1i+1,0}^{(n)} - u_{1i-1,0}^{(n)}}{2\triangle\sigma}]. \tag{2.36}$$

On the other hand, the column of mesh points bounding region R_ψ, which are on the line Γ_1, forms the boundary of the region R_{u_1}, and equation(2.31d) is used to calculated the boundary condition. The integral in this equation is approximated by using a mid-point formula;hence

$$u_{1(i,0)}^{(n+1)} = u_{1(i,2)}^{(n+1)} - 2\triangle\theta[u_{1int}(i\triangle\sigma)]\frac{e^{-i\triangle\sigma}}{q_c}, \tag{2.37}$$

where

$$\begin{aligned} u_{1int}(i\triangle\sigma) \quad &= \tfrac{\triangle\sigma}{2}[e^{(i+1)\triangle\sigma}(\psi_{i+1,n_1-2}^{(n)} - 4\psi_{i+1,n_1-1}^{(n)} + 3\psi_{i+1,n_1}^{(n)}) \\ &\quad + e^{i\triangle\sigma}(\psi_{i,n_1-2}^{(n)} - 4\psi_{i,n_1-1}^{(n)} + 3\psi_{i,n_1}^{(n)})]/(2\triangle\theta) + u_{1int}((i+1)\triangle\sigma) \end{aligned}$$

and $u_{1int}(N\triangle\sigma) = 0$, where N is the number of divisions in the σ-direction.

The iteration sequence is started by setting the boundary conditions for ψ in the region R_ψ and for u_1 in the region R_{u_1}, and a zero initial guess for the interior $\psi_{i,j}^{(0)}$ and $u_{1(i,j)}^{(0)}$. Then using equations (2.33), the $\psi_{i,j}^{(1)}$ are obtained starting from the lower left interior point and moving to the right along mesh points and upwards until all the interior points in R_ψ are covered. The next step is to set the boundary conditions for R_{u_1} making use of equation (2.37) and the newly calculated values for ψ, and then starting from the upper left point in the region until the entire mesh is covered. $u_{1(0,j)}$ on Γ_2 will be computed as the last row by using a difference scheme derived from the combination of (2.31a) and (2.31c). This provides new u_1 values for points on Γ_1, and hence new boundary conditions for ψ in region R_ψ using equation (2.36).

This alternating sweeping of the two regions continues until the conditions (2.35) are satisfied. Since numerical values for u_1 inside the boundary, Γ, are nonzero and those on the boundary and inside of it are zero, the zero points bordering non-zero points in the R_u region determine this boundary.

The velocity on the boundary of the cavity, q_c, is assumed to be known but, as stated before, the upstream velocity q_∞ is,like the boundary Γ, unknown *a priori* and is to be found as part of the solution. Therefore, different values for σ_∞, where $\sigma_\infty = -ln(q_\infty/q_c)$, are used until the best one is found. The calculation sequence assumes mesh points on the boundary $\Gamma_0 = \{(\theta,\sigma)|\theta = 0, 0 < \sigma < \sigma_u\}$, starting from the point with minimum σ and going upwards. For each assumed σ_∞ the alternating iteration sequence described above is performed and coordinates of the wake, using equation (2.32),are calculated.

In order to calculate co-ordinates of the boundary of the wake, we must first consider Γ_2 to obtain the location of CE; then consider $\Gamma_3 = \{(\theta, 0)|0 < \theta < \theta_0\}$ to obtain the location of ED.

On Γ_2, $\psi = -q_c(u_1 + u_{1\sigma})$ and $\psi = 0$, then $u_{1\sigma} = -u_1$ and

$$\psi_\sigma = -q_c(u_{1\sigma} + u_{1\sigma\sigma}) = -q_c(u_{1\sigma\sigma} - u_1).$$

Therefore, using (2.32) we have

$$x_{n1+j-1} = x_{n1+j} - \frac{\triangle\theta}{2}[t_j \cos\theta_j + t_{j-1}\cos\theta_{j-1}] \quad \text{on } \Gamma_2 \tag{2.38a}$$

and

$$y_{n1+j-1} = y_{n1+j} - \frac{\triangle\theta}{2}[t_j \cos\theta_j + t_{j-1}\cos\theta_{j-1}] \quad \text{on } \Gamma_2, \tag{2.38b}$$

where x_{n1+n2}, y_{n1+n2} are the co-ordinates of C, $1 \leq j \leq n_2$, n_2 is the number of spacing in the θ-direction in R_{u_1}, t_j is approximated by its forward difference expression

$$t_j = (2.0 * u_{1(0,j)} - 5.0 * u_{1(1,j)} + 4.0 * u_{1(2,j)} - u_{1(3,j)})/(\triangle\sigma)^2 - u_{1(0,j)}.$$

On Γ_3, equation (2.32) are integrated directly between two adjacent mesh points using the trapezoidal rule, which yields

$$x_{j-1} = x_j + \frac{\triangle\theta}{2q_c}[\frac{\partial\psi}{\partial\sigma}|_j \cos\theta_j + \frac{\partial\psi}{\partial\sigma}|_{j-1} \cos\theta_{j-1}] \quad \text{on } \Gamma_3 \tag{2.39a}$$

and

$$y_{j-1} = y_j + \frac{\triangle\theta}{2q_c}[\frac{\partial\psi}{\partial\sigma}|_j \cos\theta_j + \frac{\partial\psi}{\partial\sigma}|_{j-1} \cos\theta_{j-1}] \quad \text{on } \Gamma_3, \tag{2.39b}$$

where $1 \leq j \leq n_1$, $\frac{\partial\psi}{\partial\sigma}|_j$ is approximated by its forward difference expression

$$\frac{\partial\psi}{\partial\sigma}|_j = \frac{1}{2\triangle\sigma}[-3\psi_{0,j} + 4\psi_{1,j} - \psi_{2,j}].$$

Once the co-ordinates of the wake are determined, the cavity distance d_c,which is the distance between the boundary of the cavity and the wall at infinity (see Figure 2.4) is calculated. Then from equation (2.1) the upstream velocity q_∞ or consequently σ_∞, is calculated and is compared to the assumed value of σ_∞. The mesh point corresponding to the minimum difference between the calculated upstream velocity and assumed upstream velocities chosen for the desired value for σ_∞. It is evident that the finer the mesh points are on boundary Γ_3, the better is the accuracy in the determination of σ_∞.

Results obtained are shown below and we can see the coincidence of our results with another published numerical solution(Lesnic et al.[15]).

2.4 Computational Results

Figure 2.4 shows results for an open profile which has the shape of an arc of a circle(radius = 1.0) and for which the free streamline leaves the profile at 180 degrees (θ_1 = 180). The profile is located between walls each having a distance h = 10.0 from the axis of symmetry for one case and h = 50.0 for another. These cases are shown in Figure 2.4 along with the results given by Lesnic et al.[15] without walls. As can be seen, the shapes and locations of the free streamlines are in appropriate agreement. The velocity on the boundary of the cavity was assumed to be $q_c = 1$, the over-relaxation parameter was taken to be $\omega = 1.6$, $\sigma_u = 4.0$ and the number of divisions in the θ and σ-directions were 150 and 200, respectively, with variable $\triangle\sigma$. The upstream uniform fluid velocity was computed to be $q_\infty = 0.68147$ for the case of $h = 10.0$ with $\epsilon = 1.3 \times 10^{-4}$ and $q_\infty = 0.84713$ for the case when $h = 50.0$ with $\epsilon = 8.0 \times 10^{-5}$. The calculated and assumed upstream velocities in each of the two cases were in agreement to within 0.6%.

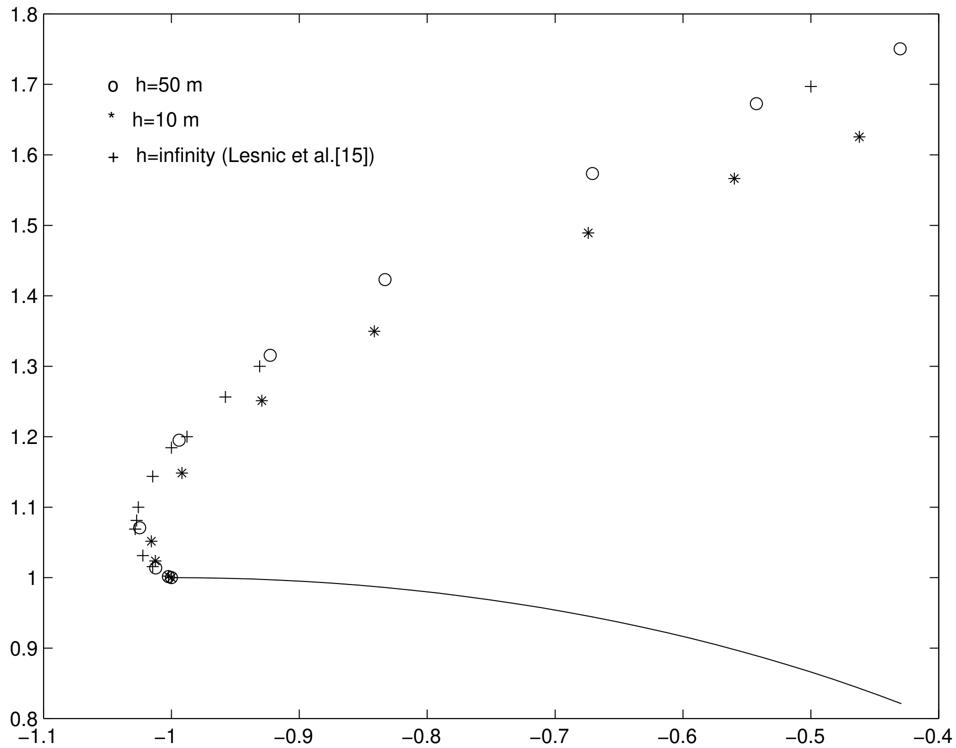

Figure 2.4: Comparison of Numerical Results

2.5 Summary and Conclusion

The numerical algorithm presented here is simple and efficient and, as seen from the comparison of results, give reasonable solutions. Thus, the solution approach can be applied to general truncated concave shaped profiles between walls. Furthermore, the numerical scheme gives the velocity along the profile which is the curve Γ in the (θ, σ)-plane as part of the solution. This is simply the line that separates the region where $u_1 > 0$ from that where $u_1 = 0$. This free boundary type of problem is different from other such problems in that the free streamline CD is a horizontal line in the (θ, σ)-plane, whereas the velocity distribution on BC becomes the unknown Γ, the boundary sought in the (θ, σ)-plane.

Chapter 3

Domain Decomposition and Heterogeneous Modeling Used in Solving a Free Streamline Flow

3.1 Introduction

The physical problem studied herein is again the flow past a concave shaped profile in a channel. Figure 2.1 shows the flow where the location of the free streamlines are unknown *a priori*. The objective is to provide a basic potential-flow solution to this problem and in particular the location of the free streamline.

Dormiani et al. [11] use an overlapping domain decomposition approach and Bruch et al. [7] use a non-overlapping domain decomposition approach in solving a problem of flow past a truncated convex profile. Also, Bruch et al. [8] used a non-overlapping domain decomposition approach to solve a similar problem to the one studied herein. In the problems solved in [3], [6], [7] and in Chapter 2, the original dependent variable, the stream function, was the solution variable that was used in the domain without the unknown boundary. On the boundary between the two solution domains a relationship between the stream function and the Baiocchi transformation variable is used. In the approach used herein the Baiocchi type variable transformation is extended across the boundary between the two domains. This assures that the dependent variables and their normal derivatives are continuous along this common boundary. The numerical results obtained herein will again be compared with those of Lesnic et al.[15] for a truncated circular arc profile.

3.2 Formulation of the Problem

The formulation and the approach to obtain the hodograph plane are given in Chapter 2. The region R of the problem in the hodograph plane is divided into two non-overlapping regions R_{u_1} and R_{u_2}(see Figure 3.1), such that $R = R_{u_1} \cup R_{u_2} \cup \Gamma_1$, where $R_{u_2} = \{(\theta,\sigma)|0 < \theta < \theta_0, \sigma > 0\}$, $R_{u_1} = \{(\theta,\sigma)|\theta_0 < \theta < \theta_1, 0 < \sigma < l(\theta)\}$, and $\Gamma_1 = \{(\theta,\sigma)|\theta = \theta_0, \sigma > 0\}$, in which θ_0 is the value of θ at the stagnation point and θ_1 is the value at the detachment point of the cavity boundary from the profile.

In Chapter 2, we defined an integrated stream function u_1 in the right subdomain only and still used ψ in the left domain. The numerical scheme then iterates between these two subdomains by computing u_1 and ψ alternatively. The numerical scheme shows the performance is good. However, it is hard to prove the convergence of that numerical scheme since the interface condition expressed by u_1 and ψ is complicated. Next, we propose a new scheme which uses only one function on the whole domain, then the expression of this problem seems more natural than the first scheme introduced in Chapter 2.

Again define an integrated stream function by using the Baiocchi type transformation

$$u_1(\theta,\sigma) = \frac{e^{-\sigma}}{q_c}\int_{\sigma}^{l(\theta)} e^{\tau}\psi(\theta,\tau)d\tau \tag{3.4}$$

in the region R_{u_1}. Note that $u_1 > 0$ in R_{u_1} since $\psi > 0$ there.

Let $D = \{(\theta,\sigma)|\sigma > 0, 0 \leq \theta \leq \theta_1\}$, and u_1 is defined in (3.4) for the region R_{u_1}, but in $R_{u_2} \cup \Gamma_1$, u_2 is defined as :

$$u_2(\theta,\sigma) = \frac{e^{-\sigma}}{q_c}\int_{\sigma}^{\infty} e^{\tau}\psi(\theta,\tau)d\tau \tag{3.5}$$

and u is defined in R as follows

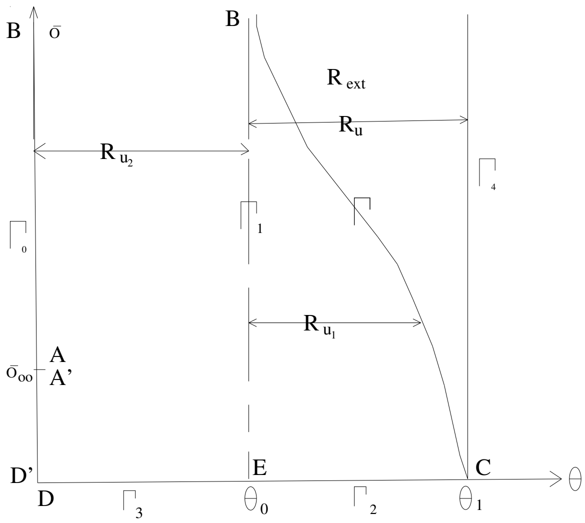

Figure 3.1: The Transformed Problem in $\theta - \sigma$ Plane Expressed by $u_1 - u_2$.

$$u = \left\{ \begin{array}{ll} u_1(\theta,\sigma), & (\theta,\sigma) \in R_{u_1} \\ u_2(\theta,\sigma), & (\theta,\sigma) \in R_{u_2} \end{array} \right\}.$$

We had proven in Chapter 2(see (2.27)) that

$$\iint_{R_{u_1}} \nabla u_1 \cdot \nabla \phi d\theta d\sigma = \iint_{R_{u_1}} [e^{-\sigma}\widetilde{R}(\theta)]\phi d\theta d\sigma. \tag{2.27}$$

Furthermore, we can easily prove $\triangle u_2(\theta,\sigma) = 0 \quad$ in R_{u_2} by using Lemma 1 and repeating the same procedure as in Chapter 2. The proof is skipped.

Lemma 1 *Suppose ψ, σ, θ have the same meaning as previously given, then*

$$\lim_{\sigma\to\infty} e^{\sigma}\psi(\theta,\sigma) = 0 \textit{ and } \lim_{\sigma\to\infty} e^{\sigma}\psi_\sigma(\theta,\sigma) = 0 \textit{ in } R_{u_2}.$$

Proof.

Let us prove the first limit. For the variable transformation $(\theta,\sigma) \to (q_1,q_2)$, we can consider $\psi(\theta,\sigma)$ and a function $\widetilde{\psi}(q_1,q_2)$. From (2.7), i.e., $q = q_c e^{-\sigma}$ where q is the scalar velocity, we have

$$\begin{aligned}
\lim_{\sigma\to\infty} e^{\sigma}\psi(\theta,\sigma) &= \lim_{q\to 0} \tfrac{q_c}{q}\widetilde{\psi}(q_1,q_2) \\
&= \lim_{q\to 0} \tfrac{q_c}{q}(\widetilde{\psi}(0,0) + \tfrac{\partial\widetilde{\psi}}{\partial q_1}(\overline{A})q_1 + \tfrac{\partial\widetilde{\psi}}{\partial q_2}(\overline{B})q_2) \\
&= \lim_{q\to 0} \tfrac{q_c}{q}(\tfrac{\partial\widetilde{\psi}}{\partial q_1}(\overline{A})q\cos(\theta) + \tfrac{\partial\widetilde{\psi}}{\partial q_2}(\overline{B})q\sin(\theta) \\
&= \lim_{q\to 0} \tfrac{\partial\widetilde{\psi}}{\partial q_1}(\overline{A})q_c\cos(\theta) + \lim_{q\to 0} \tfrac{\partial\widetilde{\psi}}{\partial q_2}(\overline{B})q_c\sin(\theta) \\
&= 0.
\end{aligned}$$

Here we have used the Taylor expansion of ψ at (0,0), the stream function $\psi(0,0) = 0$, and the fact that $\psi_x = \psi_y$ =the components of the velocity at the stagnation point B =0 gives $\widetilde{\psi}_{q_1} = \widetilde{\psi}_{q_2} = 0$ at the stagnation point B, where q_1 and q_2 are the two components of scalar velocity q. $\overline{A}$ and $\overline{B}$ denote some point close enough to (0,0) from the Taylor theorem.

Now let us prove the second limit. Since variables q and σ satisfy $\sigma = -ln\frac{q}{q_c}$, then

$$q = q_c e^{-\sigma}.$$

Therefore,

$$\frac{\partial\psi}{\partial\sigma} = \frac{\partial\widetilde{\psi}}{\partial q}\frac{\partial q}{\partial\sigma} = \frac{\partial\widetilde{\psi}}{\partial q}(-q_c e^{-\sigma}) = -q\frac{\partial\widetilde{\psi}}{\partial q},$$

then

$$\begin{aligned}
\lim_{\sigma\to\infty} e^{\sigma}\psi_\sigma(\theta,\sigma) &= \lim_{q\to 0} \tfrac{q_c}{q}(-q\tfrac{\partial\widetilde{\psi}}{\partial q}) \\
&= \lim_{q\to 0} -q_c\tfrac{\partial\widetilde{\psi}}{\partial q} \\
&= \lim_{q\to 0} -q_c(\widetilde{\psi}_{q_1}\cos(\theta) + \widetilde{\psi}_{q_2}\sin(\theta)) \\
&= -q_c(\widetilde{\psi}_{q_1}(0,0)\cos(\theta) + \widetilde{\psi}_{q_2}(0,0)\sin(\theta)).
\end{aligned}$$

Since the components of the velocity at the stagnation point B are zero, i.e.,

$$\widetilde{\psi}_{q_1}(0,0) = \widetilde{\psi}_{q_2}(0,0) = 0,$$

therefore, we have

$$\lim_{\sigma\to\infty} e^{\sigma}\psi_\sigma(\theta,\sigma) = 0.$$

Now this second approach seems more natural since the only unknown of our problem is u and on the common interface Γ_1 the connection conditions are the continuity of u and $\frac{\partial u}{\partial\theta}$. Then our original problem can be solved using the traditional non-overlapping domain decomposition method and the property of convergence can be proved in the area of DDM. In comparison, the first method uses a heterogeneous modeling using u and ψ in different regions, the interface condition is not obvious and the proof of convergence seems impossible due to the complexity of interface condition and the difference of representation of PDE's in each domain.

In fact, u_2 is simply an extension of u_1 since on Γ_1, $u_1(\theta_0,\sigma) = u_2(\theta_0,\sigma)$ and $u_{1\theta}(\theta_0,\sigma) = u_{2\theta}(\theta_0,\sigma)$. The first equality is clear from (3.4) and (3.5). The second equality can be proved as follows:

From (3.5), we have

$$u_{2\theta}(\theta,\sigma) = \frac{e^{-\sigma}}{q_c}\int_\sigma^\infty e^\tau \psi_\theta(\theta,\tau)d\tau \text{ in } R_{u_2},$$

therefore,

$$u_{2\theta}(\theta_0,\sigma) = \frac{e^{-\sigma}}{q_c}\int_\sigma^\infty e^\tau \psi_\theta(\theta_0,\tau)d\tau$$

while from (3.4)

$$\begin{aligned} u_{1\theta}(\theta,\sigma) &= \tfrac{e^{-\sigma}}{q_c}\textstyle\int_\sigma^{l(\theta)} e^\tau \psi_\theta(\theta,\tau)d\tau + \tfrac{e^{-\sigma}}{q_c}e^{l(\theta)}\psi(\theta,l(\theta)) \\ &= \tfrac{e^{-\sigma}}{q_c}\textstyle\int_\sigma^{l(\theta)} e^\tau \psi_\theta(\theta,\tau)d\tau \text{ in } R_{u_1} \end{aligned}$$

since $\psi(\theta,l(\theta)) = 0$ on Γ_1. Therefore,

$$u_{1\theta}(\theta_0,\sigma) = \frac{e^{-\sigma}}{q_c}\int_\sigma^\infty e^\tau \psi_\theta(\theta_0,\tau)d\tau.$$

Now it is clear that $u_{1\theta}(\theta_0,\sigma) = u_{2\theta}(\theta_0,\sigma)$ on Γ_1.

Therefore, the representation of our problem using u_1 and u_2 can be stated in the two non-overlapping regions as follows:

$$\triangle u_1(\theta,\sigma) = -\widetilde{R}(\theta)e^{-\sigma}\chi_{R_{u_1}} \quad \text{in } R_u \tag{3.6a}$$

$$u_1(\theta,0) + u_{1\sigma}(\theta,0) = 0, \quad \theta_0 \le \theta \le \theta_1 \tag{3.6b}$$

$$u_1(\theta_1,\sigma) = 0, \quad \sigma > 0 \tag{3.6c}$$

$$u_1(\theta,\sigma) = u_2(\theta,\sigma) \text{ on } \Gamma_1 \tag{3.6d}$$

and

$$\triangle u_2(\theta,\sigma)=0 \quad \text{in } R_{u_2} \tag{3.7a}$$

$$u_2(\theta,0)+u_{2\sigma}(\theta,0)=0, \quad 0\le\theta\le\theta_0 \tag{3.7b}$$

$$u_2(0,\sigma)=h[e^{-\sigma}-e^{-\sigma_\infty}], \quad 0\le\sigma\le\sigma_\infty; \quad u_2(0,\sigma)=0, \quad \sigma>\sigma_\infty \tag{3.7c}$$

$$u_{2\theta}(\theta,\sigma)=u_{1\theta}(\theta,\sigma) \text{ on } \Gamma_1. \tag{3.7d}$$

Expressions describing the co-ordinates of the physical plane in terms of the co-ordinates of the hodograph plane, i.e., $x=x(\theta,\sigma)$ and $y=y(\theta,\sigma)$ are needed for calculating the co-ordinates of the wake boundary. These expressions are:

$$dx=-\frac{1}{q_c}\frac{\partial\psi}{\partial\sigma}\cos\theta d\theta \quad \text{and} \quad dy=-\frac{1}{q_c}\frac{\partial\psi}{\partial\sigma}\sin\theta d\theta \text{ on } \Gamma_2\cup\Gamma_3. \tag{3.8}$$

3.3 Numerical Procedure

The problem posed in the hodograph plane by equations(3.6) and (3.7) is formulated in a region which is unbounded in the positive σ- direction. For numerical computations, the region will be truncated. Toward this end, a σ_u is chosen which is sufficiently large so that for all practical purposes the values of ψ and u_1 for $\sigma>\sigma_u$ are approximately zero. The solution algorithm is a finite difference successive over-relaxation scheme for both u_1 and u_2 with projection for the u_1-problem only. A grid of mesh points is superimposed on the bounded region, where each node is specified by row i and column j. Therefore, the field equation for u_2, (3.7a), is:

$$\begin{aligned} u_{2(i,j)}^{(n+\frac{1}{2})} &= \tfrac{\alpha\beta(\triangle\theta)^2(\triangle\sigma)^2}{2[\alpha(\triangle\theta)^2+\beta(\triangle\sigma)^2]}\big[\tfrac{2}{(1+\alpha)(\triangle\theta)^2}u_{2(i,j-1)}^{(n+1)}+\tfrac{2}{\alpha(1+\alpha)(\triangle\theta)^2}u_{2(i,j+1)}^{(n)} \\ &\quad +\tfrac{2}{(1+\beta)(\triangle\sigma)^2}u_{2(i-1,j)}^{(n+1)}+\tfrac{2}{\beta(1+\beta)(\triangle\sigma)^2}u_{2(i+1,j)}^{(n)}\big] \end{aligned} \tag{3.9a}$$

and

$$u_{2(i,j)}^{(n+1)}=u_{2(i,j)}^{(n)}+\omega(u_{2(i,j)}^{(n+\frac{1}{2})}-u_{2(i,j)}^{(n)}), \tag{3.9b}$$

where $\triangle\theta$ and $\triangle\sigma$ are the spacings in θ and σ- direction, respectively, α and β provide for unequal divisions for mesh points, ω is the over-relaxation parameter and $u_{2(i,j)}^{(n)}$

is the value of u_2 at node i, j for the nth iteration. Similarly, for u_1 in the region R_{u_1} (see equation(3.6a)):

$$\begin{aligned} u_{1(i,j)}^{(n+\frac{1}{2})} &= \tfrac{\alpha\beta(\triangle\theta)^2(\triangle\sigma)^2}{2[\alpha(\triangle\theta)^2+\beta(\triangle\sigma)^2]}[\tfrac{2}{(1+\alpha)(\triangle\theta)^2}u_{1(i,j-1)}^{(n+1)} + \tfrac{2}{\alpha(1+\alpha)(\triangle\theta)^2}u_{1(i,j+1)}^{(n)} \\ &\quad + \tfrac{2}{(1+\beta)(\triangle\sigma)^2}u_{1(i-1,j)}^{(n+1)} + \tfrac{2}{\beta(1+\beta)(\triangle\sigma)^2}u_{1(i+1,j)}^{(n)} + \widetilde{R}(\theta_j)e^{-\sigma_i}] \end{aligned} \tag{3.10a}$$

and

$$u_{1(i,j)}^{(n+1)} = max\{0, u_{1(i,j)}^{(n)} + \omega(u_{1(i,j)}^{(n+\frac{1}{2})} - u_{1(i,j)}^{(n)})\}, \tag{3.10b}$$

where $u_{1(i,j)}^{(n)}$ is the value of u_1 at node i, j for the nth iteration. These iterations are stopped when

$$\max_{i,j} |u_{2(i,j)}^{(n+1)} - u_{2(i,j)}^{(n)}| < \epsilon \quad \text{and} \quad \max_{i,j} |u_{1(i,j)}^{(n+1)} - u_{1(i,j)}^{(n)}| < \epsilon, \tag{3.11}$$

where ϵ is some fixed positive constant.

Note that on the boundary $\{(\theta,\sigma)|0 \leq \theta \leq \theta_0, \sigma = \sigma_u\}$, which corresponds to the stagnation point in the physical plane , $u_2 \approx 0$, for the region R_{u_2}. On the boundary $\{(\theta,\sigma)|\theta_0 \leq \theta \leq \theta_1, \sigma = \sigma_u\}$, $u_1 \approx 0$, which provides the appropriate boundary condition.

The iteration sequence is started by setting the known boundary conditions for u_2 for the region R_{u_2}, and a zero initial guess for the interior $u_{2(i,j)}^{(0)}$ and on Γ_1. Then using equations (3.7b). Using equations (3.9), the $u_{2(i,j)}^{(1)}$ are obtained starting in the first row from the lower right interior point and moving to the left along mesh points and upwards until all the interior points in R_u are covered. The next step is to set the boundary conditions for R_u making use of equation (3.6c) and the newly calculated values for u_2 and set a zero initial guess for the interior $u_{1(i,j)}^0$. Then using equations (3.10) starting from the upper left point in the region R_u and move to the right and downward until the entire mesh is covered and finally use equation (3.6b). This provides new values for u_1 in R_u, and hence new boundary conditions for u_2 for region R_{u_2} are set using equation (3.7d). This alternating sweeping of the two regions continues until the conditions (3.11) are satisfied. Since numerical values for u_1 inside the boundary, Γ, are nonzero and those on the boundary and inside of it are zero, the zero points bordering non-zero points in the R_u region determine this boundary.

The velocity on the boundary of the cavity, q_c, is assumed to be known but, as stated before, the upstream velocity q_∞ is, like the boundary Γ, unknown *a priori* and is to be found as part of the solution. Therefore, different values for σ_∞, where $\sigma_\infty = -ln(q_\infty/q_c)$, are used until the best one is found. The calculation sequence assumes mesh points on the boundary

$$\Gamma_0 = \{(\theta,\sigma)|\theta = 0, 0 < \sigma < \sigma_u\}$$

starting from the point with minimum σ and going upwards. For each assumed σ_∞ the alternating iteration sequence described above is performed and coordinates of the wake, using equation (2.32), are calculated.

For calculating co-ordinates of the boundary of the wake, see equations (2.38) and (2.39). Once the co-ordinates of the wake are determined, the cavity distance d_c, which is the distance between the boundary of the cavity and the wall at infinity is calculated. Then from equation (2.1) the upstream velocity q_∞ or consequently σ_∞, is calculated and is compared to the assumed value of σ_∞. The mesh point corresponding to the minimum difference between the calculated upstream velocity and assumed upstream velocities chosen for the desired value for σ_∞. It is evident that the finer the mesh points are on boundary Γ_0, the better is the accuracy in the determination of σ_∞.

3.4 Computational Results

Figure 3.2 shows results for an open profile which has the shape of an arc of a circle(radius = 1.0) and for which the free streamline leaves the profile at 180 degrees (θ_1 = 180). The profile is located between walls each having a distance h = 10.0 from the axis of symmetry for one case and h = 50.0 for another. These cases are shown in Figure 3.2 along with the results given by Lesnic et al.[15] without walls. As can be seen, the shapes and locations of the free streamlines are in appropriate agreement. The velocity on the boundary of the cavity was assumed to be q_c = 1, the over-relaxation parameter was taken to be $\omega = 1.6$, $\sigma_u = 4.0$ and the number of divisions in the θ and σ-directions were 150 and 200, respectively, with variable $\triangle\sigma$. The upstream uniform fluid velocity was computed to be $q_\infty = 0.69037$ for the case of $h = 10.0$ with $\epsilon = 1.0 \times 10^{-5}$ and $q_\infty = 0.85419$ for the case when $h = 50.0$ with $\epsilon = 1.0 \times 10^{-4}$. The calculated and assumed upstream velocities in each of the two cases were in agreement to within 0.24%.

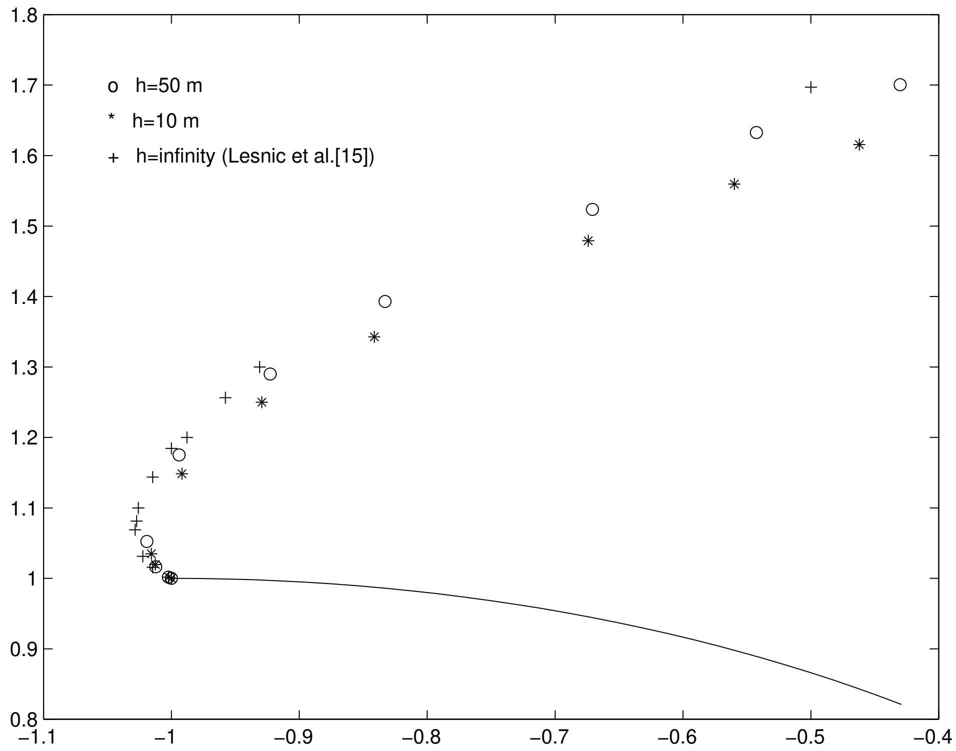

Figure 3.2: Comparison of Numerical Results

Chapter 4

A Parallel Domain Decomposition Iteration Scheme For a Heterogeneously Modeled Free Boundary Problem

4.1 Introduction

A new domain decomposition technique for the free boundary problems is considered. We propose a parallel iterative scheme that reduced the original free boundary problem to a sequence of problems on both subdomains, one of which includes the free boundary and is described by a variational inequality and the other includes the remainder of the problem and is described by a second order PDE. At each step of the iteration, we solve these two subproblems simultaneously either by using a Dirichlet condition on the interface or using a Neumann condition on the interface. Since these two subproblems can be solved simultaneously in this parallel scheme, the convergence speed is faster than the old scheme which can only iterate one subproblem at one time. Furthermore, this new parallel scheme can be extended to multi-subdomain problems very easily.

Many problems, involving free boundaries, can be reduced to the study of variational inequalities. The flow past the concave-shaped problem considered in the previous two chapters belongs to this type. There are some other problems which also involve free boundaries and will be considered in Chapter 5. In Papadopolous et al.[19], [20], the authors proposed several domain decomposition methods, trying to split the domain into two or more subdomains, one of which includes the free boundary and is described by a variational inequality and the others will be described by the PDE. Then by iterating between these subdomains we can solve the whole problem and find the free boundary. However, the above schemes solve one subproblem at one time. Now we propose a new DDM scheme which can solve these two or more subproblems simultaneously.

In this chapter, we will use this new parallel scheme to solve the model problem, i.e., flow past a concave-shaped profile. This new scheme is based on the domain decomposition scheme derived in Chapter 3.

A successive over-relaxation approach is applied over the whole problem domain with use of a projection-operation over only the fixed domain formulated part(the part containing the unknown boundary). The scheme starts by assuming Dirichlet data on the boundary between the two domains for both domains in the first iteration. Neumann data is then obtained from each domain on this boundary, averaged and used as the boundary data for the next iteration. The iterations in each domain are done in parallel on different processors and continued until the preset error criteria are satisfied. Numerical results are given for the case of a truncated circular profile. These results are again compared with other published results and are found to be in good agreement.

4.2 Formulation of the Problem

We will use u_1 and u_2 as before. The problem can be stated in the two non-overlapping regions as follows:

$$\triangle u_1(\theta, \sigma) = -\widetilde{R}(\theta) e^{-\sigma} \chi_{R_{u_1}} \quad \text{in } R_u \tag{4.1a}$$

$$u_1(\theta,0)+u_{1\sigma}(\theta,0)=0, \quad \theta_0\le\theta\le\theta_1 \tag{4.1b}$$

$$u_1(\theta_1,\sigma)=0, \quad \sigma>0 \tag{4.1c}$$

$$u_1(\theta,\sigma)=u_2(\theta,\sigma) \text{ on } \Gamma_1 \tag{4.1d}$$

and

$$\triangle u_2(\theta,\sigma)=0 \quad \text{ in } R_{u_2} \tag{4.2a}$$

$$u_2(\theta,0)+u_{2\sigma}(\theta,0)=0, \quad 0\le\theta\le\theta_0 \tag{4.2b}$$

$$u_2(0,\sigma)=h[e^{-\sigma}-e^{-\sigma_\infty}], \quad 0\le\sigma\le\sigma_\infty; \quad u_2(0,\sigma)=0, \quad \sigma>\sigma_\infty \tag{4.2c}$$

$$u_{2\theta}(\theta,\sigma)=u_{1\theta}(\theta,\sigma) \text{ on } \Gamma_1. \tag{4.2d}$$

Expressions describing the co-ordinates of the physical plane in terms of the co-ordinates of the hodograph plane, i.e., $x = x(\theta,\sigma)$ and $y = y(\theta,\sigma)$ are needed for calculating the co-ordinates of the wake boundary. These expressions are (2.32).

4.3 Numerical Procedure

The problem posed in the hodograph plane by equations (4.1) and (4.2) is formulated in a region which is unbounded in the positive σ- direction. For numerical computations, the region will be truncated. Toward this end, a σ_u is chosen which is sufficiently large so that for all practical purposes the values of u_1 and u_2 for $\sigma>\sigma_u$ are approximately zero. The solution algorithm is a finite difference successive over-relaxation scheme for both u_1 and u_2 with projection for the u_1-problem only. A grid of mesh points is superimposed on the bounded region. The new numerical iteration procedure is shown below:

1. Let $\lambda^{(0)}=0$ be given on Γ_1. We consider the two functions $u_2^{(n+\frac{1}{2})}$ and $u_1^{(n+\frac{1}{2})}$, $n\ge 0$ satisfying, respectively, the problems:

$$\triangle u_1^{(n+\frac{1}{2})} = -\widetilde{R}(\theta)e^{-\sigma}\chi_{R_{u_1^{(n+\frac{1}{2})}}} \text{ in } R_u$$

$$u_1^{(n+\frac{1}{2})}(\theta,0)+u_{1\sigma}^{(n+\frac{1}{2})}(\theta,0)=0$$

$$u_1^{(n+\frac{1}{2})}(\theta_1,\sigma)=0$$

$$u_1^{(n+\frac{1}{2})}=\lambda^{(n)} \text{ on } \Gamma_1 \tag{4.3a}$$

$$R_{u_1^{(n+\frac{1}{2})}}=\{(\theta,\sigma)|u_1^{(n+\frac{1}{2})}(\theta,\sigma)>0\}$$

and

$$\triangle u_2^{(n+\frac{1}{2})}=0 \quad \text{ in } R_{u_2}$$

$$u_2^{(n+\frac{1}{2})}(\theta,0)+u_{2\sigma}^{(n+\frac{1}{2})}(\theta,0)=0$$

$$u_2^{(n+\frac{1}{2})}(0,\sigma)= \text{ preassigned as in (4.2c)}$$

$$u_2^{(n+\frac{1}{2})}=\lambda^{(n)} \text{ on } \Gamma_1. \tag{4.3b}$$

2. Let $\mu^{(n)}=0.5*\frac{\partial u_1^{(n+\frac{1}{2})}}{\partial\theta}+0.5*\frac{\partial u_2^{(n+\frac{1}{2})}}{\partial\theta}$ on Γ_1. Then solve for $u_1^{(n+1)}$ and $u_2^{(n+1)}$ as follows:

$$\triangle u_1^{(n+1)}=-\widetilde{R}(\theta)e^{-\sigma}\chi_{R_{u_1^{(n+1)}}} \text{ in } R_u$$

$$u_1^{(n+1)}(\theta,0)+u_{1\sigma}^{(n+1)}(\theta,0)=0$$

$$u_1^{(n+1)}(\theta_1,\sigma)=0$$

$$\frac{\partial u_1^{(n+1)}}{\partial\theta}=\mu^{(n)} \quad \text{ on } \Gamma_1 \tag{4.4a}$$

$$R_{u_1^{(n+1)}}=\{(\theta,\sigma)|u_1^{(n+1)}(\theta,\sigma)>0\}$$

and

$$\triangle u_2^{(n+1)}=0 \quad \text{ in } R_{u_2}$$

$$u_2^{(n+1)}(\theta,0)+u_{2\sigma}^{(n+1)}(\theta,0)=0$$

$$u_2^{(n+1)}(0,\sigma)= \text{ preassigned as in (4.2c)}$$

$$\frac{\partial u_2^{(n+1)}}{\partial\theta}=\mu^{(n)} \text{ on } \Gamma_1. \tag{4.4b}$$

Then let $\lambda^{(n+1)}=0.5*u_1^{(n+1)}+0.5*u_2^{(n+1)}$ on Γ_1.

3. Repeat Step 1 with $n+1$ replacing n.

These iterations are stopped when

$$\max_{i,j} |u_{2(i,j)}^{(n+1)} - u_{2(i,j)}^{(n)}| < \epsilon \quad \text{and} \; \max_{i,j} |u_{1(i,j)}^{(n+1)} - u_{1(i,j)}^{(n)}| < \epsilon, \tag{4.5}$$

where ϵ is some fixed positive constant.

Note that on the boundary $\{(\theta,\sigma)|0 \leq \theta \leq \theta_0, \sigma = \sigma_u\}$, which corresponds to the stagnation point in the physical plane , $u_2 \approx 0$, for the region R_{u_2}. On the boundary $\{(\theta,\sigma)|\theta_0 \leq \theta \leq \theta_1, \sigma = \sigma_u\}$, $u_1 \approx 0$, which provides the appropriate boundary condition.

The iteration sequence is started by setting the known Dirichlet boundary conditions for u_2 in the region R_{u_2} and u_1 in the region R_u, and a zero initial guess for the interior $u_{2(i,j)}^{(0)}$ and $u_{1(i,j)}^{(0)}$. Then using equations (4.2b) and (4.3), the $u_{2(i,j)}^{(1)}$ are obtained starting in the first row from the lower right interior and moving to the left along the mesh points and upwards until all the interior in R_{u_2} are covered, while at the same time, using (4.4) $u_{1(i,j)}^{(1)}$ are obtained starting from the upper left point in R_u and move to the right and downwards in this region until the entire mesh is covered and finally use boundary condition (4.1b). The next step is to set the Neumann boundary conditions for both R_u and R_{u_2} making use of (4.2d) and taking the average of normal derivative of the newly computed u_1 and u_2. Then we can use (4.2ab) to compute u_2 and (4.4) and (4.1b) to compute u_1 simultaneously as before, but on the common boundary Γ_1, the averaged normal derivative is used. This provides new values for u_1 in R_{u_1} and for u_2 in R_{u_2}. Then we repeat the step of using a Dirichlet BC on Γ_1 and using the average of u_1 and u_2 as the boundary condition on Γ_1. Repeat alternating in this way, until conditions (4.5) are satisfied. Since numerical values for u_1 inside the boundary, Γ, are nonzero and those on the boundary and outside of it are zero, the zero points bordering non-zero points in the R_u region determine this boundary.

The velocity on the boundary of the cavity, q_c, is assumed to be known but, as stated before, the upstream velocity q_∞ is, like the boundary Γ, unknown *a priori* and is to be found as part of the solution. Therefore, different values for σ_∞, where $\sigma_\infty = -ln(q_\infty/q_c)$, are used until the best one is found. The calculation sequence assumes mesh points on the boundary

$$\Gamma_0 = \{(\theta, \sigma) | \theta = 0, 0 < \sigma < \sigma_u\},$$

starting from the point with minimum σ and going upwards. For each assumed σ_∞ the alternating iteration sequence described above is performed and coordinates of the wake, using equation (2.32), are calculated.

For calculating co-ordinates of the boundary of the wake, see equations (2.38) and (2.39). Once the co-ordinates of the wake are determined, the cavity distance d_c, which is the distance between the boundary of the cavity and the wall at infinity is calculated. Then from equation (2.1) the upstream velocity q_∞ or consequently σ_∞, is calculated and is compared to the assumed value of σ_∞. The mesh point corresponding to the minimum difference between the calculated upstream velocity and assumed upstream velocities is chosen for the desired value for σ_∞. It is evident that the finer the mesh points are on the boundary Γ_0, the better is the accuracy in the determination of σ_∞.

4.4 Computational Results

Figure 4.1 shows results for an open profile which has the shape of an arc of a circle (radius = 1.0) and for which the free streamline leaves the profile at 180 degrees ($\theta_1 = 180$). The profile is located between walls each having a distance $h = 10.0$ from the axis of symmetry for one case and $h = 50.0$ for another. These cases are shown in Figure 4.1 along with the results given by Lesnic et al.[15] without walls. As can be seen, the shapes and locations of the free streamlines are in appropriate agreement. The velocity on the boundary of the cavity was assumed to be $q_c = 1$, the over-relaxation parameter was taken to be $\omega = 1.6$, $\sigma_u = 4.0$ and the number of divisions in the θ and σ-directions were 150 and 200, respectively, with variable $\triangle\sigma$. The upstream uniform fluid velocity was computed to be $q_\infty = 0.69040$ for the case of $h = 10.0$ with $\epsilon = 2.0 \times 10^{-5}$ and $q_\infty = 0.85420$ for the case when $h = 50.0$ with $\epsilon = 1.0 \times 10^{-4}$. The calculated and assumed upstream velocities in each of the two cases were in agreement to within 0.24%.

The new parallel version in this chapter uses less time for convergence than the traditional DDM methods used in Chapter 3. The required iteration number for convergence of the new parallel method when $h = 10.0$ is 4603 while the iteration number for the traditional method in Chapter 3 is 8872. Therefore, the parallel version saved almost half of the time for convergence.

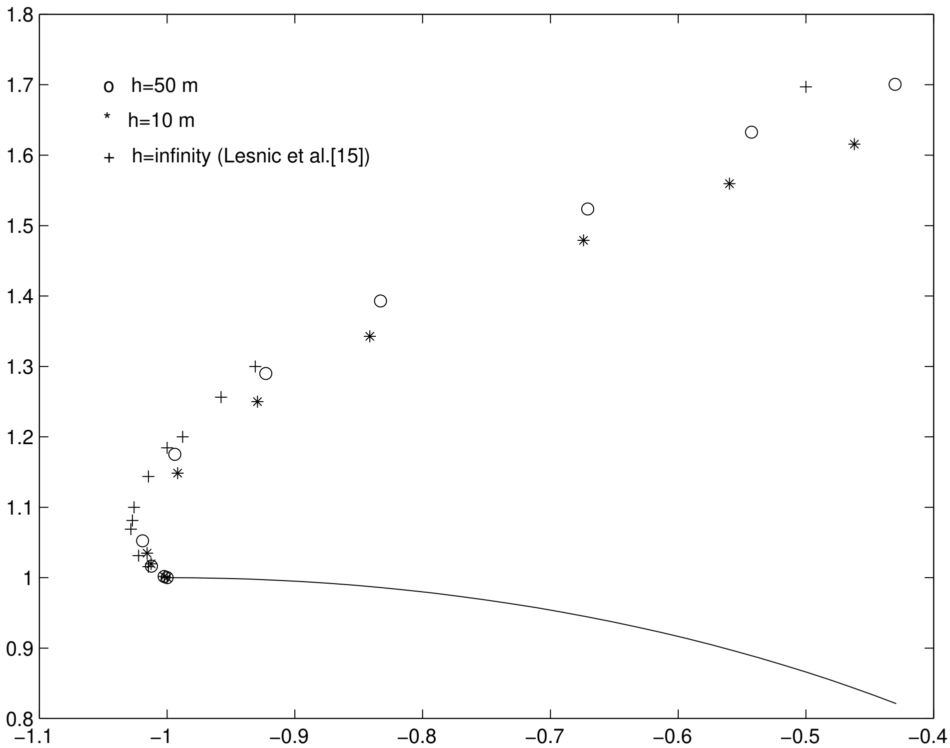

Figure 4.1: Comparison of Parallel Numerical Result with Lesnic et al.[15]

Chapter 5

Application of the New Parallel Version Scheme to Other Free Boundary Problems.

5.1 Introduction

From the last Chapter, we can see that the new parallel scheme based on non-overlapping shows good performance for the flow past a concave profile problem. In fact, we can extend this new idea to some other free boundary problems that have been considered. In this Chapter, we will reconsider two of these problems and show that the performance of the parallel version is better than the traditional DDM method.

The first example is to find the free surface in a steady, two-dimensional seepage through a rectangular dam. The second example is a free boundary seepage problem of flow through a porous dam with a toe drain.

Before we consider these problems, we propose the general idea of our parallel scheme applied to a general free boundary problem as follows:

Consider the following free boundary value problem on the open bounded connected set D in R^2:

$$\text{Find} \quad \{w, \Omega\}, \quad w(x_1, x_2) \in H^1(D), \quad \Omega \subset D \text{ such that}$$

$$Lw = f(x_1, x_2)\chi_\Omega \quad \text{in } D,$$

$$w = g \quad \text{on } \partial D$$

$$\Omega = \{(x_1, x_2) \in D :\ w(x_1, x_2) > 0\}$$

with

$$Lw = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x_1, x_2)\frac{\partial w}{\partial x_j}) + \lambda w, \quad \lambda > 0$$

$$f > 0 \text{ in } D, \quad g \geq 0 \text{ on } \partial D,$$

where $\{a_{ij}\}$ are symmetric, bounded, smooth and satisfying

$$\sum_{i,j=1}^{1} a_{ij}x_i x_j \geq M(x_1^2 + x_2^2), \quad M > 0$$

for all $x = (x_1, x_2) \in R^2$, a, b are piecewise smooth and

$$\chi_\Omega = 1 \text{ on } \Omega, \quad \chi_\Omega = 0 \text{ on } D - \Omega.$$

Examples of this arise in the filtration of a liquid through a porous dam, convex and concave profile wake problems, etc.

Before we introduce the new parallel scheme, let us take a look at an earlier scheme proposed by Papadopoulos et al.[19]. The following non-overlapping domain decomposition scheme is widely used to solve free boundary problems:

Split the domain D into two subdomains D_1 and D_2. Γ is the interface between D_1 and D_2. The free boundary problem is stated above. We iterate between these two subdomains as follows:

Choose $\phi(x) \geq 0$ on Γ. Let $\lambda^{(1)} = \phi(x)$ and for $n = 1$:

1. Solve the following Dirichlet subproblem for $\{u_1^{(n)}, \Omega_1^{(n)}\}$ in D_1:

$$\begin{aligned} Lu_1^{(n)} &= f\chi_{\Omega_1^{(n)}} \quad \text{on} \quad D_1 \\ u_1^{(n)} &= g \quad \text{on} \quad \partial D_1 - \Gamma \\ u_1^{(n)} &= \lambda^{(n)} \quad \text{on} \quad \Gamma \\ \Omega_1^{(n)} &= \{(x_1, x_2) | u_1^{(n)}(x_1, x_2) > 0\}. \end{aligned}$$

Then let $\mu^{(n)} = \frac{\partial u_1^{(n)}}{\partial n_1}$ on Γ, where n_1 is the exterior normal of Γ.

2. Solve the following Neumann subproblem u_2^n in D_2:

$$\begin{aligned} Lu_2^{(n)} &= f \quad \text{on} \quad D_2 \\ u_2^{(n)} &= g \quad \text{on} \quad \partial D_2 - \Gamma \\ \frac{\partial u_2^{(n)}}{\partial n_1} &= \mu^{(n)} \quad \text{on} \quad \Gamma. \end{aligned}$$

Then let $\lambda^{(n+1)} = \overline{\theta} * \lambda^{(n)} + (1-\overline{\theta}) * u_2^{(n)}$, where $0 < \overline{\theta} < 1$.

3. Repeat step 1 with $n+1$ replacing n. These iterations are stopped when

$$\max_{i,j} |u_{2(i,j)}^{(n+1)} - u_{2(i,j)}^{(n)}| < \epsilon \quad \text{and} \quad \max_{i,j} |u_{1(i,j)}^{(n+1)} - u_{1(i,j)}^{(n)}| < \epsilon,$$

where ϵ is some fixed positive constant.

There are some other variants of the above method which show good performance in numerical computation. However, in order to solve these problems, we must solve only one subproblem at one time while the other is waiting. With the advent of parallel computers, the demands for parallel computing are increasing. Therefore, we devised a new parallel scheme based on the above scheme as follows:

Choose $\phi(x)$ on Γ, let $\lambda^{(1)} = \phi(x)$. ($\phi(x) = 0$ is acceptable)

1. Solve the following two Dirichlet subproblems for $\{u_1^{(n+\frac{1}{2})}, \Omega_1^{(n+\frac{1}{2})}\}$ and $u_2^{(n+\frac{1}{2})}$ simultaneously:

$$\begin{aligned} Lu_1^{(n+\frac{1}{2})} &= f\chi_{\Omega_1^{(n+\frac{1}{2})}} \quad \text{in} \quad D_1 \\ u_1^{(n+\frac{1}{2})} &= g \quad \text{on } \partial D_1 - \Gamma \\ u_1^{(n+\frac{1}{2})} &= \lambda^{(n)} \quad \text{on } \Gamma \\ \Omega_1^{(n+\frac{1}{2})} & = \{(x_1,x_2) | u_1^{(n+\frac{1}{2})}(x_1,x_2) > 0\} \end{aligned}$$

and

$$\begin{aligned} Lu_2^{(n+\frac{1}{2})} &= f \quad \text{in} \quad D_2 \\ u_2^{(n+\frac{1}{2})} &= g \quad \text{on } \partial D_2 - \Gamma \\ u_2^{(n+\frac{1}{2})} &= \lambda^{(n)} \quad \text{on } \Gamma. \end{aligned}$$

Then let $\mu^{(n)} = \overline{\theta} * \frac{\partial u_1^{(n+\frac{1}{2})}}{\partial n_1} + (1-\overline{\theta}) * \frac{\partial u_2^{(n+\frac{1}{2})}}{\partial n_1}$ on Γ, where n_1 is the exterior normal on Γ.

2. Solve the following two Neumann subproblems for $\{u_1^{(n+1)}, \Omega_1^{(n+1)}\}$ and $u_2^{(n+1)}$ simultaneously:

$$\begin{aligned} Lu_1^{(n+1)} &= f\chi_{\Omega_1^{(n+1)}} \quad \text{in} \quad D_1 \\ u_1^{(n+1)} &= g \quad \text{on } \partial D_1 - \Gamma \\ \frac{\partial u_1^{(n+1)}}{\partial n_1} &= \mu^{(n)} \quad \text{on } \Gamma \\ \Omega_1^{(n+1)} &= \{(x_1,x_2)|u_1^{(n+1)}(x_1,x_2) > 0\} \end{aligned}$$

and

$$\begin{aligned} Lu_2^{(n+1)} &= f \quad \text{in} \quad D_2 \\ u_2^{(n+1)} &= g \quad \text{on } \partial D_2 - \Gamma \\ \frac{\partial u_2^{(n+1)}}{\partial n_1} &= \mu^{(n)} \quad \text{on } \Gamma. \end{aligned}$$

Then let $\lambda^{(n+1)} = \overline{\theta} * u_1^{(n+1)} + (1-\overline{\theta}) * u_2^{(n+1)} \quad$ on Γ.

3. Repeat Step 1 with $n+1$ replacing n.

These iterations are stopped when

$$\max_{i,j} |u_{2(i,j)}^{(n+1)} - u_{2(i,j)}^{(n)}| < \epsilon \quad \text{and} \quad \max_{i,j} |u_{1(i,j)}^{(n+1)} - u_{1(i,j)}^{(n)}| < \epsilon,$$

where ϵ is some fixed positive constant.

Numerical results show that the parallel scheme is better than the old one since it makes use of the parallel properties of the problem and solves these two subproblems at one time by using two processors on parallel machines. Thus the speed is increased. It is not hard to see that the new scheme can be easily extended to a problem split into more subdomains.

5.2 Example Problem 1

5.2.1 Numerical scheme

Consider the following free boundary value problem: Find the free surface in a steady, two-dimensional seepage through a rectangular dam. For simplicity, the soil in the flow field is assumed to be homogeneous and isotropic, and the capillary and evaporation effects are neglected. In addition, the flow follows Darcy's Law:

$$\overrightarrow{q} = -k\nabla h = -k\nabla[(\frac{p}{\rho g}) + y], \tag{5.1}$$

where $\overrightarrow{q}$ is the velocity vector, p is the pressure, k is the permeability of the soil, ρ is the density of the soil, g is the gravitational acceleration, y is the vertical coordinate(positive upward), and h is the piezometric head. The seepage velocity has a potential:

$$\phi = k[(\frac{p}{\rho g}) + y]. \tag{5.2}$$

In this example, the free surface, whose position is not known *a priori*, is to be found. On the free surface two boundary conditions have to be satisfied:

$$\phi = ky \tag{5.3}$$

and

$$\phi_\eta = 0, \tag{5.4}$$

where η is the outward normal direction. Either Neumann or Dirichlet data are given on the remainder of the boundaries. The location of the free surface $y = \overline{f}(x)$ and the seepage domain Ω need to be found. As shown in Figure 5.1, the seepage region is defined as:

$$\Omega = \{(x, y) : 0 < x < x_1, 0 < y < \overline{f}(x))\}, \tag{5.5}$$

where x_1 is the horizontal distance from point a to b.

In the domain Ω, setting $k = 1$ for simplicity, the following conditions must hold:

$$\begin{aligned}
\triangle\phi = &\quad 0 \quad \text{in } \Omega \\
\phi = &\quad y_1 \quad \text{on } [af] \\
\phi = &\quad y_2 \quad \text{on } [bc] \\
\phi = &\quad y \quad \text{on } [cd] \\
\phi = &\quad y \quad \text{on } \widehat{fd} \\
\phi_\eta = &\quad 0 \quad \text{on } \widehat{fd} \\
\phi_\eta \leq &\quad 0 \quad \text{on } [cd],
\end{aligned}$$

where y_1 and y_2 is the height of the water on the left and right side, respectively. The flow domain is not known *a priori* since the location of the free surface is unknown.

A new known region D is defined as:

$$D = \{(x, y) : 0 < x < x_1, 0 < y < y_1\}. \tag{5.6}$$

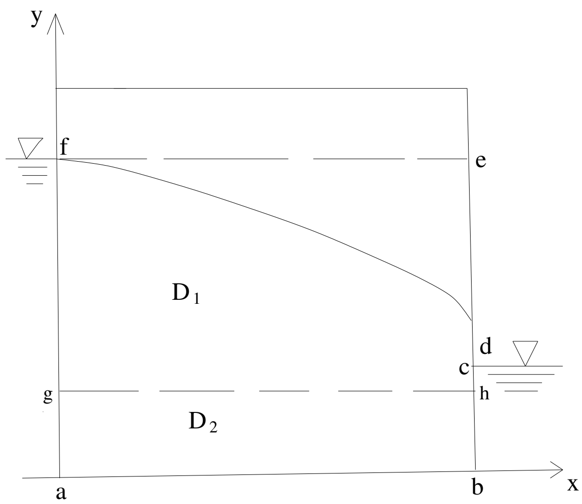

Figure 5.1: The First Example Problem.

A new variable $\overline{\phi}$ is defined:

$$\overline{\phi} = \begin{cases} \phi(x,y) & \text{in } \Omega \\ y & \text{in } \overline{D} - \Omega = \Omega_{ext}. \end{cases}$$

which extends $\phi(x,y)$ into D. It follows that

$$-\triangle\overline{\phi} = \frac{\partial(\chi_\Omega)}{\partial y}. \tag{5.7}$$

By using a Baiocchi transformation, a new dependent variable w is defined as:

$$w(x,y) = \int_y^{y_1} [\overline{\phi}(x,\overline{\eta}) - \overline{\eta}]d\overline{\eta}. \tag{5.8}$$

Then w satisfies:

$$\begin{aligned} \triangle w = &\quad \chi_\Omega \quad \text{in } D \\ w(0,y) = &\quad \tfrac{1}{2}(y_1 - y)^2 \quad \text{on } [af] \\ w(x,0) = &\quad \tfrac{y_1^2}{2} - \tfrac{y_1^2 - y_2^2}{2x_1}x \quad \text{on } [ab] \\ w(x_1,y) = &\quad \tfrac{1}{2}(y_2 - y)^2 \quad \text{on } [bc] \\ w = &\quad 0 \quad \text{in } \overline{D} - \overline{\Omega}. \end{aligned}$$

Once we obtain w satisfying the above, then

$$\begin{aligned}
\Omega = \quad & \{(x,y) : (x,y) \in D, w(x,y) > 0\} \\
\text{graph } \overline{f} = \quad & \partial\Omega - \partial D = \text{ points of } \partial\Omega \text{ not in } \partial D \\
\phi = \quad & y - w_y \text{ in } \Omega.
\end{aligned}$$

Next we use our new scheme to solve this problem. Decompose D into two non-overlapping regions D_1 and D_2 with common boundary Γ such that D_2 is the region not containing the free surface(Figure 5.1). Now consider the following scheme:

Choose $\phi(x)$ on Γ, let $\lambda^{(1)} = \phi(x)$. ($\phi(x) = 0$ is acceptable)

1. Solve for $n = 1$ the following two Dirichlet subproblems for $\{w_1^{(n+\frac{1}{2})}, \Omega_1^{(n+\frac{1}{2})}\}$ and $w_2^{(n+\frac{1}{2})}$ simultaneously:

$$\begin{aligned}
\triangle w_1^{(n+\frac{1}{2})} &= \chi_{\Omega_1^{(n+\frac{1}{2})}} \quad \text{in} \quad D_1 \\
w_1^{(n+\frac{1}{2})} &= \tfrac{1}{2}(y_1 - y)^2 \quad \text{on } [gf] \\
w_1^{(n+\frac{1}{2})} &= \tfrac{1}{2}(y_2 - y)^2 \quad \text{on } [hc] \\
w_1^{(n+\frac{1}{2})} &= 0 \quad \text{on } [fe] \text{ on } [ce] \\
w_1^{(n+\frac{1}{2})} &= \lambda^{(n)} \quad \text{on } \Gamma \\
\Omega_1^{(n+\frac{1}{2})} &= \{(x,y) | w_1^{(n+\frac{1}{2})}(x,y) > 0\}
\end{aligned}$$

and

$$\begin{aligned}
\triangle w_2^{(n+\frac{1}{2})} &= 1 \quad \text{in} \quad D_2 \\
w_2^{(n+\frac{1}{2})} &= \tfrac{1}{2}(y_1 - y)^2 \quad \text{on } [ag] \\
w_2^{(n+\frac{1}{2})} &= \tfrac{1}{2}(y_2 - y)^2 \quad \text{on } [bh] \\
w_2^{(n+\frac{1}{2})} &= \tfrac{y_1^2}{2} - \tfrac{y_1^2 - y_2^2}{2x_1} x \quad \text{on } [ab] \\
w_2^{(n+\frac{1}{2})} &= \lambda^{(n)} \quad \text{on } \Gamma.
\end{aligned}$$

Then let $\mu^{(n)} = \overline{\theta} * \frac{\partial w_1^{(n+\frac{1}{2})}}{\partial y} + (1 - \overline{\theta}) * \frac{\partial w_2^{(n+\frac{1}{2})}}{\partial y}$ on Γ.

2. Solve the following two Neumann subproblems for $\{w_1^{(n+1)}, \Omega_1^{(n+1)}\}$ and $w_2^{(n+1)}$ simultaneously:

$$\triangle w_1^{(n+1)} = \chi_{\Omega_1^{(n+1)}} \quad \text{in} \quad D_1$$

$$w_1^{(n+1)} = \frac{1}{2}(y_1 - y)^2 \quad \text{on } [gf]$$

$$w_1^{(n+1)} = \frac{1}{2}(y_2 - y)^2 \quad \text{on } [hc]$$

$$w_1^{(n+1)} = 0 \quad \text{on } [fe] \text{ and } [ce]$$

$$\frac{\partial w_1^{(n+1)}}{\partial y} = \mu^{(n)} \quad \text{on } \Gamma$$

$$\Omega_1^{(n+1)} = \{(x,y) | w_1^{(n+1)}(x,y) > 0\}$$

and

$$\begin{aligned}
\triangle w_2^{(n+1)} &= 1 \quad \text{in} \quad D_2 \\
w_2^{(n+1)} &= \tfrac{1}{2}(y_1 - y)^2 \quad \text{on } [ag] \\
w_2^{(n+1)} &= \tfrac{1}{2}(y_2 - y)^2 \quad \text{on } [bh] \\
w_2^{(n+1)} &= \tfrac{y_1^2}{2} - \tfrac{y_1^2 - y_2^2}{2x_1}x \quad \text{on } [ab] \\
\tfrac{\partial w_2^{(n+1)}}{\partial y} &= \mu^{(n)} \quad \text{on } \Gamma.
\end{aligned}$$

Then let $\lambda^{(n+1)} = \overline{\theta} * w_1^{(n+1)} + (1 - \overline{\theta}) * w_2^{(n+1)} \quad \text{on } \Gamma.$

3. Repeat Step 1 with $n+1$ replacing n.

These iterations are stopped when

$$\max_{i,j} |w_{2(i,j)}^{(n+1)} - w_{2(i,j)}^{(n)}| < \epsilon \quad \text{and } \max_{i,j} |w_{1(i,j)}^{(n+1)} - w_{1(i,j)}^{(n)}| < \epsilon,$$

where ϵ is some fixed positive constant.

5.2.2 Discretization and results

This numerical example uses the following data: $y_1 = 20.0$, $y_2 = 10.0$, $x_1 = 15.0$, $\omega = 1.85$, $\epsilon = 0.001$, $\triangle x = \triangle y = 0.3333$ and $\overline{\theta} = 0.5$. $D = \{(x,y) : 0 < x < 15.0,\ 0 < y < 20.0\}$ is subdivided as shown with $D_1 = \{(x,y) : 0 < x < 15.0,\ 10.0 < y < 20.0\}$,$D_2 = \{(x,y) : 0 < x < 15.0,\ 0 < y < 10.0\}$. To determine a point (x_0, y_0) of the free surface, choose the smallest y_0 so that $0 < w(x_0, y_0) < 0.01$. The numerical scheme proceeds by solving the two Dirichlet subproblems simultaneously using 2 CPU. These two solutions produce input data for the two Neumann subproblems which we solve simultaneously using the same 2 CPU. The process continues until convergence.

The new parallel version converged in 4 loops with total number of iterations being 368, while the old version needed 8 loops to converge with the total number of iterations being 4639. Most of the iterations occurred in the first 2 loops for the old method. The final error of the old method is 7×10^{-4} and the final error of the new method is 9×10^{-4}. The new parallel method reduces the number of iterations by running the program on two CPU.

5.3 Example Problem 2

5.3.1 Numerical scheme

The second example is a free boundary seepage problem of flow through a porous dam with a toe drain. For simplicity, the soil in the flow field is again assumed to be homogeneous and isotropic, and the capillary and evaporation effects are neglected. In addition, the flow follows Darcy's law:

$$\overrightarrow{q} = -k\nabla h = -k\nabla[(\frac{p}{\rho g}) + y], \tag{5.9}$$

where $\overrightarrow{q}$ is the velocity vector, p is the pressure, k is the permeability of the soil, ρ is the density of the fluid, g is the gravitational acceleration, y is the vertical coordinate(positive upward), and h is the piezometric head. For homogeneous, isotropic soil, the permeability k is constant. The seepage velocity has a potential:

$$\phi = k[(\frac{p}{\rho g}) + y]. \tag{5.10}$$

In this study, the free surface, whose position is not known *a priori*, is to be found. On the free surface, the boundary condition is

$$\phi = ky \tag{5.11}$$

while on the other boundaries, either Neumann or Dirichlet data are given. The location of the free surface $\Gamma_1 = \{x, \overline{f}(x)\}$y and the seepage domain Ω need to be found, see Figure 5.2. The seepage region is defined as:

$$\Omega = \{(x,y) : 0 < x \leq x_F,\ 0 < y < \alpha(x); \quad x_F < x < x_C,\ 0 < y < \overline{f}(x))\}, \tag{5.12}$$

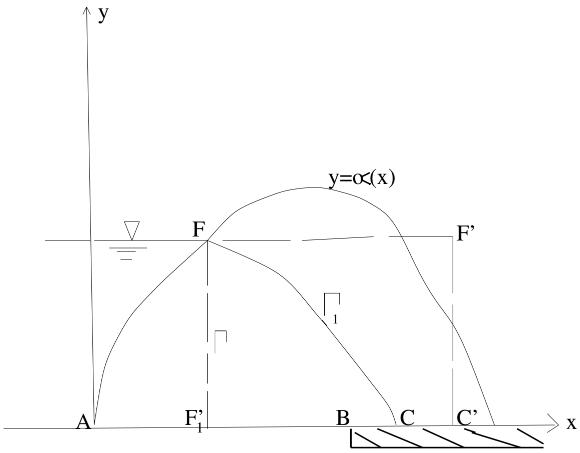

Figure 5.2: The Second Example Problem.

where $\alpha(x)$ is the shape function of the dam profile.

The functions $\phi(x, y)$ and $\psi(x, y)$ are defined on $\overline{\Omega}$ and are to be in $H^1(\Omega) \cap C^0(\overline{\Omega})$. Further, for the problem shown in Figure 5.2:

$$\Omega = \{(x, y) : 0 < x \leq x_{F_1'}, \quad 0 < y < \alpha(x); \quad x_{F_1'} < x < x_C, \quad 0 < y < \overline{f}(x))\}$$

$$\phi_x - \psi_y = 0 \text{ in } \Omega$$

$$\phi_y + \psi_x = 0 \text{ in } \Omega$$

$$\phi = y_F \text{ on } \widehat{AF}$$

$$\phi = 0 \text{ on } [BC]$$

$$\psi = q \text{ on } [AB]$$

$$\psi = 0 \text{ on } \Gamma_1$$

$$\phi = y \text{ on } \Gamma_1, \tag{5.13}$$

where y_F is the height at F, $\alpha(x)$ is the shape function of the dam profile, and q is the flow rate through the flowfield. Let the solution domain Ω be extended to the known region $D = \{(x, y) : 0 < x \leq x_{F_1'}, \quad 0 < y < \alpha(x); \quad x_{F_1'} < x < x_{C'}, 0 < y < y_F\}$ in Figure 5.2. Then extend ϕ and ψ continuously to be defined on $\overline{D}$ by setting

$$\overline{\phi}(x,y) = \begin{cases} \phi(x,y) & \text{in } \overline{\Omega} \\ y & \text{in } \overline{D} - \overline{\Omega} \end{cases}$$

and

$$\overline{\psi}(x,y) = \begin{cases} \psi(x,y) & \text{in } \overline{\Omega} \\ 0 & \text{in } \overline{D} - \overline{\Omega}. \end{cases}$$

This yields

$$\begin{aligned} \overline{\phi}_x - \overline{\psi}_y = &\ \ 0 \text{ in } D \\ \overline{\phi}_y + \overline{\psi}_x = &\ \ \chi_{D-\overline{\Omega}} \text{ in } D \end{aligned} \tag{5.14}$$

in the sense of distributions where $\chi_{D-\overline{\Omega}} = 1$ in $D - \overline{\Omega}$, $\chi_{D-\overline{\Omega}} = 0$ in Ω.

Next define a new dependent variable w using the Baiocchi transformation

$$w(P) = \int_{\overline{FP}} -\overline{\psi}dx + (y - \overline{\phi})dy, \tag{5.15}$$

where $\overline{FP}$ is a smooth path in D joining F to P in D in Figure 5.2. The integration is independent of the path due to (5.14). Then for all w in $H^2(D) \bigcap C^1(\overline{D})$ (See [3]):

$$\triangle w = \chi_\Omega \quad \text{in D} \tag{5.16}$$

$$\begin{gathered} w_y = y - y_F \text{ on } \widehat{AF} \\ w = (\frac{q^2}{6}) + q(x_B - x) \text{ on } [AB] \\ w_y = 0 \text{ on [BC]} \\ w = 0 \text{ in } \overline{D} - \overline{\Omega} (\text{ also on } \Gamma_1) \\ w > 0 \text{ in } \Omega \quad (w \geq 0 \text{ in D}). \end{gathered} \tag{5.17}$$

Hence

$$w(x,y) \geq 0, \quad 1 - \triangle(x,y) \geq 0, \quad w(1 - \triangle w) = 0 \text{ in D}.$$

If w is found satisfying (5.16) subject to conditions (5.17), then the following quantities can be obtained:

$$\begin{aligned} \Omega = &\quad \{(x,y) : (x,y) \text{ in } D, \quad w > 0\} \\ \text{graph } \overline{f} = &\quad \partial\Omega - \partial D = \text{ points of } \partial\Omega \text{ not in } \partial D \\ \phi = &\quad y - w_y \text{ in } \Omega \\ \psi = &\quad -w_x \text{ in } \Omega \\ q = &\quad \psi(x,0) \text{ on } [AB]. \end{aligned} \tag{5.18}$$

Next we use our new scheme to solve this problem. First, decompose D into subsets $D_1 = \{(x,y) : 0 < x < x_{F_1'}, 0 < y < \alpha(x)\}$ and $D_2 = \{(x,y) : x_{F_1'} < x < x_{C'}, 0 < y < y_F\}$ with the boundary of D denoted by ∂D and the interface between D_1 and D_2 denoted by $\Gamma = \{(x,y) : x = x_F, 0 < y < y_F\}$. If w_1 denotes the restriction of w to D_1 and w_2 the one to D_2, then we can write down the following iterative procedure:

1. Let $\lambda^{(1)}$ be given on Γ. We solve the two Dirichlet subproblems for $w_1^{(n+\frac{1}{2})}$ and $\{w_2^{(n+\frac{1}{2})}, \Omega_2^{(n+\frac{1}{2})}\}$, $n \geq 1$ respectively as follows:

$$\begin{aligned}
\triangle w_2^{(n+\frac{1}{2})} &= \chi_{\Omega_2^{(n+\frac{1}{2})}} \quad \text{in } D_2 \\
w_2^{(n+\frac{1}{2})} &= (\tfrac{q^2}{6}) + q(x_B - x) \text{ on } [F_1'B] \\
(w_2^{(n+\frac{1}{2})})_y &= 0 \text{ on } [BC'] \\
w_2^{(n+\frac{1}{2})} &= 0 \text{ in } \overline{D_2} - \Omega \\
w_2^{(n+\frac{1}{2})} &= \lambda^{(n)} \text{ on } \Gamma \\
\Omega_1^{(n+\frac{1}{2})} &= \{(x,y) | w_2^{(n+\frac{1}{2})}(x,y) > 0\}
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
\triangle w_1^{(n+\frac{1}{2})} &= 1 \quad \text{in } D_1 \\
w_1^{(n+\frac{1}{2})} &= (\tfrac{q^2}{6}) + q(x_B - x) \text{ on } [AF_1'] \\
(w_1^{(n+\frac{1}{2})})_y &= y - y_F \text{ on } \widehat{AF} \\
w_1^{(n+\frac{1}{2})} &= \lambda^{(n)} \text{ on } \Gamma.
\end{aligned} \tag{5.20}$$

2. Let $\mu^{(n)} = \overline{\theta} * \frac{\partial w_1^{(n+\frac{1}{2})}}{\partial x} + (1-\overline{\theta}) * \frac{\partial w_2^{(n+\frac{1}{2})}}{\partial x}$ on Γ. Then solve two Neumann subproblems for $\{w_2^{(n+1)}, \Omega_2^{(n+1)}\}$ and $w_1^{(n+1)}$ respectively as follows:

$$\begin{aligned}
\triangle w_2^{(n+1)} &= \chi_{\Omega_2^{(n+1)}} \quad \text{in } D_2 \\
w_2^{(n+1)} &= (\tfrac{q^2}{6}) + q(x_B - x) \text{ on } [F_1'B] \\
(w_2^{(n+1)})_y &= 0 \text{ on } [BC'] \\
w_2^{(n+1)} &= 0 \text{ in } \overline{D_2} - \Omega \\
\frac{\partial w_2^{(n+1)}}{\partial x} &= \mu^{(n)} \quad \text{on } \Gamma \\
\Omega_2^{(n+1)} &= \{(x,y) | w_2^{(n+1)}(x,y) > 0\}
\end{aligned} \tag{5.21}$$

and

$$\begin{aligned} \triangle w_1^{(n+1)} &= \quad 1 \quad \text{in } D_1 \\ w_1^{(n+1)} &= \quad (\tfrac{q^2}{6}) + q(x_B - x) \text{ on } [AF_1'] \\ (w_1^{(n+1)})_y &= \quad y - y_F \text{ on } \widehat{AF} \\ \frac{\partial w_1^{(n+1)}}{\partial x} &= \quad \mu^{(n)} \quad \text{on } \Gamma. \end{aligned} \tag{5.22}$$

Then let $\lambda^{(n+1)} = \overline{\theta} * w_1^{(n+1)} + (1 - \overline{\theta}) * w_2^{(n+1)} \quad$ on Γ.

3. Repeat Step 1 with $n+1$ replacing n.

These iterations are stopped when

$$\max_{i,j} |w_{2(i,j)}^{(n+1)} - w_{2(i,j)}^{(n)}| < \epsilon \quad \text{and } \max_{i,j} |w_{1(i,j)}^{(n+1)} - w_{1(i,j)}^{(n)}| < \epsilon,$$

where ϵ is some fixed positive constant.

5.3.2 Discretization and results

The flow rate through the flow field, Ω is also unknown *a priori*. Thus, in addition to the inner iteration to solve for w with a given q, there is also an outer iteration on the q to determine the flow rate. A compatibility condition is necessary for the outer iteration. The condition used herein is similar to that given by Sloss and Bruch[3], i.e.,

$$f_h(q^{(r)}) = (w_2(x_F, y_F - \triangle y))_{q^{(r)}} - \frac{\triangle y^2}{2}, \quad r = 0, 1, 2, \ldots \tag{5.23}$$

Then $f_h(q^{(r)}) = 0$ represents a compatibility condition, which if imposed on the set of solutions $(w_1)_{q,h}$ and $(w_2)_{q,h}$, permits the determination of a unique $\overline{q}$ such that $(w_1)_{\overline{q},h}$ and $(w_2)_{\overline{q},h}$ will be the a solution of (5.19), (5.20), (5.21), (5.22).

The numerical example has the following data: $\alpha(x) = x \quad$ where $0 < x < x_{F_1'}$, $y_F = 30$ft, $x_F = 30$ft, $x_B = 60$ft, $\overline{\theta} = 0.5$, $\triangle x = \triangle y = 2.5$ft, with stopping error estimates:

$$\max_{i,j} |(w_1)_{q^{(r)},i,j}^{(n+1)} - (w_1)_{q^{(r)},i,j}^{(n)}| < \epsilon_1$$

$$\max_{i,j} |(w_2)_{q^{(r)},i,j}^{(n+1)} - (w_2)_{q^{(r)},i,j}^{(n)}| < \epsilon_1$$

and

$$|f_h(q^{(r)})| < \epsilon_2,$$

where ϵ_1 and ϵ_2 are preset constants.

The iteration approach presented here is that: the Dirichlet subproblems in both regions D_1 and D_2 are solved, then using the average of the normal derivatives on the interface as input, solve the Neumann subproblems in D_1 and D_2 at the same time. This forms one step. Then using the average of the function values on the interface as input, solve the Dirichlet subproblems in both regions as before. The error was checked after each step, if it meets the criteria, then stop, otherwise move on to next step.

The old method took 120 steps of iteration before reaching the error criteria, while the new method only needed 88 steps of iteration to satisfy the error criteria. The error criteria for these two methods was the same, however, the difference between the necessary number of iterations shows the advantage of the parallel algorithm.

Chapter 6

Convergence Analysis of DDM Schemes for the Concave Profile Flow Problem

6.1 Introduction

In this chapter we should consider the convergence and existence analysis of the model problem considered in Chapter 2. The problem of flow past a concave shaped profile can be transformed into a variational inequality problem on the whole domain with mixed boundary conditions. In the following sections, we shall prove the existence and uniqueness of the solution to our problem as a variational inequality and then prove the convergence of our numerical solution using the DDM scheme by assuming some convergence property on the common interface of two subdomains. The approached used in this Chapter are similar to Bourgat and Duvaut[2] and Papadopolous[21].

6.2 Existence and Uniqueness of Solution to Model Problem

Suppose $D = \{(\theta,\sigma)|\sigma > 0, 0 \le \theta \le \theta_1\}$, and u is defined

$$u = \left\{ \begin{array}{ll} u_1(\theta,\sigma) = \frac{e^{-\sigma}}{q_c}\int_\sigma^{l(\theta)} e^\tau \psi(\theta,\tau)d\tau, & \theta_0 \le \theta \le \theta_1 \\ u_2(\theta,\sigma) = \frac{e^{-\sigma}}{q_c}\int_\sigma^{\infty} e^\tau \psi(\theta,\tau)d\tau, & 0 \le \theta \le \theta_0 \end{array} \right\}. \tag{6.1}$$

Then as in (2.28), we have

$$\triangle u = \frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial\sigma^2} = -\widetilde{R}(\theta)e^{-\sigma}\chi_{R_{u_1}} \quad \text{in } D, \tag{6.2}$$

where $\widetilde{R}(\theta) = 0, \quad 0 \le \theta \le \theta_0; \quad \widetilde{R}(\theta)$ is defined as before, $\theta_0 \le \theta \le \theta_1$, and u satisfies the boundary conditions(see Figure 3.1)

$$\begin{array}{ll} u + u_\sigma = 0 & \text{on } \Gamma_2 \cup \Gamma_3 \\ u = 0 & \text{on } \Gamma_4 \\ u = h(\sigma) & \text{on } \Gamma_0 \\ u(\theta,\infty) = 0, & \end{array} \tag{6.3}$$

where $h(\sigma) = h(e^{-\sigma} - e^{-\sigma_\infty}), \quad 0 \le \sigma \le \sigma_\infty; \quad h(\sigma) = 0, \quad \sigma \ge \sigma_\infty$. If we can prove there exists a unique $u(\theta,\sigma)$ satisfying (6.2) and (6.3), then we can find the stream function in R as

$$\psi = -q_c(u + u_\sigma). \tag{6.4}$$

Let us introduce the function space V :

$$V = \{w|w \in H^1(D), w|_{\Gamma_4} = 0, w(\theta,\infty) = 0\} \tag{6.6}$$

which is a Hilbert space when it is supplied with the norm

$$||w|| = \int_D \nabla w \cdot \nabla w d\theta d\sigma + \int_{\theta=0}^{\theta_1} w^2 d\theta. \tag{6.7}$$

Let the closed convex K_H contained in V be given by

$$K_H = \{w|w \in V, \quad w \ge 0, \quad w|_{\Gamma_0} = h(\sigma)\}.$$

Furthermore, let

$$a(u,w) = \int_D \nabla u \cdot \nabla w d\theta d\sigma + \int_{\Gamma_2 \cup \Gamma_3} uwd\theta. \qquad (6.8)$$

If $u \in H^1(D)$ and satisfies (6.3), we have

$$u \in K_H.$$

Notice that R is defined in Chapter 2 as $R = R_{u_1} \cup R_{u_2} \cup \Gamma_1$. For any $v \in K_H$, we have

$$\begin{aligned} \textstyle\int_R -(\triangle u)(v-u)d\theta d\sigma &= \textstyle\int_R \widetilde{R}(\theta)e^{-\sigma}(v-u)d\theta d\sigma \\ &\geq \textstyle\int_D \widetilde{R}(\theta)e^{-\sigma}(v-u)d\theta d\sigma \end{aligned} \qquad (6.9)$$

since $\widetilde{R}(\theta) < 0$, $u = 0$ on $D - R$ and $v \geq 0$ on $D - R$. Green's formula gives

$$\begin{aligned} \textstyle\int_R -(\triangle u)(v-u)d\theta d\sigma &= \textstyle\int_{\partial R} -\frac{\partial u}{\partial n}(v-u)d\Gamma + \int_R \nabla u \cdot \nabla(v-u)d\theta d\sigma \\ &= \textstyle\int_R \nabla u \cdot \nabla(v-u)d\theta d\sigma + \int_{\Gamma_2 \cup \Gamma_3} u(v-u)d\theta, \end{aligned}$$

where n is the exterior normal of ∂R. Then (6.9) becomes

$$a(u, v-u) \geq \int_D \widetilde{R}(\theta)e^{-\sigma}(v-u)d\theta d\sigma, \quad \forall v \in K_H. \qquad (6.10)$$

Lemma 2 *If the mapping $\theta \to \widetilde{R}(\theta)$ is square integrable on $(0, \theta_1)$, then*

$$v \to \int_D \widetilde{R}(\theta)e^{-\sigma} v d\theta d\sigma$$

is a continuous linear form on V.

Proof. Note that

$$\left|\int_D \widetilde{R}(\theta)e^{-\sigma} v d\theta d\sigma\right| \leq c\left(\int_D v^2 d\theta d\sigma\right)^{\frac{1}{2}}$$

since $(\theta, \sigma) \to \widetilde{R}(\theta)e^{-\sigma}$ is square integrable from the assumption. From $v|_{\Gamma_4} = 0$, we have

$$v(\theta, \sigma) = -\int_\theta^{\theta_1} \frac{\partial v}{\partial \theta'}(\theta', \sigma)d\theta'$$

which implies

$$|v(\theta, \sigma)| \leq |\theta_1 - \theta|^{\frac{1}{2}}\left(\int_\theta^{\theta_1} \left(\frac{\partial v}{\partial \theta'}\right)^2 d\theta'\right)^{\frac{1}{2}}.$$

Consequently,

$$\begin{aligned} \int_D v^2(\theta,\sigma)d\theta d\sigma &\le \int_D [\theta_1 \int_0^{\theta_1} (\tfrac{\partial v}{\partial \eta})^2 d\eta] d\theta d\sigma \\ &= \theta_1^2 \int_0^{\infty} \int_0^{\theta_1} (\tfrac{\partial v}{\partial \theta})^2 d\theta d\sigma \\ &\le \theta_1^2 \int_D \nabla v \cdot \nabla v d\theta d\sigma \end{aligned}$$

which proves Lemma 2.

Theorem 1 *Under the assumptions in Lemma 2, there exists a unique u satisfying (6.10).*

Proof. The bilinear form $a(u,v)$ is coercive. Further it is continuous. From Lemma 2, we know the second member of (6.9) is a linear continuous form on V. From the classical existence theorem for variational inequality(Kinderlehrer et al.[14]), the result follows.

Theorem 2 *The wake boundary $(x(\theta), y(\theta))$ is given by*

$$x(\theta) = x_0 + \int_0^{\theta} (u_\sigma(\eta,0) + u_{\sigma\sigma}(\eta,0)) \cos \eta d\eta$$

and

$$y(\theta) = y_0 + \int_0^{\theta} (u_\sigma(\eta,0) + u_{\sigma\sigma}(\eta,0)) \sin \eta d\eta.$$

Proof. Derived directly from (3.8) in Chapter 3 and the following fact:

$$\psi = -q_c(u + u_\sigma) \text{ in } R.$$

6.3 Convergence of Numerical Scheme

6.3.1 Variational form of domain decomposition scheme

From the numerical results in Chapter 2, we can see the convergence of $\psi^{(n)}$ and $u^{(n)}$ towards the exact physical solution ψ and u. Also in Chapter 3, we see the convergence of $u_1^{(n)}$ and $u_2^{(n)}$ towards the exact physical solution u. However, it is very difficult to prove the convergence of the numerical solution. The very complicated boundary condition on the common boundary Γ_1 of the two subdomains R_{u_1} and R_{u_2}

makes the convergence analysis even much difficult. How to prove the convergence of the approximate solutions when the complicated common boundary conditions are present is our objective. Note that the numerical approach for determining $\psi^{(n)}$ and $u^{(n)}$ taken in Chapter 2 is equivalent to the approach for determining $u_1^{(n)}$ and $u_2^{(n)}$ taken in Chapter 3. However in the second approach the heterogeneous model was used, that is, u_1 and u_2 are considered to be C^1 functions, since the equality of their function value and their normal derivative on the common interface must hold. Therefore, the second approach can handle the same function in both regions and on the common boundary, while the first approach has no such advantage. In the following, we shall try to prove the convergence of our u_1-u_2 approach towards the true solution described in Section 6.2.

When we take iterations on both regions consecutively, we have the following numerical problem:

Given $u_2^{(n-1)}$ on Γ_1, solve for $u_1^{(n)}$, $R_{u_1^{(n)}}$ in region $D_1 = R_u$:

$$\begin{aligned} \triangle u_1^{(n)} &= -\widetilde{R}(\theta)e^{-\sigma}\chi_{R_{u_1^{(n)}}} \text{ in } D_1 \\ u_1^{(n)}(\theta,0) + u_{1\sigma}^{(n)}(\theta,0) &= 0 \text{ on } \Gamma_3 \\ u_1^{(n)}(\theta_1,\sigma) &= 0 \\ u_1^{(n)} &= u_2^{(n-1)} \text{ on } \Gamma_1 \\ R_{u_1^{(n)}} &= \{(\theta,\sigma) | u_1^{(n)}(\theta,\sigma) > 0\}. \end{aligned} \tag{6.11}$$

Given $\frac{\partial u_1^{(n)}}{\partial \theta}$ on Γ_1, solve for $u_2^{(n)}$ in region $D_2 = R_{u_2}$:

$$\begin{aligned} \triangle u_2^{(n)} &= 0 \quad \text{ in } D_2 \\ u_2^{(n)}(\theta,0) + u_{2\sigma}^{(n)}(\theta,0) &= 0 \text{ on } \Gamma_2 \\ u_2^{(n)}(0,\sigma) &= h[e^{-\sigma} - e^{-\sigma_\infty}] \quad 0 \le \sigma \le \sigma_\infty \\ u_2^{(n)}(0,\sigma) &= 0 \quad \sigma > \sigma_\infty \\ \frac{\partial u_2^{(n)}}{\partial \theta} &= \frac{\partial u_1^{(n)}}{\partial \theta} \text{ on } \Gamma_1. \end{aligned} \tag{6.12}$$

Now, we consider the variational approach for the above two subproblems:

Subproblem 1

Let $g \in H^{\frac{1}{2}}(\Gamma_1)$, $g > 0$ on Γ_1, and define

$$\begin{aligned} U^{(1)}(g) = \quad & \{v : v \in H^1(D_1), \quad v \geq 0 \text{ in } D_1, \quad v(\theta, \infty) = 0, \\ & v(\theta_1, \sigma) = 0, \quad \gamma_0 v = g \text{ on } H^{\frac{1}{2}}(\Gamma_1)\}, \end{aligned} \tag{6.13}$$

where $\gamma_0 : H^1(D_1) \rightarrow H^{\frac{1}{2}}(\Gamma_1)$ is the trace function.

Now we can define the strong and variational forms of the free boundary subproblem in region D_1.

Strong Form:

Find $\{u_1, \Omega_1\}$ such that $u_1 \in U^{(1)}(g) \cap H^2(D_1)$, $\quad \Omega_1 = \{(\theta, \sigma) \in D_1 : u_1(\theta, \sigma) > 0\}$, and

$$-\triangle u_1 = f \chi_{\Omega_1} \text{ in } D_1 \tag{6.14a}$$

$$u_1 + \frac{\partial u_1}{\partial \sigma} = 0 \text{ on } \Gamma_2. \tag{6.14b}$$

Variational Form:

Find $\{u_1, \Omega_1\}$ such that $u_1 \in U^{(1)}(g)$, $\quad \Omega_1 = \{(\theta, \sigma) \in D_1 : u_1(\theta, \sigma) > 0\}$, and

$$a_1(u_1, v - u_1) \geq < f, v - u_1 >_1 \quad \forall v \in U^{(1)}(g), \tag{6.15}$$

where

$$< f, v >_k = \int_{D_k} f v d\theta d\sigma, \quad k = 1, 2,$$

$$a_1(u, v) = \int_{D_1} \nabla u \cdot \nabla v d\theta d\sigma + \int_{\Gamma_2} u v d\theta.$$

It is known that equations (6.14) and variational inequality (6.15) are equivalent.

Subproblem 2

Let

$$U^{(2)} = \{v : v \in H^1(D_2), \quad v(\theta, \infty) = 0, \quad v = l(\sigma) \text{ on } \Gamma_0\},$$

where

$$l(\sigma) = h[e^{-\sigma} - e^{-\sigma_\infty}], \quad 0 \leq \sigma \leq \sigma_\infty; \quad l(\sigma) = 0, \quad \sigma > \sigma_\infty.$$

The following formulations are the strong and variational forms of the second subproblem.

Strong Form:

Let $h \in H^2(D_1)$. Find $u_2 \in U^{(2)} \cap H^2(D_2)$ such that

$$-\triangle u_2 = f \text{ in } D_2 \tag{6.16a}$$

$$\frac{\partial u_2}{\partial \theta} = \frac{\partial h}{\partial \theta} \in H^{\frac{1}{2}}(\Gamma_1) \tag{6.16b}$$

$$u_2 + \frac{\partial u_2}{\partial \sigma} = 0 \text{ on } \Gamma_3. \tag{6.16c}$$

Variational Form 1:

Let $h \in H^2(D_1)$. Find $u_2 \in U^{(2)}$ such that

$$a_2(u_2, v - u_2) =< f, v - u_2 >_2 + \int_{\Gamma_1} (v - u_2)\frac{\partial h}{\partial \theta} d\sigma \quad \forall v \in U^{(2)}, \tag{6.17}$$

where

$$a_2(u, v) = \int_{D_2} \nabla u \cdot \nabla v d\theta d\sigma + \int_{\Gamma_3} uv d\theta.$$

The equivalence of the strong form equations (6.16) and variational form 1 (6.17) of the above subproblem 2 can be shown as follows. Suppose (6.16) is true, then for any $v \in U^{(2)}$,

$$\begin{aligned} -\textstyle\int_{D_2} \triangle u_2(v - u_2) d\theta d\sigma &= \textstyle\int_{D_2} \nabla u_2 \cdot \nabla(v - u_2) d\theta d\sigma - \int_{\partial D_2} (v - u_2)\frac{\partial u_2}{\partial n} ds \\ &= a_2(u_2, v - u_2) - \textstyle\int_{\Gamma_1} (v - u_2)\frac{\partial u_2}{\partial \theta} d\sigma \\ &= a_2(u_2, v - u_2) - \textstyle\int_{\Gamma_1} (v - u_2)\frac{\partial h}{\partial \theta} d\sigma, \end{aligned}$$

i.e.,

$$\begin{aligned} a_2(u_2, v - u_2) &= \textstyle\int_{D_2} (-\triangle u_2)(v - u_2) d\theta d\sigma + \int_{\Gamma_1} (v - u_2)\frac{\partial h}{\partial \theta} d\sigma \\ &= \textstyle\int_{D_2} f(v - u_2) d\theta d\sigma + \int_{\Gamma_1} (v - u_2)\frac{\partial h}{\partial \theta} d\sigma \end{aligned}$$

which is just (6.17). Notice that the above procedure is invertible, we can also prove (6.16) assuming (6.17) is true. Therefore the equivalence between (6.16) and (6.17) is proved.

Furthermore, if $h \in H^2(D_1)$ satisfies

$$-\triangle h = f\chi_{\Omega_1} \text{ in } D_1 \tag{6.18a}$$

$$h + \frac{\partial h}{\partial \sigma} = 0 \text{ on } \Gamma_2, \quad h(\theta, \infty) = 0, \quad h(\theta_1, \sigma) = 0, \tag{6.18b}$$

where

$$\Omega_1 = \{(\theta, \sigma) \in D_1 : h(\theta, \sigma) > 0\},$$

then the above two equivalent forms are also equivalent to the following variational form:

Variational Form 2:

Let $h \in H^2(D_1)$ satisfies (6.18). Find $u_2 \in U^{(2)}$ such that

$$\begin{aligned} a_2(u_2, v - u_2) = \quad & < f, v - u_2 >_2 - a_1(h, R_1\gamma_0(v - u_2)) \\ & + \int_{\Omega_1} f R_1 \gamma_0 (v - u_2) d\theta d\sigma \quad \forall v \in U^{(2)}, \end{aligned} \tag{6.19}$$

where $R_1(g)$ satisfies for any $g \in H^{\frac{1}{2}}(\Gamma_1)$,

$$\triangle R_1 g = 0 \text{ in } D_1 \tag{6.20a}$$

$$R_1 g = 0 \text{ on } \Gamma_4; \quad R_1 g(\theta, \infty) = 0; \quad R_1 g + \frac{\partial R_1 g}{\partial \sigma} = 0 \text{ on } \Gamma_2; \quad R_1 g = g \text{ on } \Gamma_1. \tag{6.20b}$$

Let us prove the equivalence between (6.17) and (6.19). Since h is a solution of the free boundary problem 1, i.e., (6.14), we have

$$\triangle h = -f \chi_{\Omega_1} \text{ in } D_1. \tag{6.21}$$

Therefore, for any $v \in U^{(2)}$,

$$\begin{aligned} & -a_1(h, R_1\gamma_0(v - u_2)) + \int_{\Omega_1} f R_1\gamma_0(v - u_2) d\theta d\sigma \\ & = -\int_{D_1} \nabla h \cdot \nabla (R_1\gamma_0(v - u_2)) d\theta d\sigma - \int_{\Gamma_2} h R_1 \gamma_0 (v - u_2) d\theta + \int_{\Omega_1} f R_1 \gamma_0 (v - u_2) d\theta d\sigma \\ & = \int_{D_1} (\triangle h)(R_1\gamma_0(v - u_2)) d\theta d\sigma - \int_{\partial D_1} (R_1 \gamma_0 (v - u_2)) \tfrac{\partial h}{\partial n} ds \\ & \quad - \int_{\Gamma_2} h R_1 \gamma_0 (v - u_2) d\theta + \int_{\Omega_1} f R_1 \gamma_0 (v - u_2) d\theta d\sigma \\ & = \int_{D_1} (-f \chi_{\Omega_1})(R_1 \gamma_0 (v - u_2)) d\theta d\sigma + \int_{\Gamma_1} (v - u_2) \tfrac{\partial h}{\partial \theta} d\sigma + \int_{\Omega_1} f R_1 \gamma_0 (v - u_2) d\theta d\sigma \\ & = \int_{\Gamma_1} (v - u_2) \tfrac{\partial h}{\partial \theta} d\sigma \end{aligned} \tag{6.22}$$

which shows the equivalence between (6.17) and (6.19) by comparing the expression of (6.17) and (6.19).

If we let $g = \gamma_0 u_2$ and $h = u_1$ in the above subproblems, then we get the following equivalence theorem.

Theorem 3 *If $\{u_1, u_2\}$ solves the variational form of subproblems 1 and 2, respectively, with $h = u_1$ and $g = \gamma_0 u_2$ and in addition u_1 and u_2 are sufficiently regular, then*

$$u = u_1 \text{ in } D_1; \quad u = u_2 \text{ in } D_2$$

gives a solution of

$$\triangle u = f\chi_\Omega \text{ in } D$$

$$u \geq 0 \text{ and } u = 0 \text{ on } \Gamma_4, \quad u + u_\sigma = 0 \text{ on } \Gamma_2 \cup \Gamma_3, \quad u = h(\sigma) \text{ on } \Gamma_0 \tag{6.23}$$

with suitable regularity assumptions.

Now we are ready to proceed with the iterative scheme that allows us to solve the above split problem.

6.3.2 Iterative scheme

Let $\Phi = H^{\frac{1}{2}}(\Gamma_1)$, $g^{(1)} \in \Phi, g^{(1)} \geq 0$ be given. For $n \geq 1$ construct $u_1^{(n)} \in H^1(D_1), u_2^{(n)} \in H^1(D_2)$ by:

Define the convex sets:

$$\begin{aligned} U^{(1)}(g) = \quad & \{v : v \in H^1(D_1), \quad v(\theta, \infty) = 0, \quad v = 0 \text{ on } \Gamma_4, \\ & \gamma_0 v = g \text{ on } \Gamma_1, \quad v \geq 0 \text{ on } D_1\}, \end{aligned}$$

$$U^{(2)} = \{v : v \in H^1(D_2), \quad v(\theta, \infty) = 0, \quad 0 < \theta < \theta_0, \quad v = h(\sigma) \text{on } \Gamma_0\}$$

First for $n = 1$, find $(u_1^{(n)}, \Omega_1^{(n)})$ such that $u_1^{(n)} \in U^{(1)}(g^{(n)})$,

$$a_1(u_1^{(n)}, v - u_1^{(n)}) \geq < f, v - u_1^{(n)} >_1 \quad \forall v \in U^{(1)}(g^{(n)}) \tag{6.24}$$

and

$$\Omega_1^{(n)} = \{(\theta, \sigma) : u_1^{(n)}(\theta, \sigma) > 0\}.$$

Then find $u_2^{(n)} \in U^{(2)}$ such that

$$\begin{aligned} a_2(u_2^{(n)}, v - u_2^{(n)}) = \quad & < f, v - u_2^{(n)} >_2 - a_1(u_1^{(n)}, R_1\gamma_0(v - u_2^{(n)})) \\ & + \int_{\Omega_1^{(n)}} f R_1 \gamma_0 (v - u_2^{(n)}) d\theta d\sigma \quad \forall v \in U^{(2)}. \end{aligned} \tag{6.25}$$

Next define

$$g^{(n+1)} = \gamma_0(u_1^{(n+1)}) = \theta_n \gamma_0(u_2^{(n)}) + (1-\theta_n)g^{(n)} \tag{6.26}$$

with $0 < \theta_n < 1$. Here all $\theta_n = \frac{1}{2}$. Then repeat (6.24) with n replaced by $n+1$. where $u_{1(i,j)}^{(n)}$ is the value of u_1 at node i, j for the nth iteration and similarly for $u_{2(i,j)}^{(n)}$. These iterations are stopped when

$$||u_1^{(n+1)} - u_1^{(n)}||_1 < \epsilon \text{ and } ||u_2^{(n+1)} - u_2^{(n)}||_2 < \epsilon,$$

where $||u||_k, k = 1, 2$ are the norm defined below and ϵ is preset.

We need to define some norm notation which will be useful later. Let

$$||v||_k^2 = a_k(v, v) \quad \text{ for } k = 1, 2 \tag{6.27}$$

$$|||\psi||| := ||R_1\psi||_1^2 \tag{6.28}$$

$$((\phi, \psi)) := a_1(R_1\phi, R_1\psi) \quad \forall \phi, \psi \in \Phi. \tag{6.29}$$

6.3.3 Convergence of the iterative scheme

Once the initial guess $g^{(1)}$ is given, we shall be able to show that this sequence of subproblems converges to our original problem as long as the $g^{(n)}$'s converge along Γ_1 as $n \to \infty$. In other words:

Theorem 4 *If the sequence $\{g^{(n)} = \gamma_0 u_1^{(n)}\}$ converges as n tends to ∞ and $g^{(n)} \geq 0$ on Γ_1, then the whole sequence $\{u_1^{(n)}, u_2^{(n)}\}$ converges to the solution $\{u_1, u_2\}$ of the free boundary problem 1, i.e., (6.15) and boundary value problem 2, i.e., (6.17).*

Proof. Define $K := \{z : z \in H^1(D_1), \quad z = 0 \text{ on } \Gamma_1 \cup \Gamma_4, \quad z(\theta, \infty) = 0, \quad z \geq 0 \text{ in } D_1\}$.

Consider the following problem: Find $z \in K$ such that:

$$a_1(z, w - z) \geq \int_{D_1} f(w - z), \quad \forall w \in K. \tag{6.30}$$

We know a solution z exists(Kinderlehrer and Stampacchia[14]). Define $R_1(g)$ as in (6.20ab), then $R_1 g$ satisfies

$$a_1(R_1 g, w - z) = 0 \quad \forall w \in K, z \in K.$$

Let z be the solution to (6.30), then

$$a_1(z, w - z) \geq \int_{D_1} f(w - z) - a_1(R_1 g^{(n)}, w - z) \quad \forall w \in K \tag{6.31}$$

i.e.,

$$a_1(z + R_1 g^{(n)}, w - z) \geq \int_{D_1} f(w - z) \quad \forall w \in K. \tag{6.32}$$

Since $\forall w \in K, w + R_1 g^{(n)} \in U^{(1)}(g^{(n)})$, let $u_1^{(n)} = w + R_1 g^{(n)}$, then

$$u_1^{(n)} \in U^{(1)}(g^{(n)}) \tag{6.33}$$

and

$$\begin{aligned} w - z &= w - (u_1^{(n)} - R_1 g^{(n)}) \\ &= v - u_1^{(n)} \quad \forall v = w + R_1 g^{(n)} \text{ in } U^1(g^{(n)}). \end{aligned} \tag{6.34}$$

As a result we have

$$a_1(u_1^{(n)}, v - u_1^{(n)}) \geq \int_{D_1} f(v - u_1^{(n)}) \quad \forall v \in U^{(1)}(g^{(n)}) \tag{6.35}$$

which is problem (6.24).

So $\forall m, n \geq 1$, $\exists$ solutions $u_1^{(n)}, u_1^{(m)}$ with respect to $g^{(n)}, g^{(m)}$ such that

$$u_1^{(n)} = z + R_1 g^{(n)} \in U^{(1)}(g^{(n)})$$

and

$$u_1^{(m)} = z + R_1 g^{(m)} \in U^{(1)}(g^{(m)}). \tag{6.36}$$

It then follows that:

$$||u_1^{(n)} - u_1^{(m)}||_{H^1(D_1)} = ||R_1 g^{(n)} - R_1 g^{(m)}||_{H^1(D_1)}. \tag{6.37}$$

Since $\gamma_0 u_1^{(n)} = g^{(n)}$, then $||u_1^{(n)} - u_1^{(m)}||_{H^1(D_1)} \to 0$ as $|||\gamma u_1^{(n)} - \gamma u_1^{(m)}||| \to 0$. This implies the convergence of $\{u_1^{(n)}\}$, since $\{u_1^{(n)}\}$ is a Cauchy sequence in the Hilbert space J, where $J := \{v : v \in H^1(\Omega_1), v = 0 \text{ on } \Gamma_4, \quad v(\theta, \infty) = 0 \quad \}$. As a result, the limit of $u_1^{(n)}$ exists. Assume $\lim_{n\to\infty} u_1^{(n)} = u_1$. By (6.26)

$$\begin{aligned} \theta_{n-1}\gamma_0 u_2^{(n-1)} &= g^{(n)} - (1 - \theta_{n-1}) g^{(n-1)} \\ &= \gamma_0 u_1^{(n)} - (1 - \theta_{n-1})\gamma_0 u_1^{(n-1)} \\ &= \gamma_0 u_1^{(n)} - \gamma_0 u_1^{(n-1)} + \theta_{n-1}\gamma_0 u_1^{(n-1)}. \end{aligned} \tag{6.38}$$

Therefore,

$$\theta_{n-1}(\gamma_0 u_2^{(n-1)} - \gamma_0 u_1^{(n-1)}) = \gamma_0 u_1^{(n)} - \gamma_0 u_1^{(n-1)}. \tag{6.39}$$

Divide both sides by $\theta_{n-1} = \frac{1}{2}$, then

$$|\gamma_0 u_2^{(n-1)} - \gamma_0 u_1^{(n-1)}| \leq 2|\gamma_0 u_1^{(n)} - \gamma_0 u_1^{(n-1)}| \to 0 \tag{6.40}$$

Therefore,

$$\lim_{n\to\infty} \gamma_0 u_2^{(n)} = \lim_{n\to\infty} \gamma_0 u_1^{(n)} \tag{6.41}$$

and $\lim_{n\to\infty} \Omega_1^{(n)}$ exists, i.e.,

$$\int_{\Omega_1^{(m)}-\Omega_1^{(n)}} 1 d\theta d\sigma = 0 \text{ as m, n} \to \infty.$$

From (6.25), we have

$$\begin{aligned}
||u_2^{(n)} - u_2^{(m)}||_2^2 &\leq a_2(u_2^{(n)} - u_2^{(m)}, u_2^{(n)} - u_2^{(m)}) \\
&= a_2(u_2^{(n)}, u_2^{(n)} - u_2^{(m)}) - a_2(u_2^{(m)}, u_2^{(n)} - u_2^{(m)}) \\
&= -a_2(u_2^{(n)}, u_2^{(m)} - u_2^{(n)}) - a_2(u_2^{(m)}, u_2^{(n)} - u_2^{(m)}) \\
&= -\textstyle\int_{D_2} f(u_2^{(m)} - u_2^{(n)}) + \int_{\Omega_1^{(n)}} f R_1 \gamma_0 (u_2^{(m)} - u_2^{(n)}) \\
&\quad -a_1(u_1^{(n)}, R_1\gamma_0(u_2^{(n)} - u_2^{(m)})) - \textstyle\int_{D_2} f(u_2^{(n)} - u_2^{(m)}) \\
&\quad + \textstyle\int_{\Omega_1^{(m)}} f R_1 \gamma_0 (u_2^{(n)} - u_2^{(m)}) - a_1(u_1^{(m)}, R_1\gamma_0(u_2^{(n)} - u_2^{(m)})) \\
&= -a_1(u_1^{(n)} - u_1^{(m)}, R_1\gamma_0(u_2^{(n)} - u_2^{(m)})) \\
&\quad + \textstyle\int_{\Omega_1^{(n)}-\Omega_1^{(m)}} f R_1 \gamma_0 (u_2^{(m)} - u_2^{(n)}) \\
&\quad + \textstyle\int_{\Omega_1^{(m)}-\Omega_1^{(n)}} f R_1 \gamma_0 (u_2^{(m)} - u_2^{(n)}) \\
&\leq ||u_1^{(n)} - u_1^{(m)}|| \, ||R_1\gamma_0(u_2^{(n)} - u_2^{(m)})|| \\
&\quad + \textstyle\int_{\Omega_1^{(n)}-\Omega_1^{(m)}} |f R_1 \gamma_0 (u_2^{(n)} - u_2^{(m)})| \\
&\quad + \textstyle\int_{\Omega_1^{(m)}-\Omega_1^{(n)}} |f R_1 \gamma_0 (u_2^{(n)} - u_2^{(m)})| \\
&\to 0 \text{ as n, m } \to \infty.
\end{aligned} \tag{6.42}$$

Since $U^{(2)}$ is a closed subspace of a Hilbert space, therefore $\{u_2^{(n)}\}$ converges to some function in $U^{(2)}$, say u_2.

If we take the limit in (6.35), we have:

$$a_1(u_1, v - u_1) \geq \int_{D_1} f(v - u_1) \quad \forall v \in U^{(1)}(g). \tag{6.43}$$

Therefore, the free boundary problem 1 is satisfied by u_1. From (6.41), we have

$$\gamma_0 u_1 = \gamma_0 u_2 \quad \text{on } \Gamma_1.$$

Taking the limit in (6.25), the boundary value problem 2 is satisfied by u_2.

As a result, to assure convergence over the entire region D it is only necessary to have convergence of $u_1^{(n)}$ along Γ_1. This iterative scheme allows a simple numerical implementation with a stopping criteria of convergence on the common boundary.

Chapter 7

Convergence Analysis of DDM Schemes for Other Free Boundary Problems

7.1 Introduction

As we mentioned before, free boundary value problems sometimes are divided into two non-overlapping problems. In one region the problem is treated as an ordinary boundary value problem. In the other region, the "free boundary part" of the problem is reduced to a variational inequality. By solving these two problems successively, it is shown numerically that the successive solutions converge to a single function that gives a solution of the original problem.

Papadopoulos et al.[19], [20], Jiang et al.[13], Bruch et al.[7] used the idea of a non-overlapping domain decomposition method to handle many free boundary problems and obtained very good numerical results. However, a mathematical proof of the convergence of this method has been an open problem. Herein, we use the maximum principle to consider the non-overlapping DDM applied to the general free boundary problem and prove the convergence of this non-overlapping DDM, which shows the coincidence of theory and numerics.

In the next 2 sections, we shall prove the convergence of 2 different numerical schemes used for the solution of the two free boundary problems in Chapter 5. In Section 7.2 we shall show the convergence of the numerical scheme for the free surface in a steady, two-dimensional seepage through a rectangular dam. In Section 7.3, we shall show the convergence of the numerical scheme for the free boundary seepage problem of flow through a porous dam with a toe drain.

7.2 Convergence of Numerical Scheme For Rectangular Dam Problem

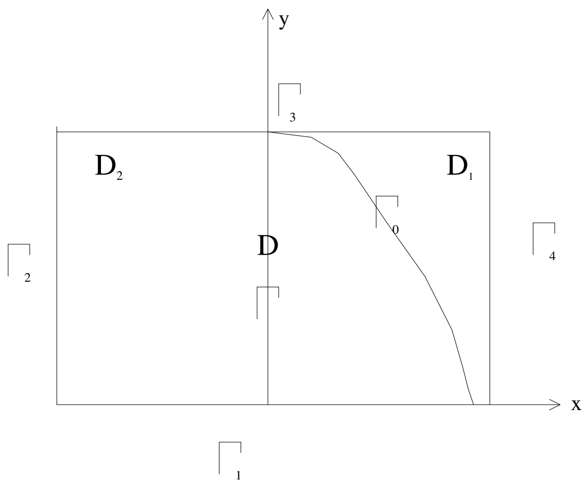

Figure 7.1: The Free Boundary Problem

The rectangular dam problem is described by the following free boundary problem:

$$(\triangle w - f)w = 0, \quad \triangle w - f \geq 0, \quad w \geq 0 \quad \text{on D} \tag{7.1}$$

with the boundary conditions:

$$w = h_1 \geq 0 \quad \text{on } \Gamma_1 \tag{7.2}$$

$$w = h_2 \geq 0 \quad \text{on } \Gamma_2 \tag{7.3}$$

$$w = 0 \text{ on } \quad \partial D - \Gamma_1 - \Gamma_2, \tag{7.4}$$

where D is an open simply connected region in the (x, y) plane for which $D = D_1 \cup D_2 \cup \Gamma$, D_1 and D_2 are open sets, $D_1 \cap D_2 = \phi$ (see Figure 7.1). $\Gamma = \partial\overline{D_1} \cap \partial\overline{D_2}$ is the common boundary of D_1 and D_2 and $f(x, y) \leq 0$.

It is assumed that if $\Omega = \{(x, y) \in D | w(x, y) > 0\}$, then $\Gamma_0 = \partial\Omega \cap \partial(D - \Omega) \subset D_1$, i.e., the free boundary Γ_0 is in D_1.(For many problems we can always split the region into two parts so that the free boundary will be contained in only one part).

In fact, (7.1)-(7.4) are equivalent to the following variational inequality formulation: find $w \in K$, so that

$$a(w, v - w) \geq < f, v - w > \quad \forall v \in K, \tag{7.5}$$

where

$$a(u, v) = \int_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) dx dy$$

$$< f, v >= \int_D f(x, y) v(x, y) dx dy$$

$$K = \{u \in H^1(D) : u|_{\Gamma_1} = h_1(x, y), \quad u|_{\Gamma_2} = h_2(x, y), \quad u|_{\partial D - \Gamma_1 - \Gamma_2} = 0, \quad u \geq 0\}.$$

The existence, uniqueness of the solution of (7.5) as well as the regularity of Γ_0 have been shown to hold(Baiocchi and Capelo[1]). As a consequence, (7.1)-(7.4) can be solved.

First, let us introduce a theorem which will be used in proving the convergence of the iteration.

Theorem 5 *Suppose the width of the right sub-domain D_1, M satisfies $0 < M < 1.732$. Then there exists a constant $\beta > \frac{1}{2}$ such that for any $u \in H^1(D_1)$,*

$$\beta ||u||_{H^1} \leq |u|_{H^1},$$

where $||u||_{H^1}$ is the complete H^1 norm of u while $|u|_{H^1}$ is the norm of the first derivatives of u.

Proof.

$$\begin{aligned} |u(x,y)| &= |\textstyle\int_{s=x}^{s=M} 1 \quad u_s(s,y)ds| \\ &\leq (\textstyle\int_{s=x}^{M} 1 \quad ds)^{\frac{1}{2}} (\textstyle\int_{s=x}^{M} u_s^2(s,y)ds)^{\frac{1}{2}} \\ &\leq M^{\frac{1}{2}} (\textstyle\int_0^M u_s^2(s,y)ds)^{\frac{1}{2}}, \end{aligned}$$

therefore,

$$u^2(x,y) \leq M \int_0^M u_s^2(s,y)ds$$

$$\int_{x=0}^{M} u^2(x,y)dx \leq M^2 \int_0^M u_s^2(s,y)ds$$

$$\begin{aligned} \textstyle\int_{y=0}^{M} \int_{x=0}^{M} u^2(x,y)dxdy &\leq M^2 \textstyle\int_{y=0}^{B} \int_0^M u_s^2(s,y)dsdy \\ &\leq M^2 \textstyle\int_{y=0}^{B} \int_0^M (u_x^2(x,y) + u_y^2(x,y))dxdy. \end{aligned}$$

Thus,

$$||u||_{L^2}^2 \leq M^2 |u|_{H^1}^2,$$

i.e.,

$$||u||_{L^2} \leq M|u|_{H^1}.$$

Consider the following equality:

$$\alpha|u|_{H^1}^2 + (1-\alpha)|u|_{H^1}^2 = |u|_{H^1}^2,$$

where $0 < \alpha < 1$ will be determined later.

From above, we have

$$\alpha|u|_{H^1}^2 + \frac{(1-\alpha)}{M^2}|u|_{L^2}^2 \leq |u|_{H^1}^2.$$

Let $\beta^2 = \min(\alpha, \frac{(1-\alpha)}{M^2})$, then

$$\beta^2 ||u||_{H^1}^2 < |u|_{H^1}^2.$$

To find the range of M so that $\beta > \frac{1}{2}$, we must have $\alpha > \frac{1}{4}$ and $\frac{(1-\alpha)}{M^2} > \frac{1}{4}$, i.e., $\alpha > \frac{1}{4}$ and $M < 2\sqrt{1-\alpha}$. M has a maximum value when $\alpha = \frac{1}{4}$ and then

$$M = 2\sqrt{1 - \frac{1}{4}} = 1.732.$$

Therefore, when $0 < M < 1.732$, we have $\beta > \frac{1}{2}$, and

$$\beta ||u||_{H^1} \leq |u|_{H^1}.$$

This completes the proof of the theorem.

Now let us set up the two different problems in D_1 and D_2 and iterate between them, and try to show these two solutions will converge to the solution of (7.3).

Problem 1. Given $g \in H^{\frac{1}{2}}(\Gamma)$, find a function u_1 defined on D_1 and $u_1 \in K_1(g)$ such that

$$a_1(u_1, v - u_1) \geq < f, v - u_1 >_1 \qquad \forall v \in K_1(g),$$

where

$$K_1(g) = \{v : v = g \text{ on } \Gamma,\ v = 0 \text{ on } \partial D_1 - \Gamma_1 - \Gamma,\ v = h_1 \text{ on } \Gamma_1,\ v \geq 0;\ v \in H^1(D_1)\}$$

$$a_j(u, v) = \int_{D_j} (\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}) dxdy \quad j = 1, 2$$

$$< f, v >_j = \int_{D_j} f(x, y) v(x, y) dxdy \quad j = 1, 2.$$

Problem 2. Given $h \in H^1(D_1)$ with $\gamma_0 \frac{\partial h}{\partial x} \in H^{\frac{1}{2}}(\Gamma)$, find a function u_2 defined on D_2 and $u_2 \in K_2$ such that

$$a_2(u_2, v - u_2) = < f, v - u_2 >_2 + \int_\Gamma (v - u_2) \frac{\partial h}{\partial x} dy \qquad \forall v \in K_2,$$

where

$$K_2 = \{v : v = 0 \text{ on } D_2 - \Gamma_2 - \Gamma,\ v = h_2 \text{ on } \Gamma_2,\ v \in H^1(D_2)\}.$$

The following iteration scheme for Problem 1 and 2 is used:

Step 1. Guess $g^{(1)} \in H^{\frac{1}{2}}(\Gamma)$ on Γ. Extend $g^{(1)}$ to $G^{(1)} \in H^1(D_1)$, i.e.,

$$\gamma_0 G^{(1)}|_{\partial D_1 - \Gamma} = 0, \quad \gamma_0 G^{(1)}|_\Gamma = g^{(1)}.$$

Set $u_1 = u_1^{(1)} + G^{(1)}$, and solve for $u_1^{(1)} \in H_0^1(D_1)$,

$$a_1(u_1^{(1)} + G^{(1)}, v - u_1^{(1)} - G^{(1)}) \geq < f, v - u_1^{(1)} - G^{(1)} >_1 \qquad \forall v \in K_1(g^{(1)}).$$

Therefore,

$$a_1(u_1^{(1)}, v_1 - u_1^{(1)}) \geq -a_1(G^{(1)}, v_1 - u_1^{(1)}) + < f, v_1 - u_1^{(1)} >_1,$$

where $v_1 = v - G^{(1)}$, $v_1 \in H_0^1(D_1)$.

Step 2. With $h = u_1^{(1)}$, solve Problem 2, i.e.,

$$\frac{\partial u_2^{(1)}}{\partial x} = \gamma_0 \frac{\partial u_1^{(1)}}{\partial x} \text{ on } \Gamma$$

for $u_2^{(1)}$ in $H^1(D_2)$, where it has been assumed $\gamma_0 \frac{\partial u_1^{(1)}}{\partial x} \in H^{\frac{1}{2}}(\Gamma)$.

Then we go back to Step 1 with $g^{(2)} = \gamma_0 u_2^{(1)}$ on Γ and check that $g^{(2)} \geq 0$ on Γ, then solve for $u_1^{(2)}$. These iterations are stopped when

$$||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)} < \epsilon \text{ and } ||u_2^{(n+1)} - u_2^{(n)}||_{H^1(D_2)} < \epsilon,$$

where ϵ is some fixed positive constant.

We have the following sequence of problems:

Find $u_1^{(n+1)} \in H_0^1(D_1)$, such that $\forall v_{n+1} \in H_0^1(D_1)$

$$a_1(u_1^{(n+1)}, v_{n+1} - u_1^{(n+1)}) \geq -a_1(G^{(n+1)}, v_{n+1} - u_1^{(n+1)}) + < f, v_{n+1} - u_1^{(n+1)} >_1 .$$

Since $u_1^{(n)} \in H_0^1(D_1)$, we can choose $v_{n+1} = u_1^{(n)}$ to yield

$$a_1(u_1^{(n+1)}, u_1^{(n)} - u_1^{(n+1)}) \geq -a_1(G^{(n+1)}, u_1^{(n)} - u_1^{(n+1)}) + < f, u_1^{(n)} - u_1^{(n+1)} >_1 .$$

Similarly we can choose $u_1^{(n+1)}$ to yield

$$a_1(u_1^{(n)}, u_1^{(n+1)} - u_1^{(n)}) \geq -a_1(G^{(n)}, u_1^{(n+1)} - u_1^{(n)}) + < f, u_1^{(n+1)} - u_1^{(n)} >_1 .$$

Adding the above two inequalities, we have

$$a_1(u_1^{(n+1)} - u_1^{(n)}, u_1^{(n+1)} - u_1^{(n)}) \leq -a_1(G^{(n+1)} - G^{(n)}, u_1^{(n+1)} - u_1^{(n)})$$

i.e.,

$$\int_{D_1} \nabla(u_1^{(n+1)} - u_1^{(n)}) \cdot \nabla(u_1^{(n+1)} - u_1^{(n)}) dx \leq -\int_{D_1} \nabla(G^{(n+1)} - G^{(n)}) \cdot \nabla(u_1^{(n+1)} - u_1^{(n)}) dx.$$

Therefore, by the Cauchy-Schwarz inequality,

$$|u_1^{(n+1)} - u_1^{(n)}|_{H^1(D_1)} \leq |G^{(n+1)} - G^{(n)}|_{H^1(D_1)}$$

and we can choose for $n > 2$, $G^{(n)} = u_1 - u_1^{(n)}$ and $G^{(n+1)} = u_1 - \frac{1}{2}(u_1^{(n)} + u_1^{(n-1)})$, then

$$|u_1^{(n+1)} - u_1^{(n)}|_{H^1(D_1)} \leq \frac{1}{2}|u_1^{(n)} - u_1^{(n-1)}|_{H^1(D_1)}.$$

Under the assumption that the conditions of Theorem 5 hold, we have

$$\begin{aligned}\beta||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)} &\leq |u_1^{(n+1)} - u_1^{(n)}|_{H^1(D_1)}\\ &\leq \tfrac{1}{2}|u_1^{(n)} - u_1^{(n-1)}|_{H^1(D_1)}\\ &\leq \tfrac{1}{2}||u_1^{(n)} - u_1^{(n-1)}||_{H^1(D_1)}.\end{aligned}$$

Therefore,

$$||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)} \leq \widetilde{\theta}||u_1^{(n)} - u_1^{(n-1)}||_{H^1(D_1)},$$

where $\widetilde{\theta} = \frac{1}{2\beta} < 1$. Now, we have

$$\begin{aligned}||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)} &\leq \widetilde{\theta}||u_1^{(n)} - u_1^{(n-1)}||_{H^1(D_1)}\\ &\leq \cdots\\ &\leq (\widetilde{\theta})^{n-1}||u_1^{(2)} - u_1^{(1)}||_{H^1(D_1)}.\end{aligned}$$

Therefore,

$$||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)} \to 0 \text{ as } n \to \infty.$$

Since $u_2^{(n)}$ satisfies

$$a_2(u_2^{(n)}, v - u_2^{(n)}) =< f, v - u_2^{(n)} >_2 + \int_\Gamma (v - u_2^{(n)})\frac{\partial u_1^{(n)}}{\partial x}dy, \quad \forall v \in K_2$$

then by choosing $v = u_2^{(n+1)}$, we have

$$a_2(u_2^{(n)}, u_2^{(n+1)} - u_2^{(n)}) =< f, u_2^{(n+1)} - u_2^{(n)} >_2 + \int_\Gamma (u_2^{(n+1)} - u_2^{(n)})\frac{\partial u_1^{(n)}}{\partial x}dy.$$

Similarly

$$a_2(u_2^{(n+1)}, u_2^{(n)} - u_2^{(n+1)}) =< f, u_2^{(n)} - u_2^{(n+1)} >_2 + \int_\Gamma (u_2^{(n)} - u_2^{(n+1)})\frac{\partial u_1^{(n+1)}}{\partial x}dy.$$

Adding the above two equations, we obtain

$$a_2(u_2^{(n+1)} - u_2^{(n)}, u_2^{(n+1)} - u_2^{(n)}) = \int_\Gamma (u_2^{(n+1)} - u_2^{(n)})(\frac{\partial(u_1^{(n+1)} - u_1^{(n)})}{\partial x}dy.$$

Therefore, by the Cauchy-Schwarz inequality,

$$|u_2^{(n+1)} - u_2^{(n)}|^2_{H^1(D_2)} \leq ||u_2^{(n+1)} - u_2^{(n)}||_{H^{\frac{1}{2}}(\Gamma)}||\frac{\partial(u_1^{(n+1)} - u_1^{(n)})}{\partial x}||_{H^{-\frac{1}{2}}(\Gamma)}$$

and then

$$\begin{aligned}||u_2^{(n+1)} - u_2^{(n)}||^2_{H^1(D_2)} &\leq C||u_2^{(n+1)} - u_2^{(n)}||_{H^1(D_2)}||\frac{\partial(u_1^{(n+1)} - u_1^{(n)})}{\partial x}||_{H^0(D_1)} \\ &\leq C||u_2^{(n+1)} - u_2^{(n)}||_{H^1(D_2)}||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)},\end{aligned}$$

where C depends on the domain D. Hence,

$$||u_2^{(n+1)} - u_2^{(n)}||_{H^1(D_2)} \leq C||u_1^{(n+1)} - u_1^{(n)}||_{H^1(D_1)},$$

i.e.,

$$||u_2^{(n+1)} - u_2^{(n)}||_{H^1(D_2)} \to 0 \text{ as } n \to \infty.$$

Since $u_1^{(n)}$ is a Cauchy sequence in $H^1(D_1)$, then there exists a function $u_1 \in H^1(D_1)$, such that $\lim_{n\to\infty} u_1^{(n)} = u_1$. Similarly, there exists a function $u_2 \in H^2(D_1)$, such that $\lim_{n\to\infty} u_2^{(n)} = u_2$. Since the function values and the normal derivatives of $u_1^{(n)}$ and $u_2^{(n)}$ are equal on the boundary Γ, then the function values and the normal derivatives of u_1 and u_2 should also be equal, then

$$u = \left\{ \begin{array}{ll} u_1(x,y), & (x,y) \in D_1 \\ u_2(x,y), & (x,y) \in D_2 \end{array} \right\}$$

is in $H^1(D)$. It has been shown(Papadopoulos[21]) that such $u(x,y)$ is the solution to (7.5). Therefore, the sequence $u_1^{(n)}$ and $u_2^{(n)}$ constructed in our numerical scheme will converge to the true solution of (7.5).

7.3 Convergence of Numerical Scheme For Porous Dam With Toe Drain

7.3.1 A PDE problem with Dirichlet condition on Γ_4

First, we consider the following Dirichlet problem:

$$\triangle u = 0 \text{ in D}, \tag{7.6a}$$

where D is a region whose four boundaries are Γ_1, Γ_2, Γ_3 and Γ_4 shown in Figure 7.2. The solution u to (7.6) will satisfy the following boundary conditions:

$$u_y = 0 \text{ on } \Gamma_1$$

$$u = 0 \text{ on } \Gamma_2 \cup \Gamma_3 \tag{7.6b}$$

$$u = h(y) \text{ on } \Gamma_4.$$

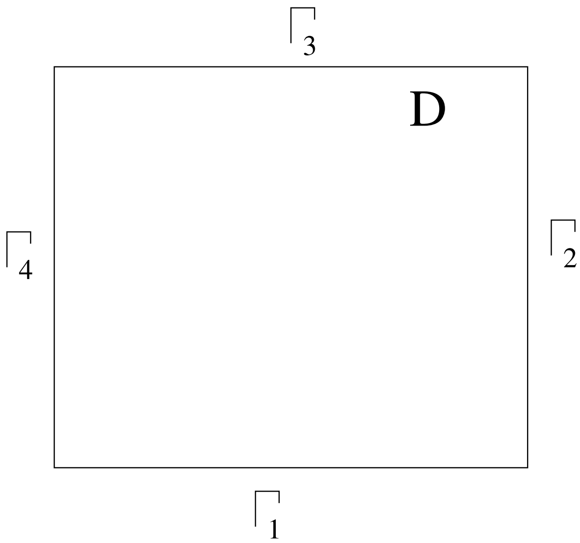

Figure 7.2: The PDE Dirichlet Problem

Theorem 6 *If $u \in H^2(D)$ satisfies (7.6a) and (7.6b), and $-\epsilon \leq h(y) \leq \epsilon$ on Γ_4, then u will satisfy*

$$-\epsilon \leq u(x,y) \leq \epsilon \text{ in } D.$$

Proof. First let us show $u \leq \epsilon$ in D. Since u satisfies (7.6a), then the maximum value of u in D should be on the boundary. Suppose the maximum value of u is u_0. If u_0 happens on Γ_1, then from the maximum principle, we have

$$u_y(p) < 0,$$

which contradicts the assumption $u_y = 0$ on Γ_1. Thus u_0 can only happen at the other boundaries, i.e.,

$$u_0 = \max(0, h(y)) \leq \epsilon.$$

Next let us show $u \geq -\epsilon$ in D. Define $v = -u$, then v will still satisfy (7.6a) with the same boundary conditions except on Γ_4, which is replaced by

$$v = -h(y) \text{ on } \Gamma_4.$$

By repeating the same procedure as before with $v = -u$, we can prove

$$v \leq \epsilon \text{ in } D$$

i.e.,

$$u \geq -\epsilon \text{ in } D.$$

This completes the proof of the theorem.

7.3.2 A PDE problem with Neumann condition on Γ_4

We consider the following problem:

$$\triangle u = 0 \text{ in D}, \tag{7.7a}$$

where D is a region whose four boundaries are Γ_1, Γ_2, Γ_3 and Γ_4 shown in Figure 7.3. The solution u to (7.7) will satisfy the following boundary conditions:

$$\begin{aligned} u_y &= 0 \text{ on } \Gamma_1 \\ u &= 0 \text{ on } \Gamma_2 \cup \Gamma_3 \\ u_x &= h(y) \text{ on } \Gamma_4. \end{aligned} \tag{7.7b}$$

Theorem 7 *If $u \in H^2(D)$ satisfies (7.7a) and (7.7b), and $-\epsilon \leq h(y) \leq \epsilon$ on Γ_4, then u will satisfy $-\delta \leq u(x,y) \leq \delta$ in D, and*

$$\delta(\epsilon) \to 0 \ \text{as} \ \epsilon \to 0.$$

Proof. Let D_1 be the reflected region of D with respect to the y-axis and define $D_0 = D \cup D_1$ to be the union shown as Figure 7.4. Then we extend u to the whole region by

$$u(x,y) = -u(-x,y) \quad x < 0.$$

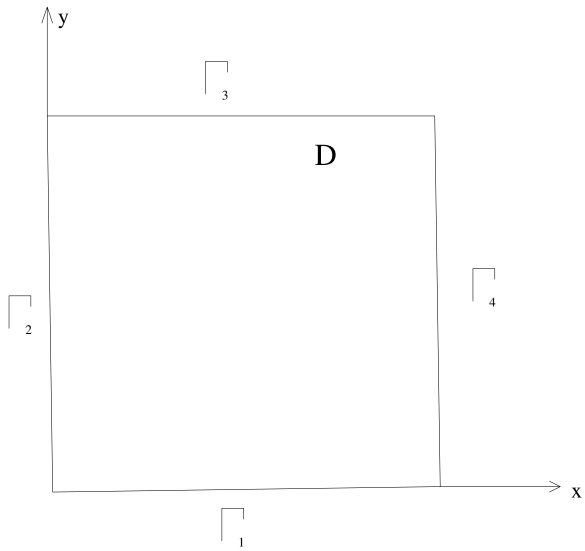

Figure 7.3: The PDE Neumann Problem

Then in D_0, u will satisfy

$$\triangle u = 0 \text{ in } D_0$$

$$u_y = 0 \text{ on } \Gamma_1$$

$$u_x = -h(y) \text{ on } \Gamma_4'$$

$$u = 0 \text{ on } \Gamma_3$$

$$u_x = h(y) \text{ on } \Gamma_4.$$

Now define $w = u_x$ in D_0, then w will satisfy

$$\triangle w = 0 \text{ in } D_0$$

$$w_y = 0 \text{ on } \Gamma_1$$

$$w = -h(y) \text{ on } \Gamma_4'$$

$$w = 0 \text{ on } \Gamma_3$$

$$w = h(y) \text{ on } \Gamma_4.$$

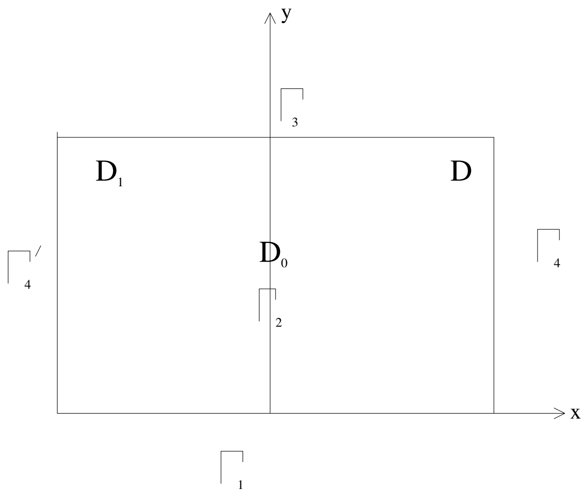

Figure 7.4: The Extended PDE Neumann Problem

Since $-\epsilon \leq h(y) \leq \epsilon$, we can repeat the same procedure for w as in Section 7.3.1, and obtain

$$-\epsilon \leq w \leq \epsilon \text{ in } D_0.$$

However, $u_x = w$ and $u(0, y) = 0$, we have in D

$$u(x, y) = \int_0^x w(\phi, y) d\phi.$$

Therefore,

$$-M\epsilon \leq u \leq M\epsilon \quad \text{ in } D,$$

where M is the width of D in Figure 7.3. This completes the proof of Theorem 7.

7.3.3 A free boundary problem with Dirichlet condition on the boundary Γ

We consider the following free boundary problem in D_1, shown as in Figure 7.5.

$$(\triangle u - f)u = 0, \quad u \geq 0, \quad \triangle u - f \geq 0 \quad \text{ in } D_1 \tag{7.8}$$

$$u_y = 0 \text{ on } \Gamma_1$$

$$u = h \text{ on } \Gamma$$

$$u = 0 \text{ on } \Gamma_3 \cup \Gamma_4.$$

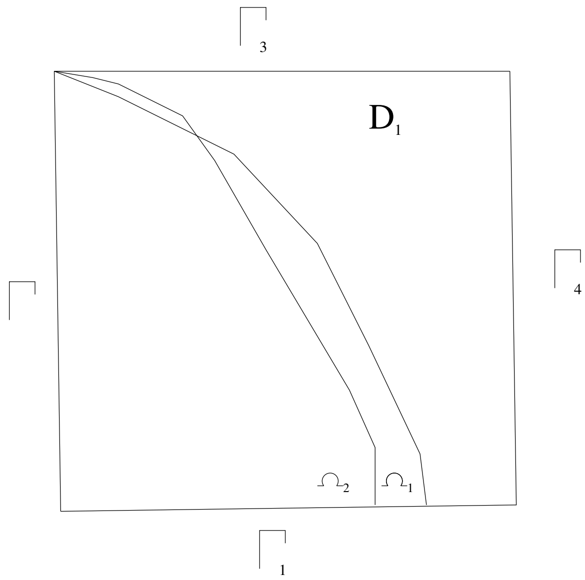

Figure 7.5: The Free Boundary Dirichlet Problem

Theorem 8 *Suppose u_1 and u_2 are solutions to (7.8) with $u_1 \geq u_2$ on Γ, then $u_1 \geq u_2$ in D_1.*

Proof. Let $\Omega_1 = \{(x,y) \in D_1 : u_1(x,y) > 0\}$ and $\Omega_2 = \{(x,y) \in D_1 : u_2(x,y) > 0\}$. Then

$$\triangle u_1 = f \text{ in } \Omega_1$$

$$\triangle u_2 = f \text{ in } \Omega_2.$$

Let $\overline{u} = u_2 - u_1$, then

$$\overline{u}_\Gamma \leq 0, \quad \overline{u}|_{\Gamma_3 \cup \Gamma_4} = 0, \quad \overline{u}_y|_{\Gamma_1} = 0.$$

Break up $\overline{D_1}$ into four regions:

$$\overline{D_1} = (\overline{\Omega_1 \cap \Omega_2}) \cup (\overline{\Omega_1 - \Omega_2}) \cup (\overline{\Omega_2 - \Omega_1}) \cup \overline{\sigma},$$

where $\overline{\sigma}$ is the complement in D_1 of the first three. Clearly $\overline{u} = 0$ on $\overline{\sigma}$.

$\overline{u}$ can not take on a positive maximum on D_1, except on $\overline{\Gamma}$. To see this, observe:

(1) A positive maximum can not occur on $\overline{\Omega_1 - \Omega_2}$, since $\overline{u} \leq 0$ there.

(2) A positive maximum can not occur on $\overline{\sigma}$, since $\overline{u} = 0$ there.

(3) If a positive maximum occurs on $\overline{\Omega_1 \cap \Omega_2}$, and does not occur on Γ, then since $\triangle\overline{u} = 0$ on $\Omega_1 \cap \Omega_2$, it follows from the maximum principle that the maximum must occur either at a point P common to the boundary of $\Omega_1 \cap \Omega_2$ and to the boundary of $\Omega_2 - \Omega_1$ or at a point P on Γ_1. For the former case, since $\overline{u} \in C^1(\overline{D_1})$ it follows that $\frac{\partial \overline{u}}{\partial N}(P) > 0$ where N is the direction pointing exterior to $\Omega_1 \cap \Omega_2$ and interior to $\Omega_2 - \Omega_1$. Meanwhile, on $\Omega_2 - \Omega_1$, $\overline{u} = u_2 \geq 0$ and hence

$$\triangle\overline{u} = \triangle\overline{u_2} = f \leq 0 \text{ on } \Omega_2 - \Omega_1.$$

Again by maximum principle applied to $\overline{u} = u_2$ on $\Omega_2 - \Omega_1$ it follows that if $\overline{u} = u_2$ continuous on $\overline{\Omega}_2 - \Omega_1$, is to have a positive maximum it must occur at the same P as above. However, by the maximum principle $\frac{\partial u_2}{\partial N}(P) < 0$ gives a contradiction since $u_2 \in C^1(D_1)$. For the latter case, since $\overline{u} \in C^1(\overline{D_1})$, the maximum principle gives $\frac{\partial \overline{u}}{\partial y}(P) > 0$, which contradicts the assumption of the boundary condition on Γ_1.

Hence $\overline{u}$ can not take on a positive maximum on $\overline{D_1}$ except on $\overline{\Gamma}$. However, $\overline{u} \leq 0$ on Γ, which means $\overline{u}$ can not be positive on $\overline{D_1}$. Then $\overline{u} \leq 0$ in D_1, i.e., $u_1 \geq u_2$ in D_1.

Corollary 1 *Suppose u_1 and u_2 are solutions to (7.8), then*

$$\max_{D_1} |u_1 - u_2| = \max_{\Gamma} |u_1 - u_2|.$$

Proof. By using the same idea as Theorem 8, we can prove

(1) the positive maximum of $u_1 - u_2$ on D_1 can only happen at Γ if it has one.

(2) the negative minimum of $u_1 - u_2$ on D_1 can only happen at Γ if it has one.

By combining both statements (1) and (2), it is completed.

Theorem 9 *If u satisfies (7.8) with the given boundary conditions, and $-\epsilon \leq h(y) \leq \epsilon$ on Γ, then u will satisfy $-\epsilon \leq u(x,y) \leq \epsilon$ in D_1.*

Proof. Let $h_1(y) = h(y)$ and $h_2(y) = 0$ are two functions defined on Γ and u_1 and u_2 are the solutions to (7.8) corresponding to h_1 and h_2, respectively. Then, it is clear that $u_2 = 0$ in D_1. From Corollary 1, we have

$$\max_{D_1} |u_1 - u_2| = \max_{\Gamma} |h_1 - h_2| = \epsilon$$

i.e., $-\epsilon \le u(x,y) \le \epsilon$ in D_1.

7.3.4 Convergence result for the free boundary problem with mixed boundary conditions

We consider the following problem:

$$(\triangle u - f)u = 0, \quad \triangle u - f \ge 0, \quad u \ge 0 \quad \text{on D}, \tag{7.9}$$

where D is a region whose four boundaries are Γ_1, Γ_2, Γ_3 and Γ_4 shown in Fig 7.6. The solution u to (7.9) will satisfy the following boundary conditions:

$$u_y = 0 \text{ on } \Gamma_1$$

$$u = \widetilde{f}(y) \text{ on } \Gamma_2$$

$$u = 0 \text{ on } \Gamma_3 \cup \Gamma_4.$$

Now let us split the domain into two subdomains D_1 and D_2 with Γ as their common boundary. We will select Γ so that the free boundary will be in D_1. Now we will have 2 subproblems as below, and we will iterate between them, the successive solutions will converge to the solution to the original problem.

Subproblem 1. Given $g(y)$ on Γ, solve for u in D_1

$$(\triangle u - f)u = 0, \quad u \ge 0, \quad \triangle u - f \ge 0 \quad \text{in } D_1 \tag{7.10}$$

$$u_y = 0 \text{ on } \Gamma_1$$

$$u = g(y) \text{ on } \Gamma$$

$$u = 0 \text{ on } \Gamma_3 \cup \Gamma_4.$$

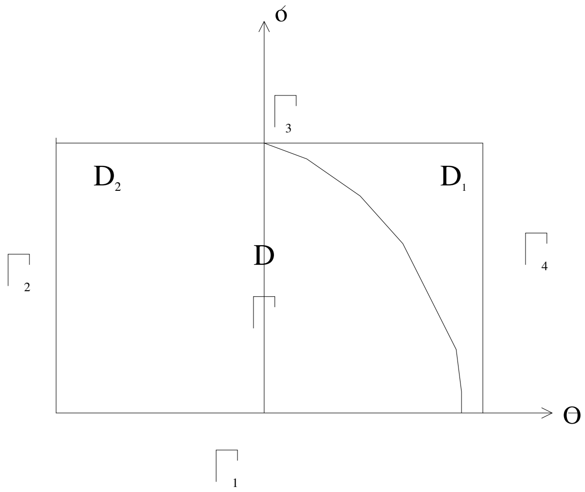

Figure 7.6: The Free Boundary Problem

Subproblem 2. Given $h(y)$ on Γ, solve for u in D_2

$$\triangle u = 0 \text{ in } D_2 \tag{7.11}$$

$$u = \widetilde{f}(y) \text{ on } \Gamma_2$$

$$u = 0 \text{ on } \Gamma_3$$

$$u_x = h(y) \text{ on } \Gamma.$$

The iteration scheme is:

Step 1. Choose $g^{(0)} = 0$ on Γ, solve Subproblem 1 for $u_1^{(0)}$ in D_1.

Step 2. Let $h^{(0)} = \frac{\partial u_1^{(0)}}{\partial x}|_\Gamma$, solve Subproblem 2 for $u_2^{(0)}$ in D_2.

Step 3. Let $g^{(1)} = u_2^{(0)}|_\Gamma$, solve Subproblem 1 for $u_1^{(1)}$ in D_1.

Generally, for $n > 1$, let $g^{(n)} = u_2^{(n-1)}|_\Gamma$, solve Subproblem 1 for $u_1^{(n)}$ in D_1. Then let $h^{(n)} = \frac{\partial u_1^{(n)}}{\partial x}|_\Gamma$, solve Subproblem 2 for $u_2^{(n)}$ in D_2. These iterations are stopped when

$$\max_{i,j} |u_{1(i,j)}^{(n+1)} - u_{1(i,j)}^{(n)}| < \epsilon \quad \text{and} \quad \max_{i,j} |u_{2(i,j)}^{(n+1)} - u_{2(i,j)}^{(n)}| < \epsilon,$$

where ϵ is preset.

Theorem 10 *Suppose $u_1^{(n)}$ and $u_2^{(n)}$ are the solutions as above. If $g^{(n)}$ converges on Γ, then $u_1^{(n)}$ and $u_2^{(n)}$ will converge to the original solution of (7.9) in D_1 and D_2, respectively, as $n \to \infty$.*

Proof. Since $g^{(n)}$ converge to g on Γ, let u_1 be the solution of Subproblem 1 in D_1 with g as the value on Γ. Then from Corollary 1, we have $u_1^{(n)} \to u_1$. Therefore, $\frac{\partial u_1^{(n)}}{\partial x}$, i.e., $h^{(n)}$ also converges, for example, to h. Let u_2 be the solution of Subproblem 2 in D_2 with h as the partial derivative value on Γ, then from Theorem 7 , we have $u_2^{(n)} \to u_2$.

It is clear u_1 and u_2 will form the solution of (7.9). The proof is completed.

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