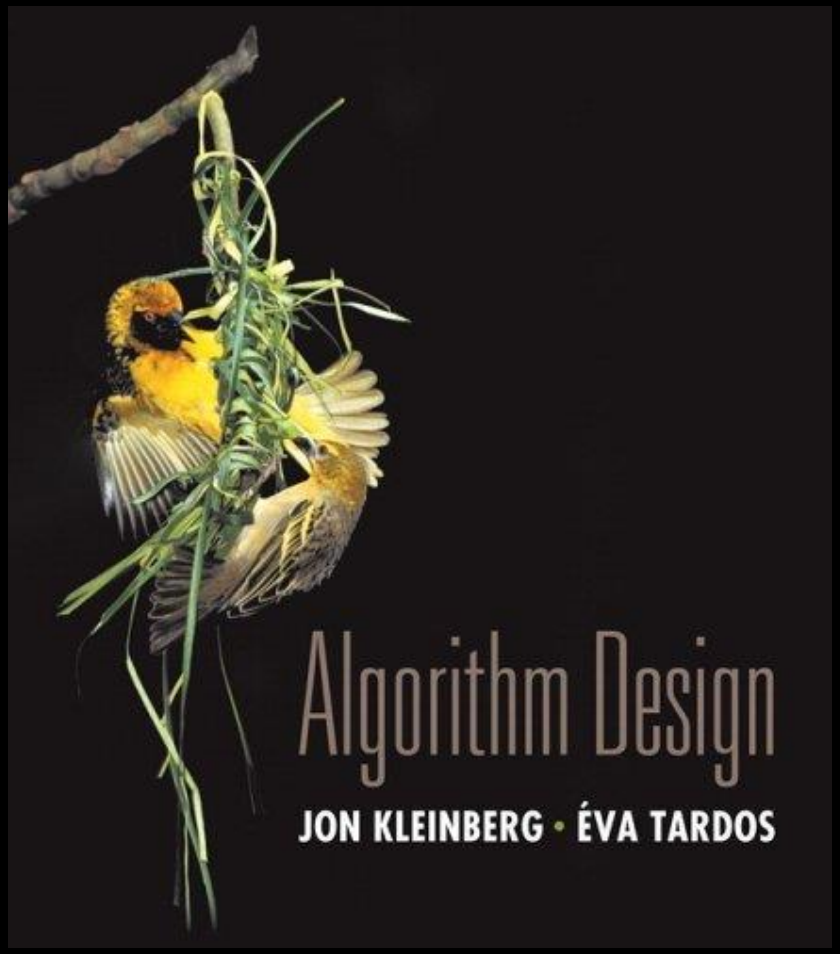


Chapter 5

Divide and Conquer

CS 350 Winter 2018

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

- Break up problem of size n into two equal parts of size $\frac{1}{2}n$.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: n^2.
- Divide-and-conquer: n log n.

Divide et impera.
Veni, vidi, vici.
- Julius Caesar

Divide-and-Conquer Example with Recurrences

Recall the classic example of divide-and-conquer for efficient integer multiplication that we encountered during the Chapter 2 Lecture:

(*)Suppose x and y are two n-bit integers, and assume, for convenience, that n is a power of 2 (the more general case is not too different).

As a first step toward multiplying x and y, split each of them into their left and right halves, which are n/2 bits long:

$$x = [x_L][x_R] = 2^{n/2}x_L + x_R$$
$$y = [y_L][y_R] = 2^{n/2}y_L + y_R$$

For instance, if x=101101102, then $x_L = 1011_2$ and $x_R = 0110_2$, and $x = 1011_2 * 2^4 + 0110_2$.

$$xy = (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

The product of x and y can thus be written:

Divide-and-Conquer Example with Recurrences

$$xy = (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

Consider the computation requirements of the RHS:

(*) The additions take linear time, as do the multiplications by powers of 2 (which are merely left-shifts).

(*) The significant operations are the four n/2-bit multiplications: $x_L y_L$, $x_L y_R$, $x_R y_L$, $x_R y_R$; these can be handled with four recursive calls.

(*) Thus our method for multipliying n-bit numbers starts by making recursive calls to multiply these four pairs of n/2-bit numbers (four subproblems of half the size), and then evaluates the preceding expression in O(n) time.

Writing T(n) for the overall running time on n-bit inputs, we get the recurrence relation:

$$T(n)=4T(n/2)+O(n)$$

Divide-and-Conquer Example with Recurrences

$$xy = (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

Writing T(n) for the overall running time on n-bit inputs, we get the recurrence relation:

$$T(n)=4T(n/2)+O(n)$$

(*) In this course we will develop general strategies for solving such equations.

(*) In the meantime, this particular equation works out to $O(n^2)$, the same running-time as the traditional grade school multiplication technique.

Q: How can we speed up this method? A: Apply Gauss' trick.

Divide-and-Conquer Example with Recurrences

$$xy = (2^{n/2} x_L + x_R)(2^{n/2} y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

$$T(n)=4T(n/2)+O(n)$$

A: Apply Gauss' trick.

(*) Although the expression xy seems to demand four n/2-bit multiplications, as before just three will do: $x_L y_L$, $x_L y_R$, and $(x_L+x_R)(y_L+y_R)$, since $x_L y_R + x_R y_L = (x_L+x_R)(y_L+y_R) - x_L y_L - x_R y_R$.

The resulting algorithm has an improved running time of:

$$T(n)=3T(n/2)+O(n)$$

(*) The point is that now the constant factor improvement, from 4 to 3, occurs at every level of the recursion, and this compounding effect leads to a dramatically lower time bound of $O(n^{1.59})$.

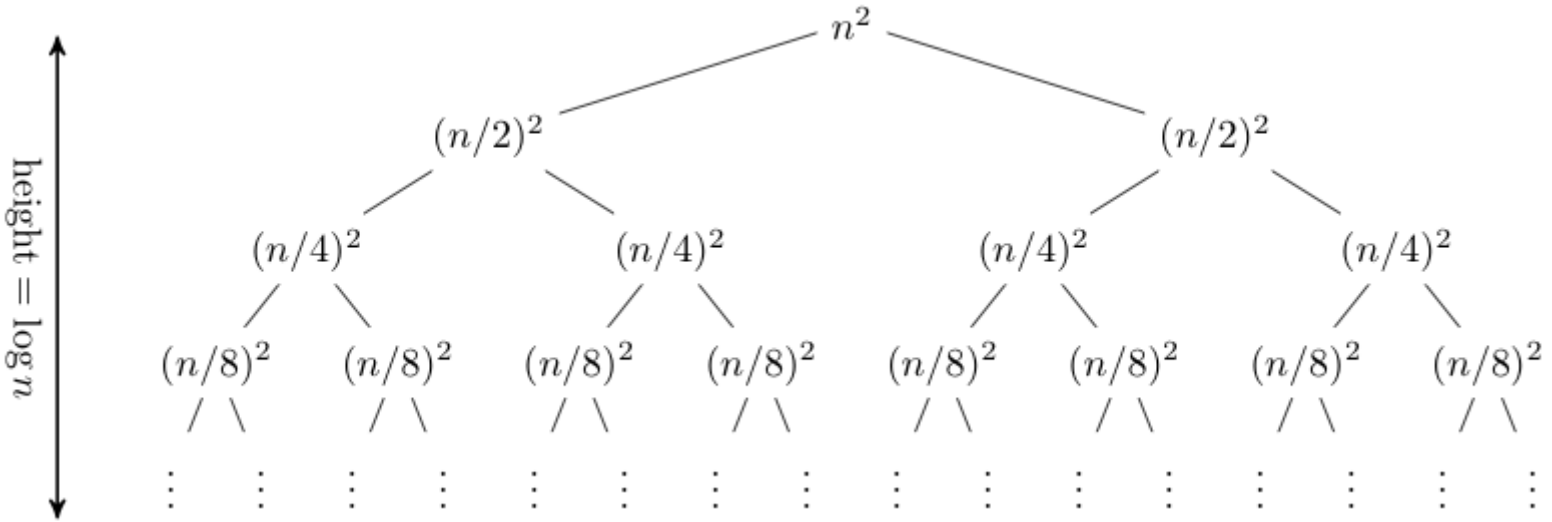
Q: How do we determine this bound (more later) – but for now, it is helpful to consider the recursive calls with respect to a tree structure (also: the "Master Theorem" can be used).

"Master Method" Example

(*) A *recursion tree* is useful for visualizing what happens when a recurrence is iterated. It diagrams the tree of recursive calls and the amount of work done at each call.

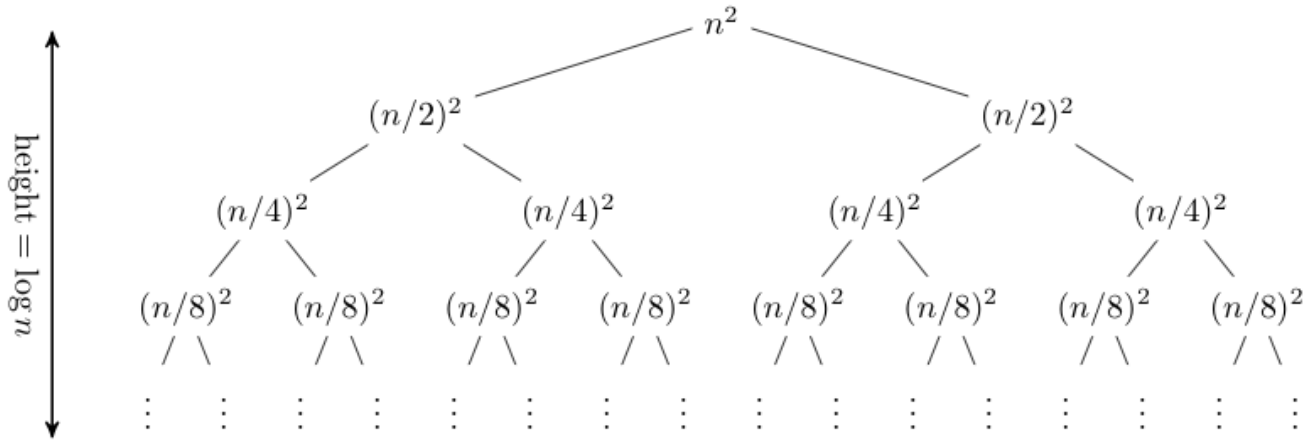
Consider the recurrence: $T(n)=2T(n/2)+n^2$

The corresponding recursion tree has the following form:

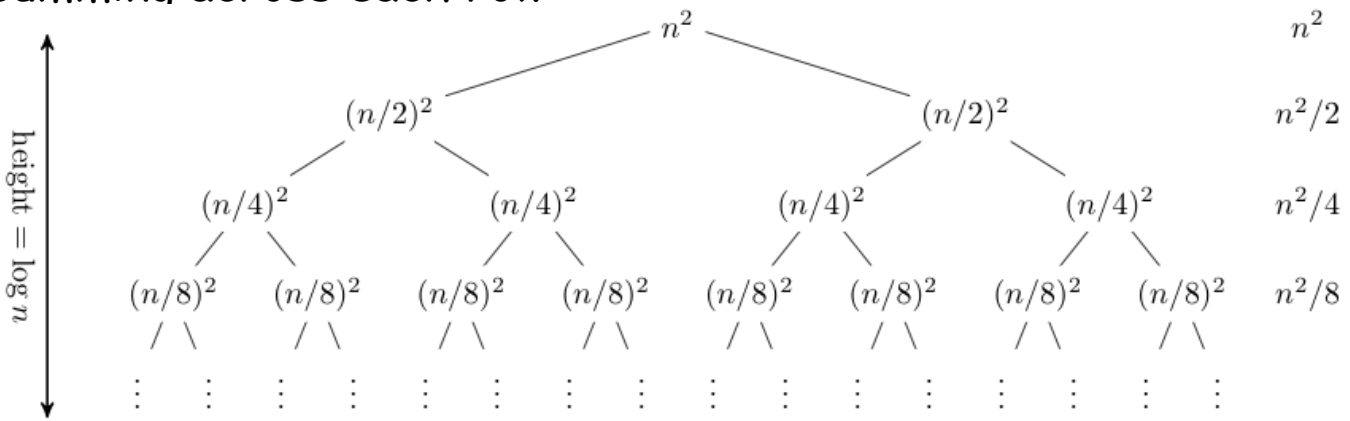


"Master Method" Example

Consider the recurrence: $T(n)=2T(n/2)+n^2$



Consider summing across each row:

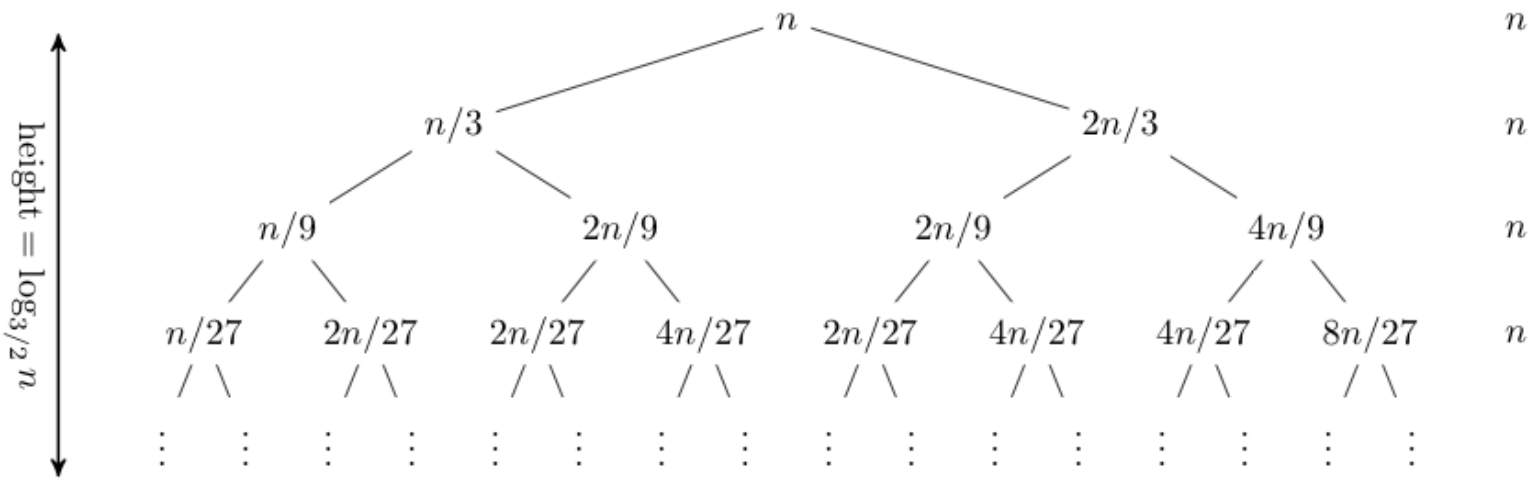


This yields a geometric series:

$$\sum_{i=1}^{n} \frac{n^2}{2^i} = ?$$

"Master Method" Example

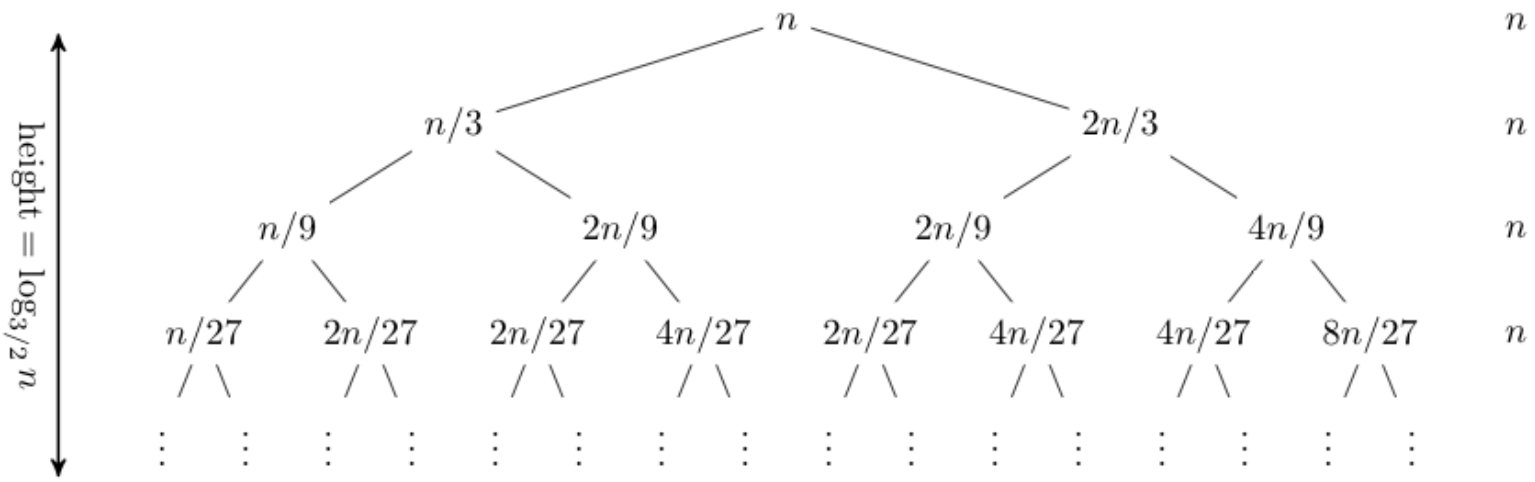
Consider the recurrence: T(n)=T(n/3)+T(2n/3)+n



Note that the recursion tree is not balance in this case, and that the longest path is the rightmost one.

"Master Method" Example

Consider the recurrence: T(n)=T(n/3)+T(2n/3)+n



Note that the recursion tree is not balance in this case, and that the longest path is the rightmost one.

Since the longest path is $O(\log_{3/2}(n))$, our guess for the closed form solution to the recurrence is: $O(n \log n)$.

5.1 Mergesort

Mergesort

6 5 3 1 8 7 2 4

An example of merge sort. First divide the list into the smallest unit (1 element), then compare each element with the adjacent list to sort and merge the two adjacent lists. Finally all the elements are sorted and merged.

Sorting

Sorting. Given n elements, rearrange in ascending order.

Applications.

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

obvious applications

- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

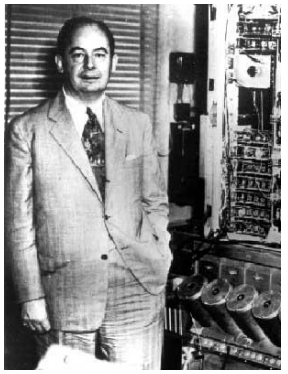
- Data compression.
- Computer graphics.
- Computational biology.
- Supply chain management.
- Book recommendations on Amazon.
- Load balancing on a parallel computer.
- . . .

non-obvious applications

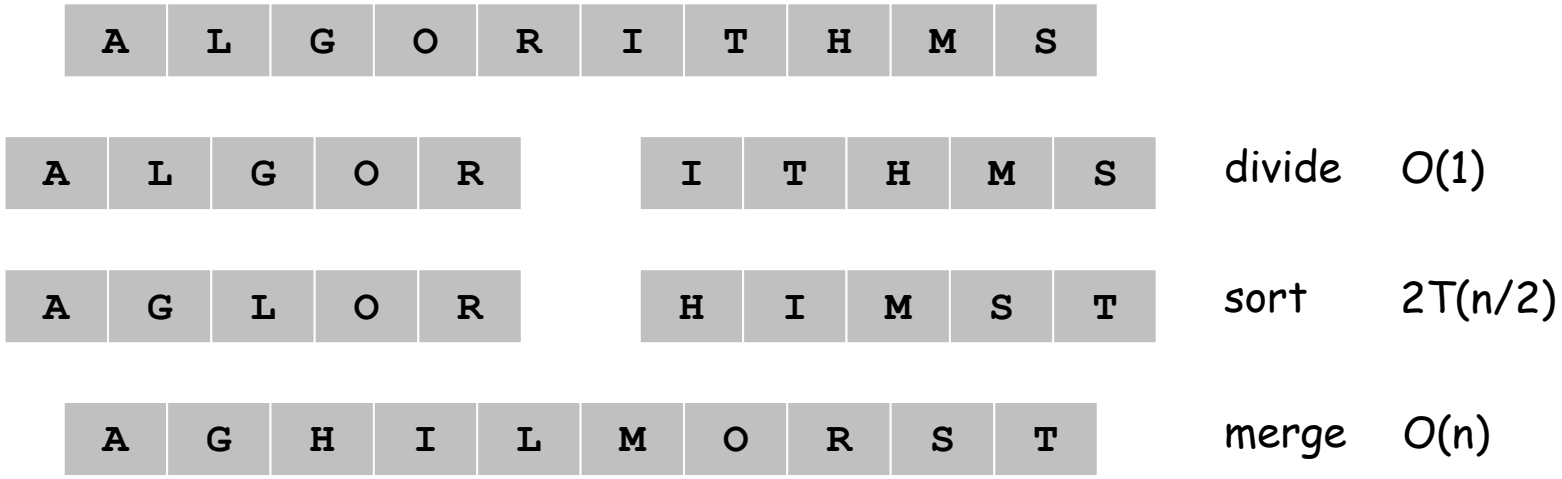
Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



Jon von Neumann (1945)

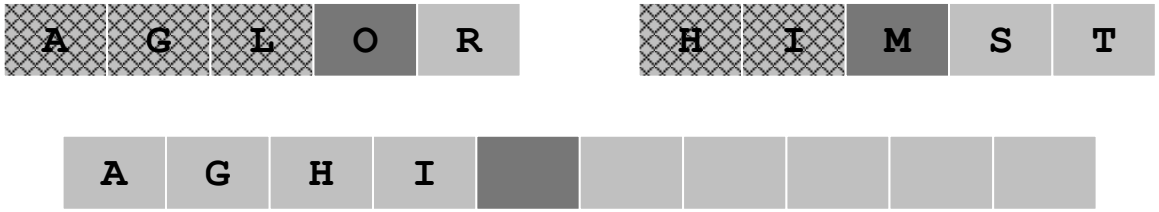


Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?

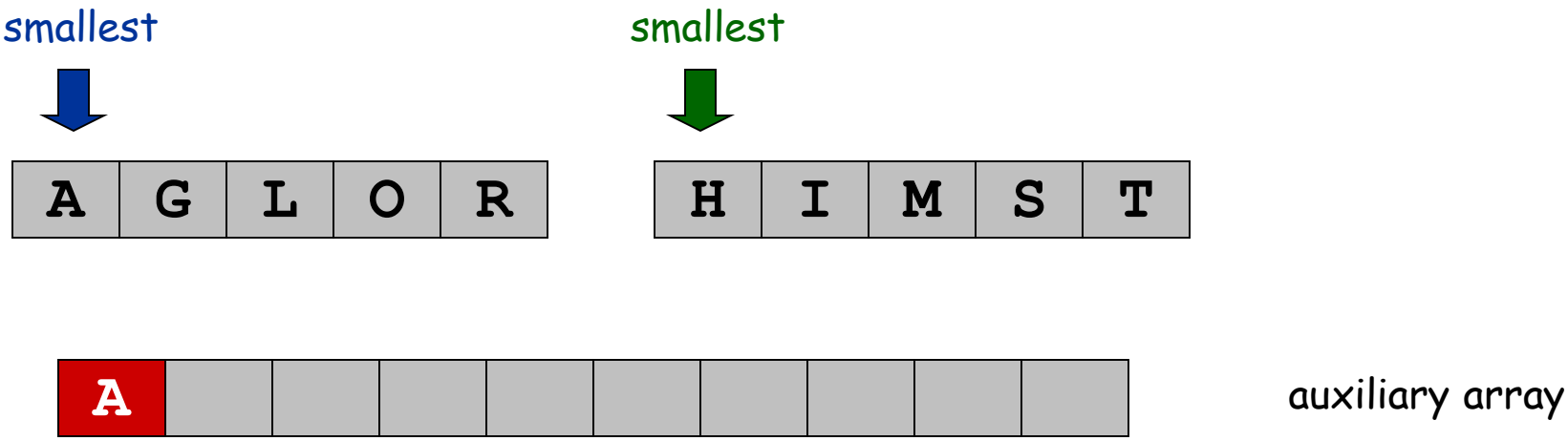
- Linear number of comparisons.
- Use temporary array.



Merging

Merge.

- Keep track of smallest element in each sorted half.
- Insert smallest of two elements into auxiliary array.
- Repeat until done.



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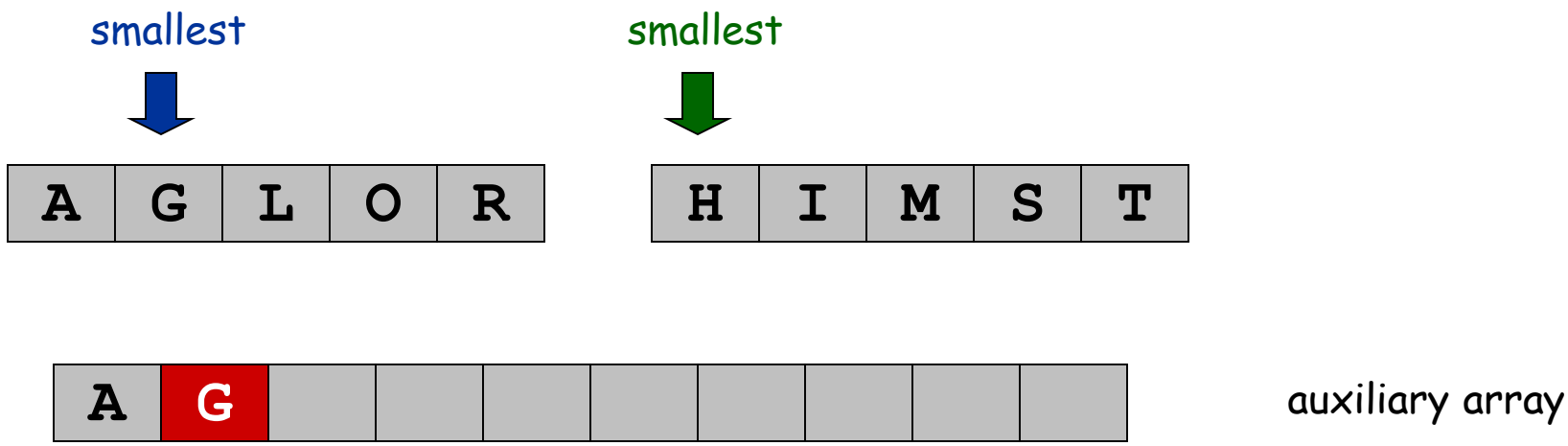

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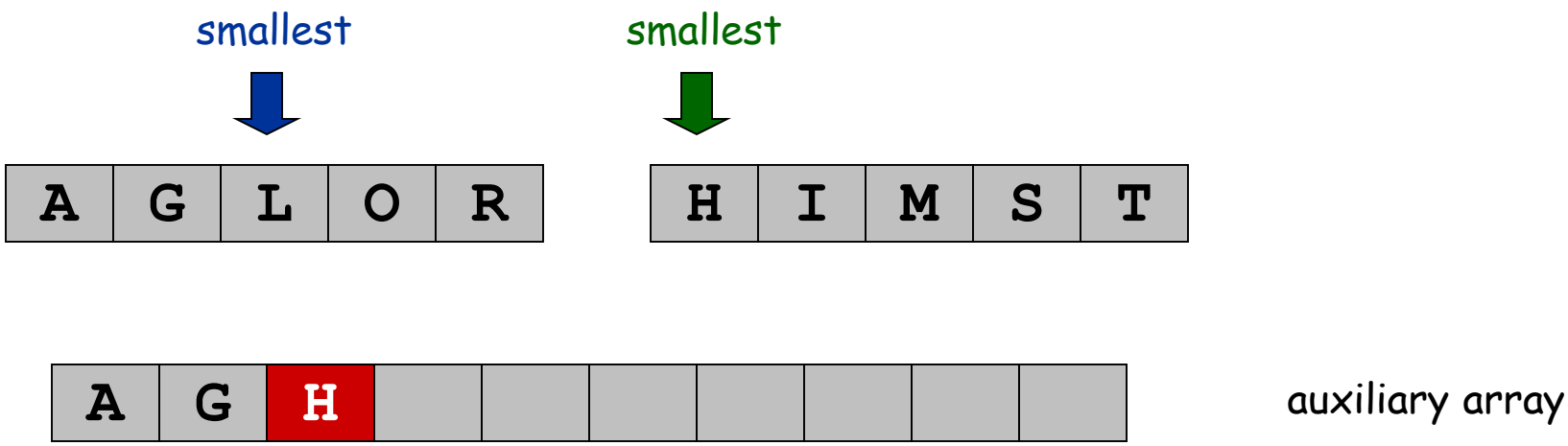

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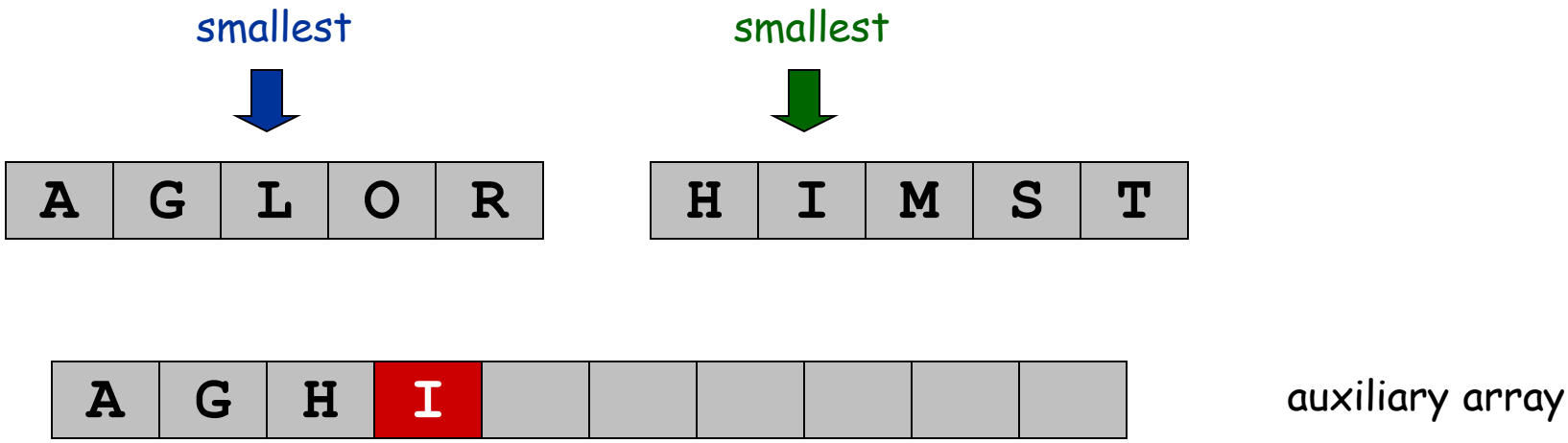

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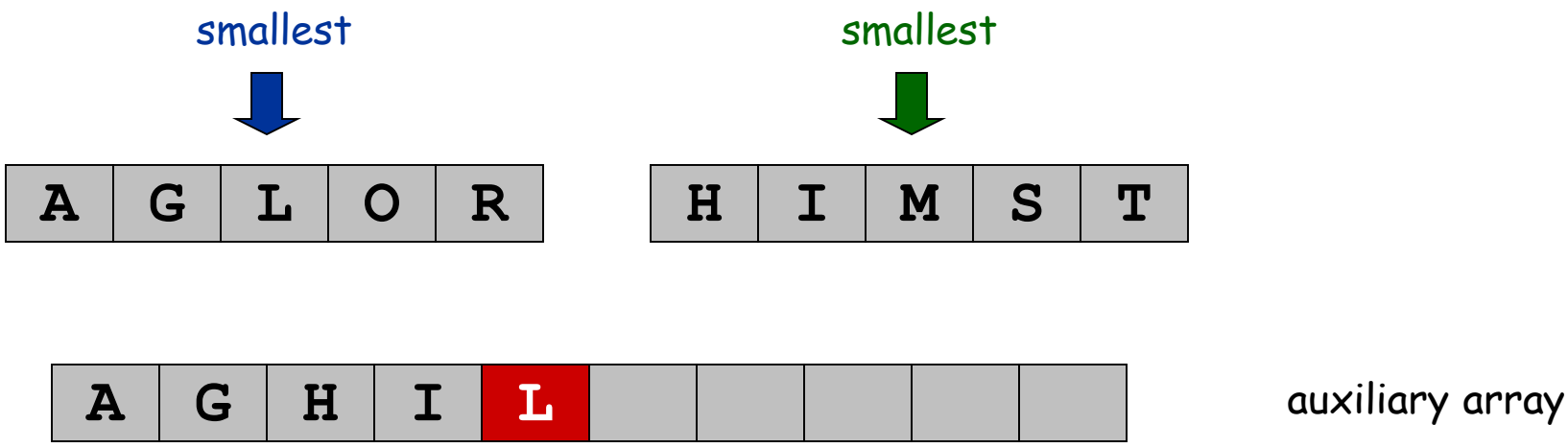

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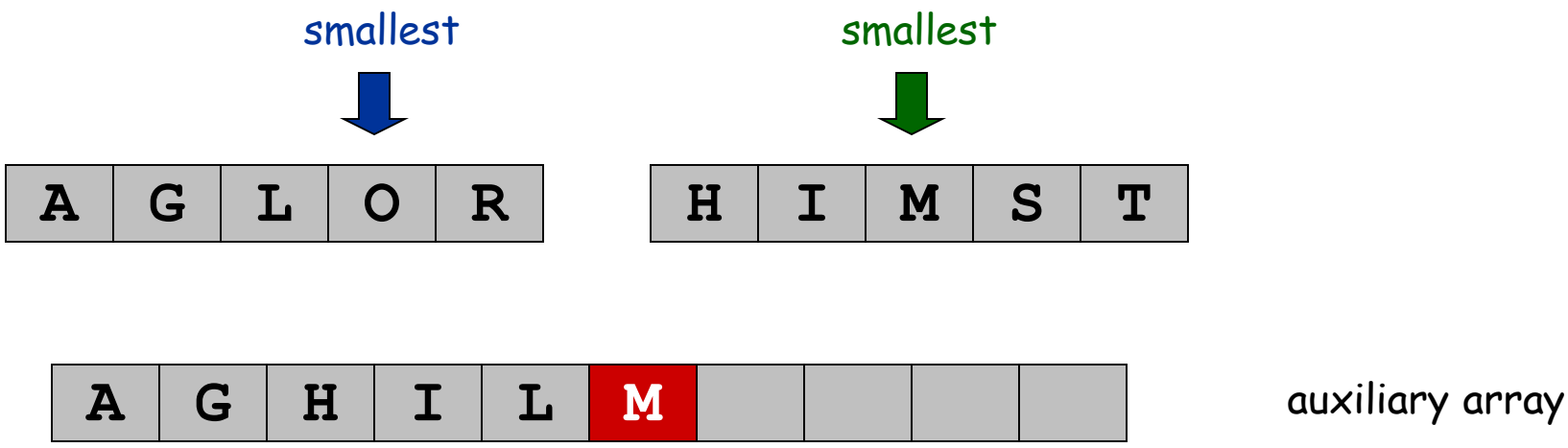

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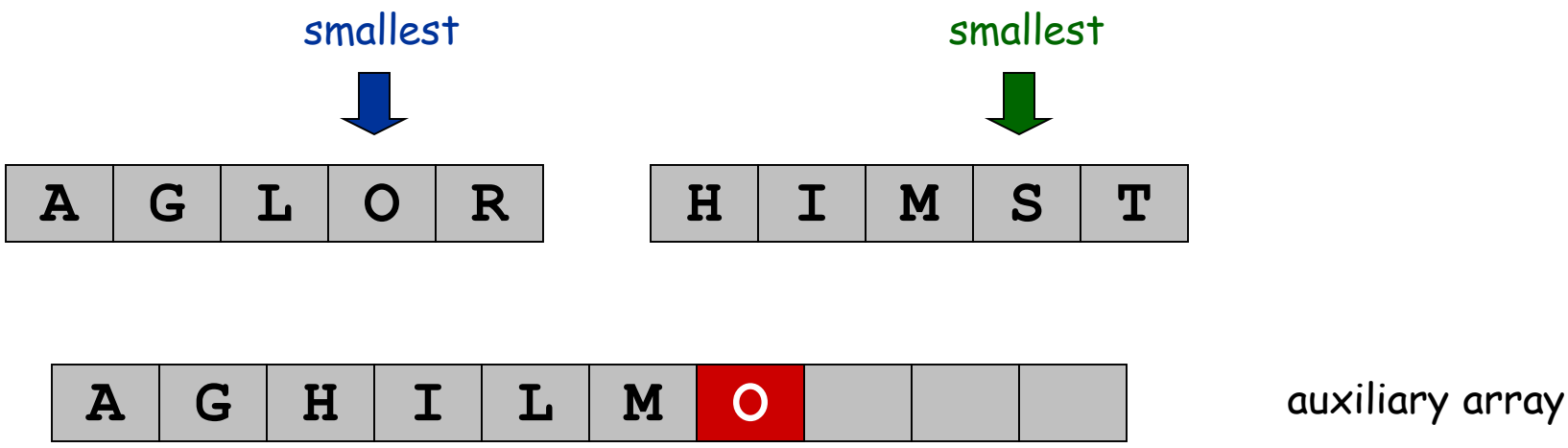

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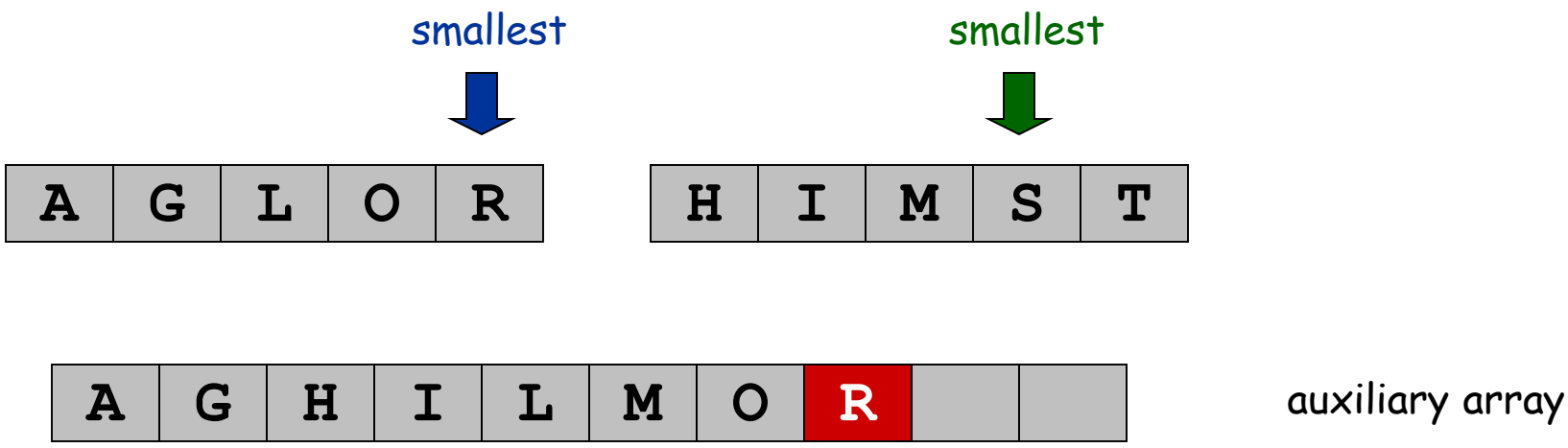

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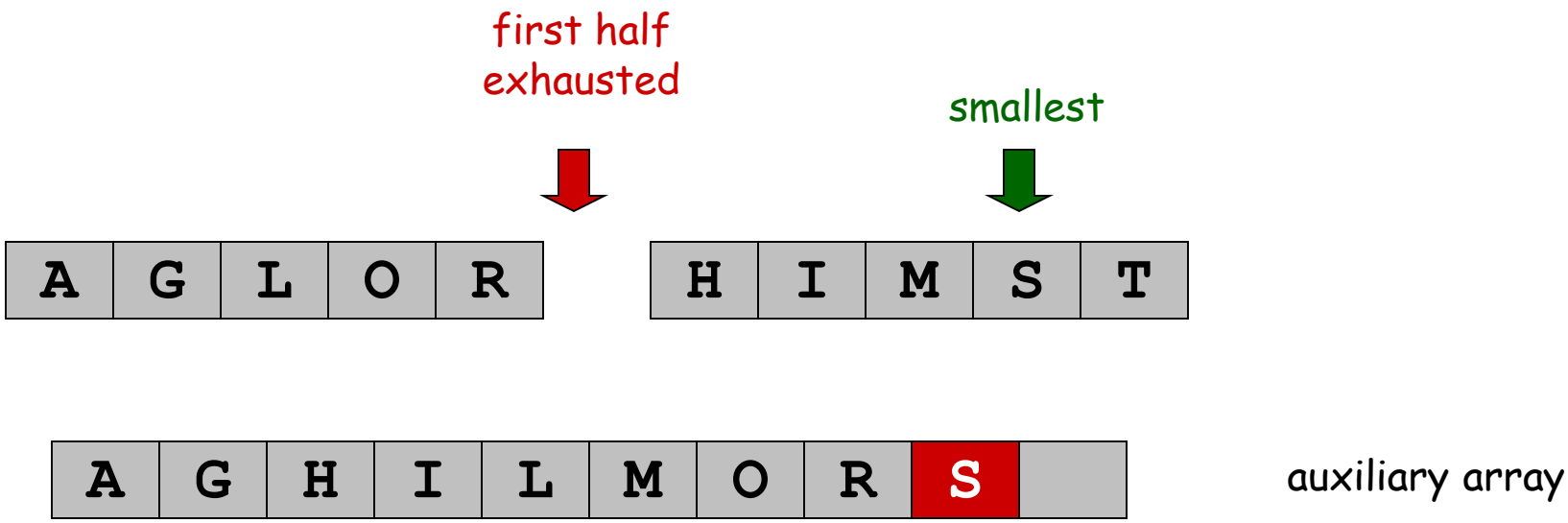

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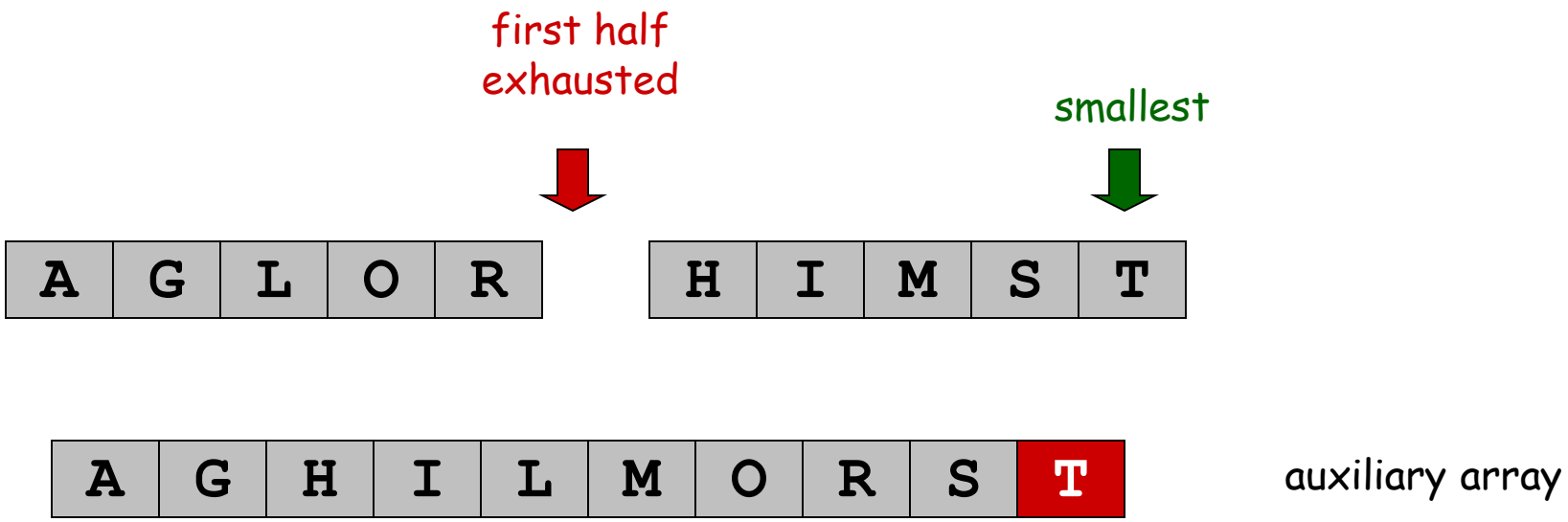

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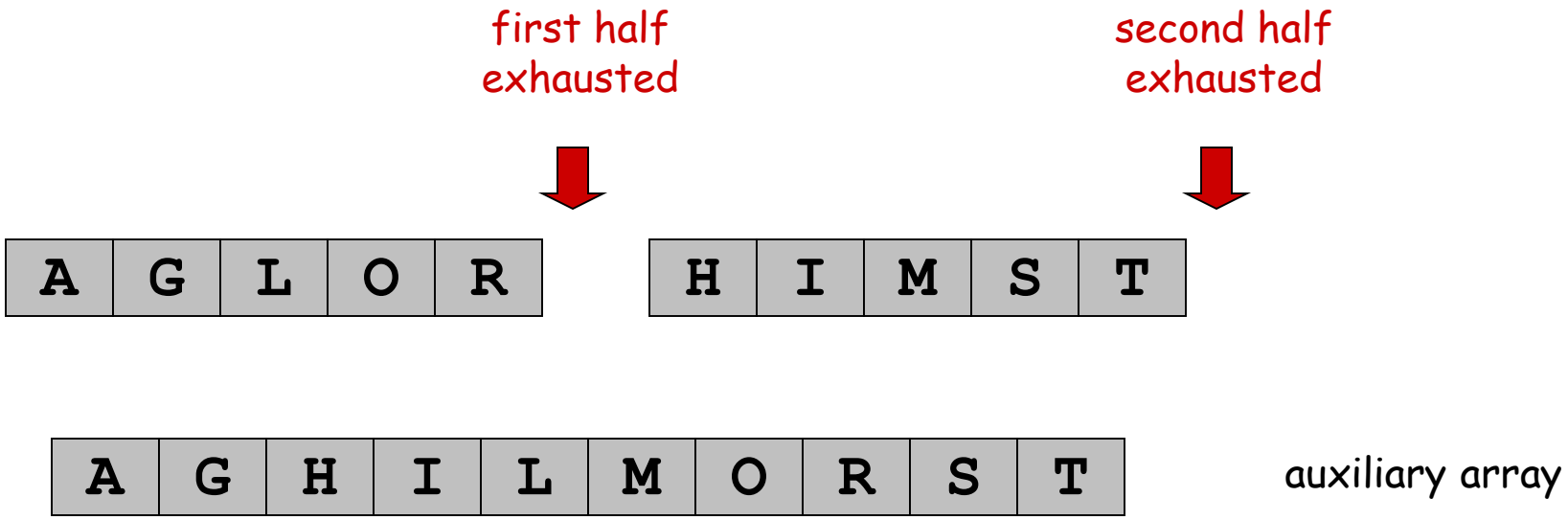


A Useful Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

Mergesort recurrence.

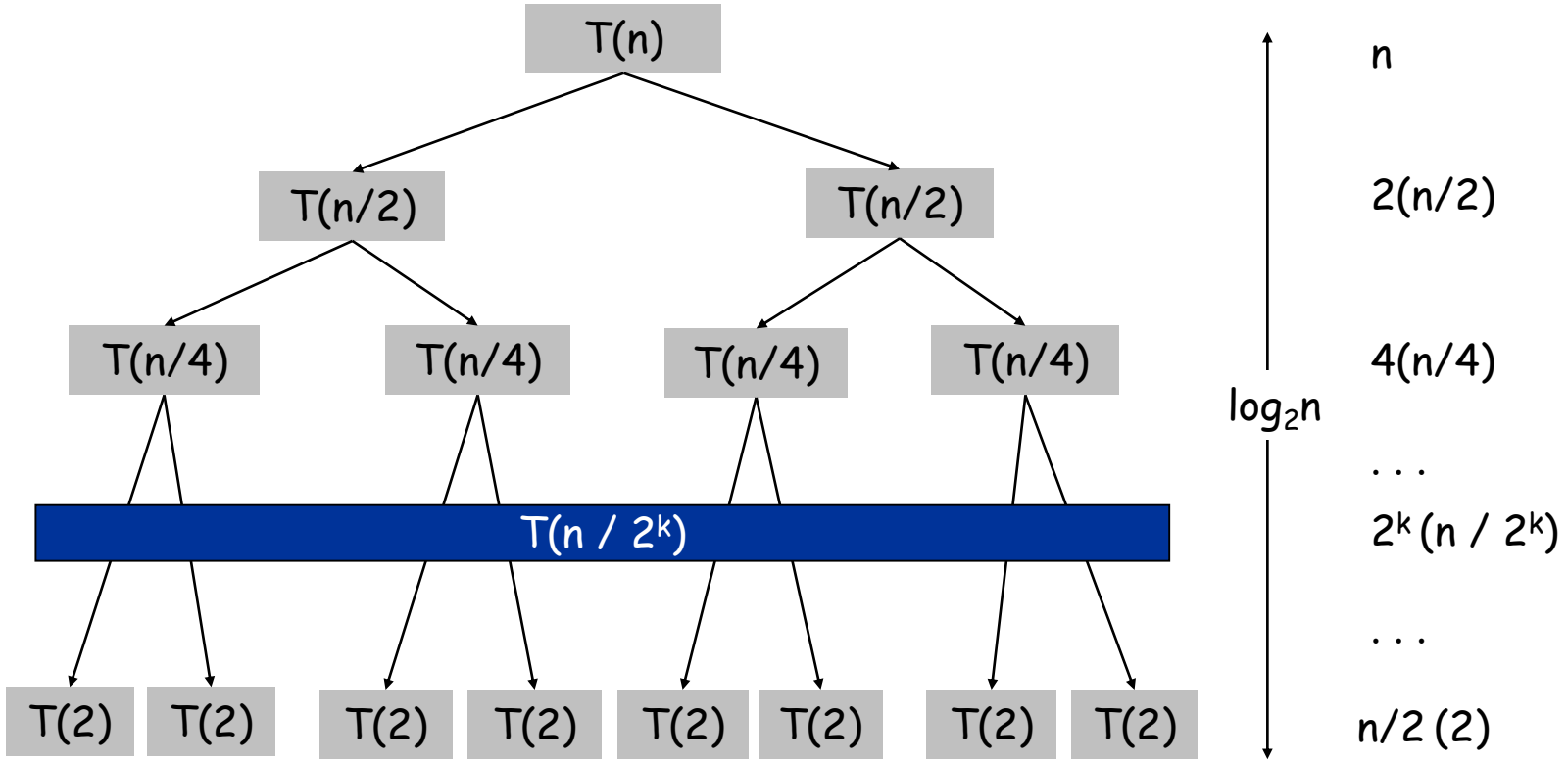
$$T(n) \leq \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. T(n) = $O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace $\leq$ with =.

Proof by Recursion Tree

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$



Proof by Telescoping

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:

$$\begin{aligned} \frac{T(n)}{n} &= \frac{2T(n/2)}{n} + 1 \\ &= \frac{T(n/2)}{n/2} + 1 \\ &= \frac{T(n/4)}{n/4} + 1 + 1 \\ &\cdots \\ &= \frac{T(n/n)}{n/n} + \underbrace{1 + \cdots + 1}_{\log_2 n} \\ &= \log_2 n \end{aligned}$$

Proof by Induction

Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: T(n) = n $\log_2$ n.
- Goal: show that T(2n) = 2n $\log_2$ (2n).

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n\log_2 n + 2n \\ &= 2n(\log_2(2n) - 1) + 2n \\ &= 2n\log_2(2n) \end{aligned}$$

Analysis of Mergesort Recurrence

Claim. If T(n) satisfies the following recurrence, then $T(n) \le n \lceil \lg n \rceil$.

$\lg n$ ↑ $\log_2 n$

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\le n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\le n(\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \lceil 2^{\lceil \lg n \rceil} / 2 \rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\le \lceil \lg n \rceil - 1 \end{aligned}$$

5.2 (aside) "Master Theorem"

Master Theorem

(*) Divide-and-conquer algorithms commonly follow a generic pattern: they tackle a problem of size n by recursively solving, say, a subproblem of size n/b and then combining these answers.

There exists a closed-form solution to this general recurrence so that we no longer need to solve it explicitly in each new instance. This approach is called the Master Theorem.

Master Theorem

(*) Divide-and-conquer algorithms commonly follow a generic pattern: they tackle a problem of size n by recursively solving, say, a subproblem of size n/b and then combining these answers.

Master Theorem. If $T(n)=aT(n/b)+O(n^d)$ for some constants, $a > 0$, $b > 1$, and $d \geq 0$, then:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \quad \text{case(1)} \\ O(n^d \log n) & \text{if } d = \log_b a \quad \text{case(2)} \\ O(n^{\log_b a}) & \text{if } d < \log_b a \quad \text{case(3)} \end{cases}$$

This lone theorem tells us the running times of most of the divide-and-conquer procedures we will use.

Intuition: Case 1 – recursion tree is "leaf heavy"
Case 2 – work to split/recombine a problem is comparable in subproblems
Case 3 – recursion tree is "root heavy"

Master Theorem

(*) Master Theorem. If $T(n)=aT(n/b)+O(n^d)$ for some constants, $a > 0$, $b > 1$, and $d \geq 0$, then:

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Example #1: Mergesort.

$$T(n)=2T(n/2)+O(n)$$

Master Theorem

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Example #1: Mergesort.

$$T(n)=2T(n/2)+O(n)$$

Here $a=2$, $b=2$ and $d=1$. Since $d=1=\log_b a=\log_2 2$, the Master Theorem asserts:

$T(n)=O(n \log n)$, as was previously shown.

Master Theorem

(*) Master Theorem. If $T(n)=aT(n/b)+O(n^d)$ for some constants, $a > 0$, $b > 1$, and $d \geq 0$, then:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Example #2:

$$T(n)=8T(n/2)+1000n^2$$

Here $a=8$, $b=2$ and $d=2$. Since $d=2<\log_b a=\log_2 8=3$, the Master Theorem asserts:

$T(n)=O(n^3)$.

Master Theorem

(*) Master Theorem. If $T(n)=aT(n/b)+O(n^d)$ for some constants, $a > 0$, $b > 1$, and $d \geq 0$, then:

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Example #3:

$$T(n)=2T(n/2)+n^2$$

Here a=2, b=2 and d=2. Since $d=2>\log_b a=\log_2 2=1$, the Master Theorem asserts:

$T(n)=O(n^2)$.

5.3 Counting Inversions

Counting Inversions

Music site tries to match your song preferences with others.

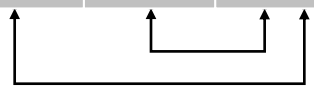
- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j inverted if i < j, but $a_i > a_j$.

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5



Inversions

3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs i and j.

Applications

Applications.

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's Tau distance).

Counting Inversions: Divide-and-Conquer

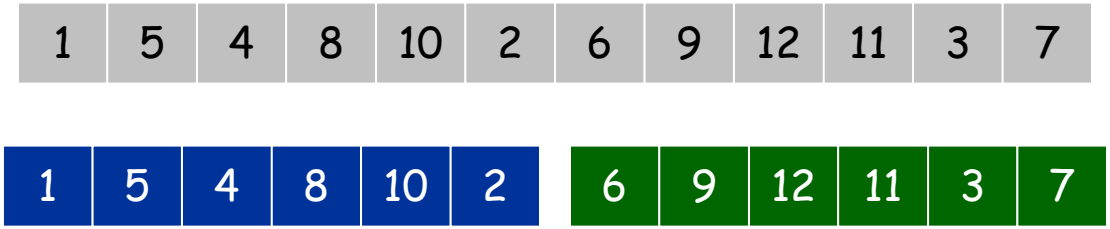
Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.



Divide: $O(1)$.

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.

Divide: $O(1)$.

Conquer: $2T(n / 2)$

5 blue-blue inversions

5-4, 5-2, 4-2, 8-2, 10-2

8 green-green inversions

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.

Divide: $O(1)$.

Conquer: $2T(n / 2)$

5 blue-blue inversions

8 green-green inversions

9 blue-green inversions
5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

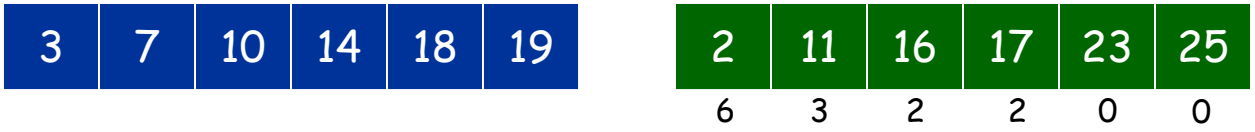
Total = 5 + 8 + 9 = 22.

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.

to maintain sorted invariant



13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

Count: O(n)

Merge: O(n)

$$T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) \Rightarrow T(n) = O(n \log n)$$

Counting Inversions: Implementation

Pre-condition. [Merge-and-Count] A and B are sorted.

Post-condition. [Sort-and-Count] L is sorted.

```
Sort-and-Count(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   (r_A, A) ← Sort-and-Count(A)
   (r_B, B) ← Sort-and-Count(B)
   (r , L) ← Merge-and-Count(A, B)

   return r = r_A + r_B + r and the sorted list L
}
```

Merge and Count

Merge and count step.

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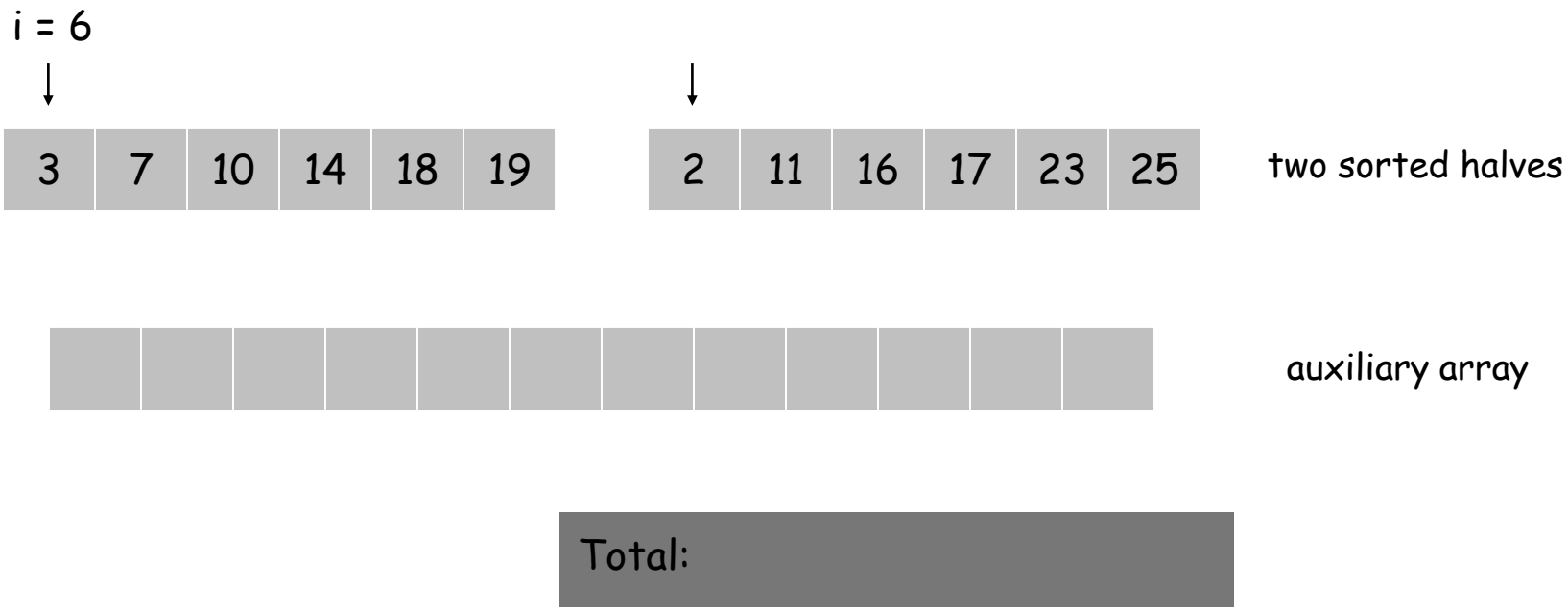

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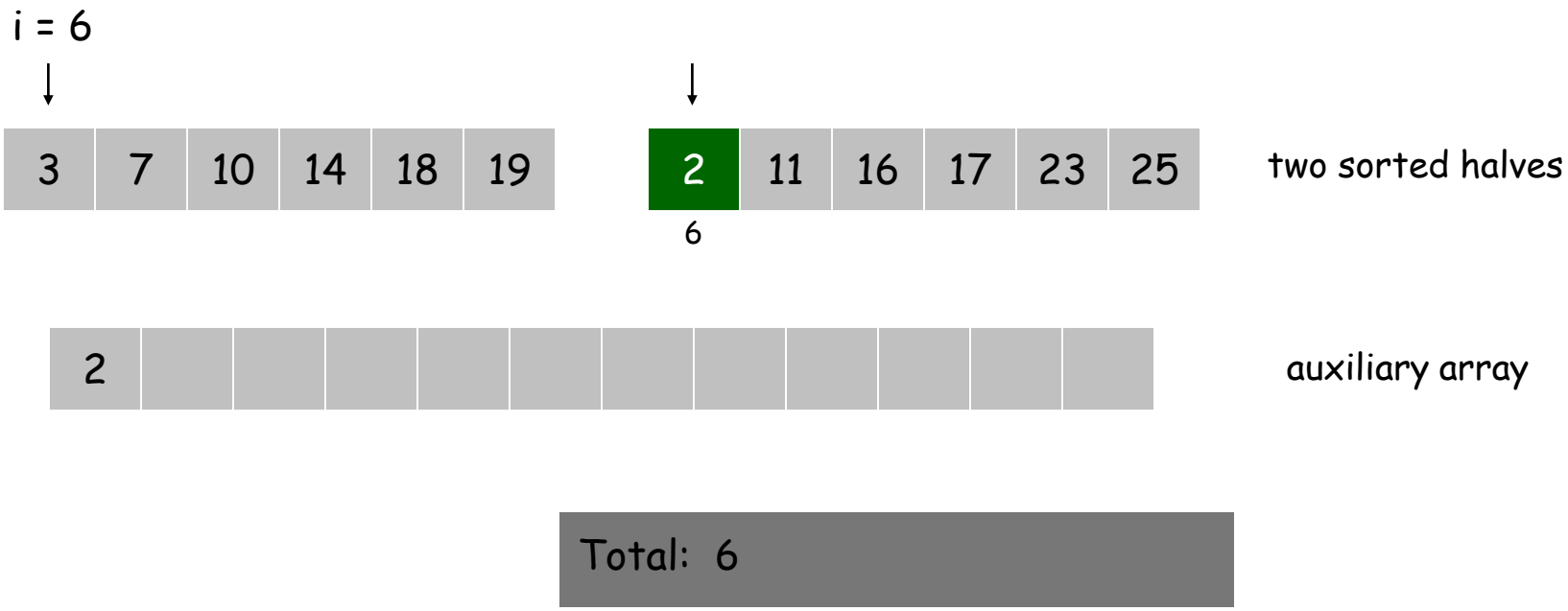

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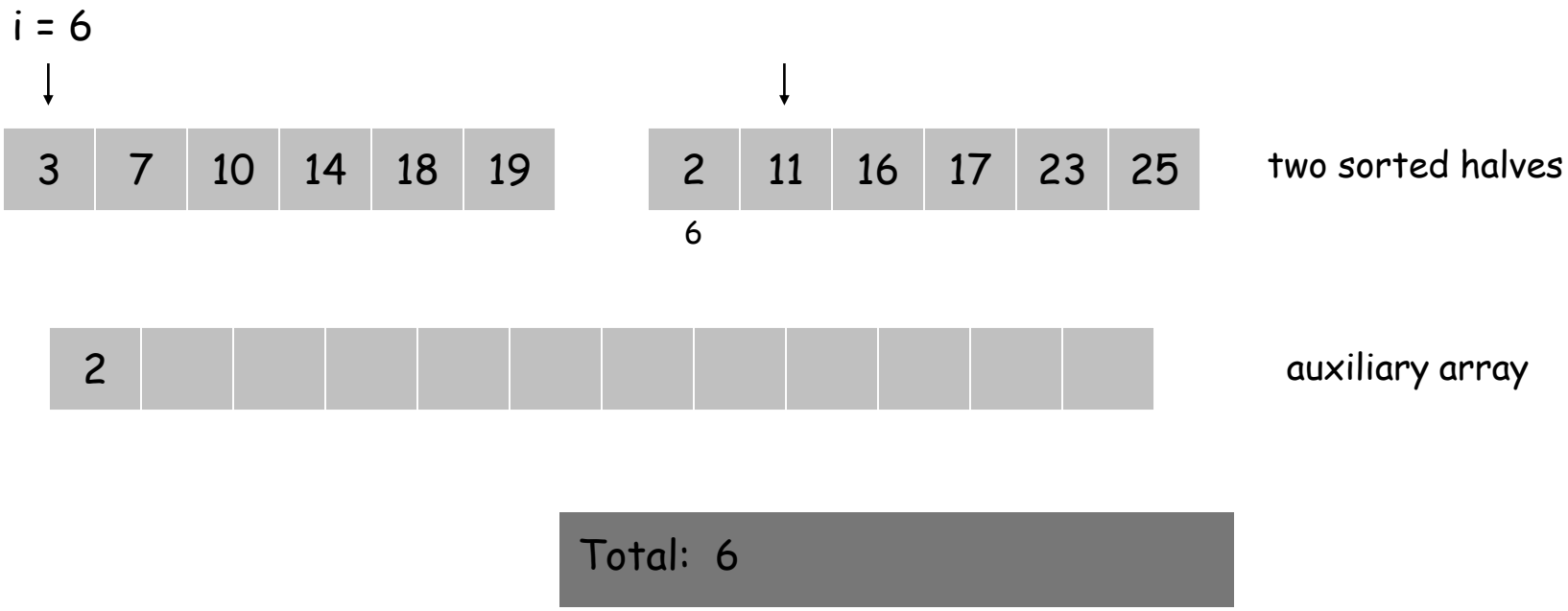

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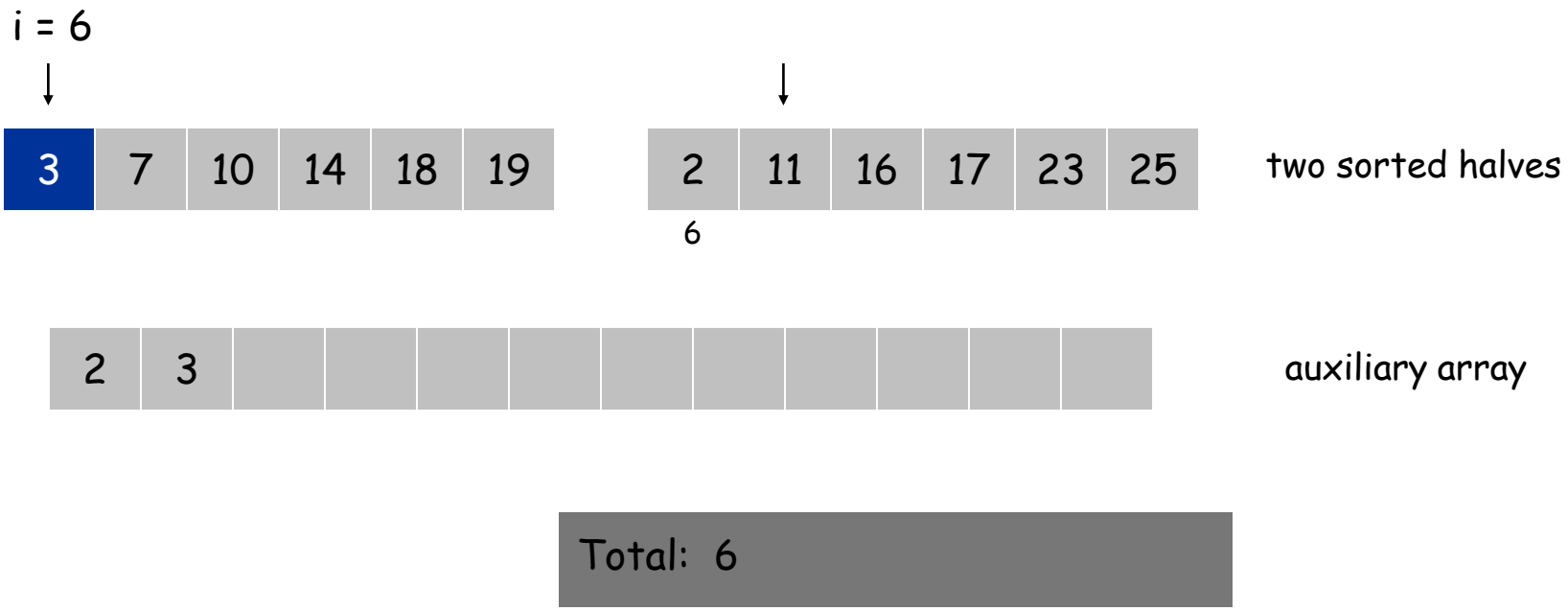

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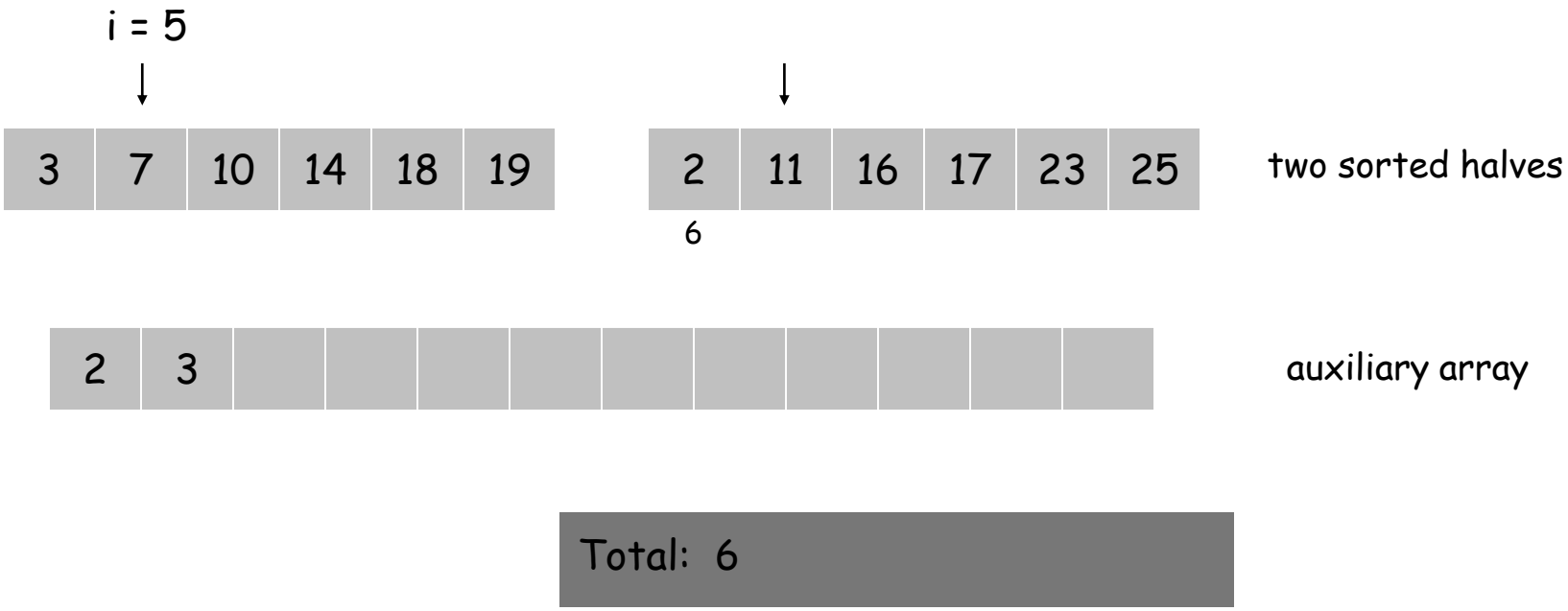

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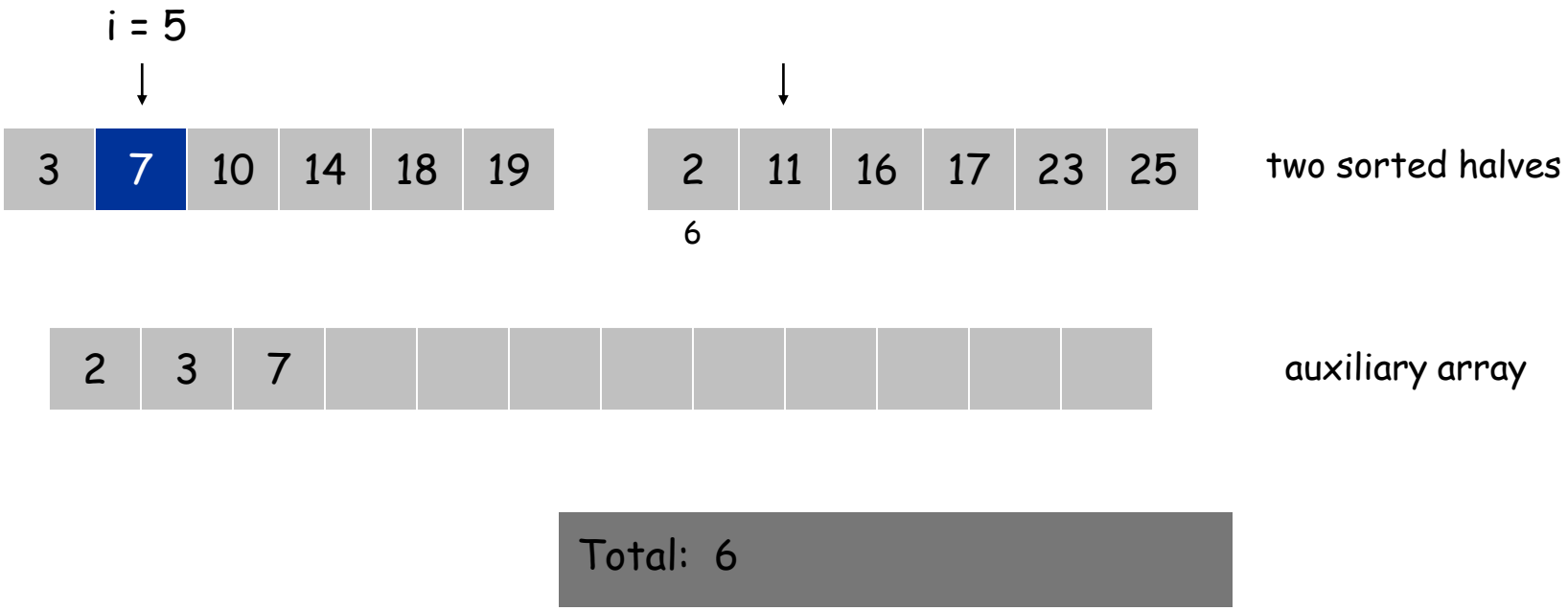

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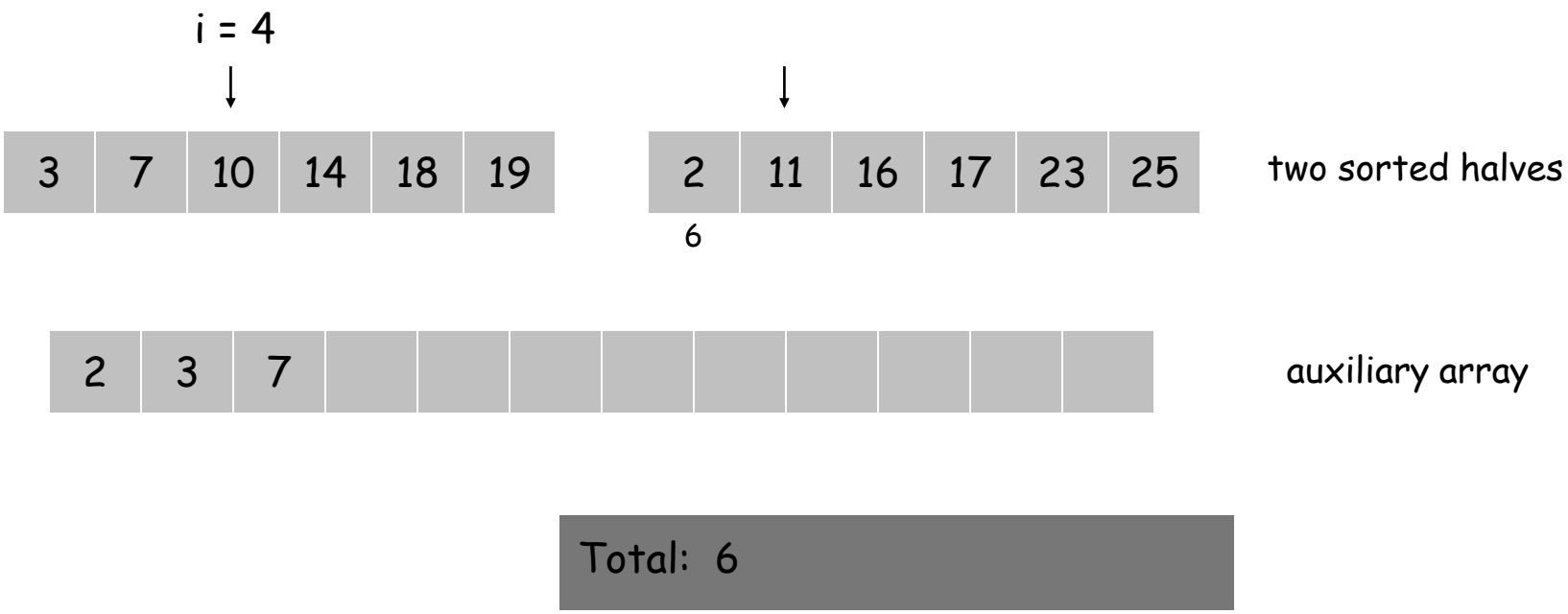

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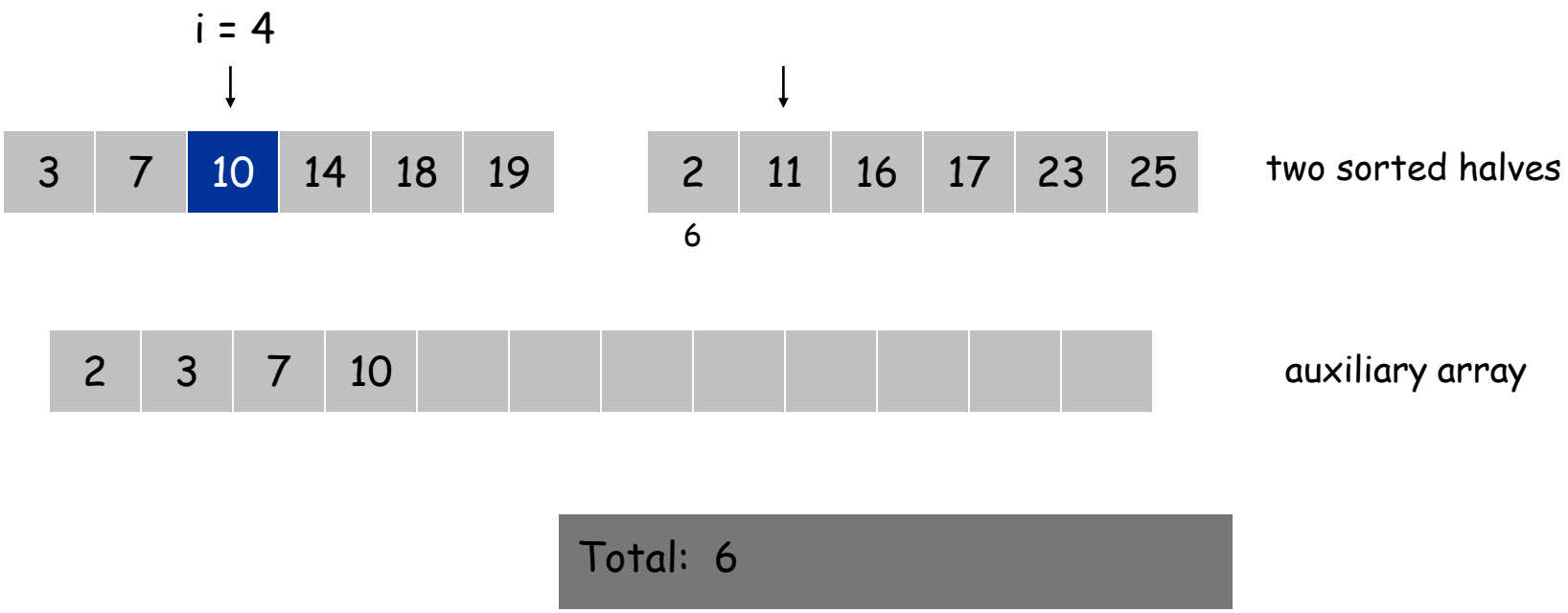

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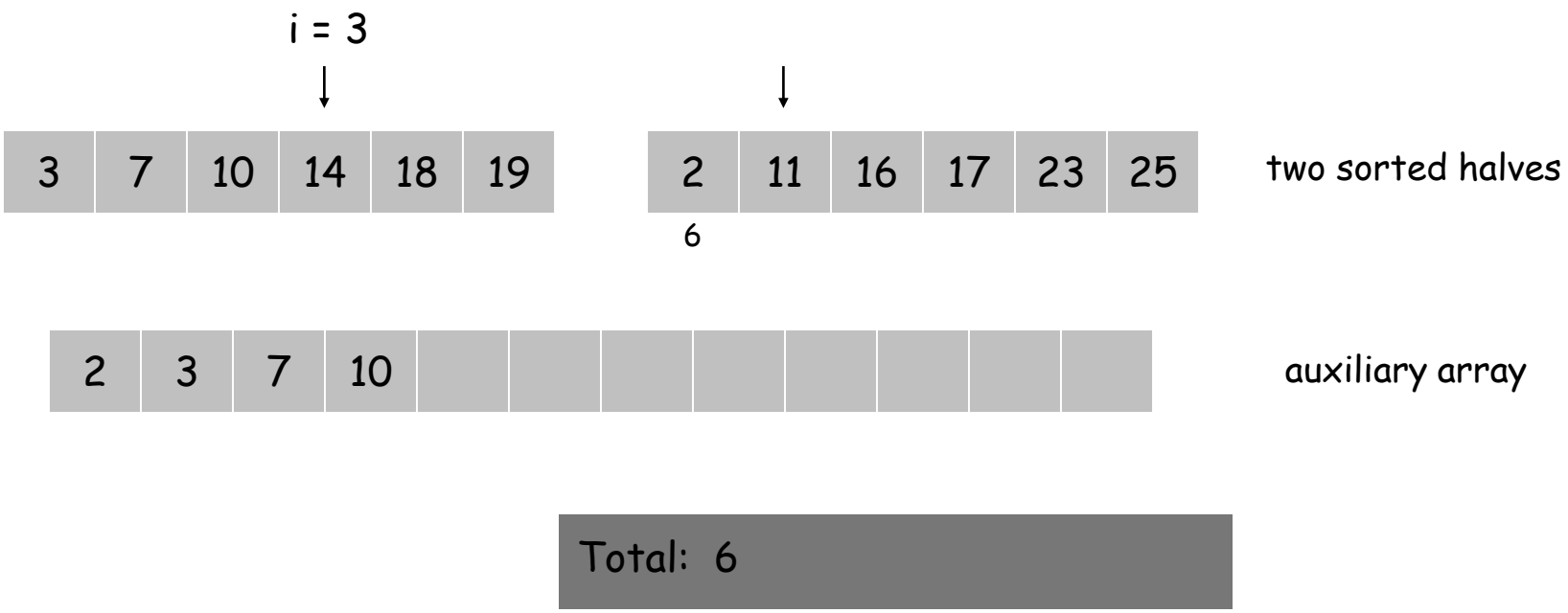

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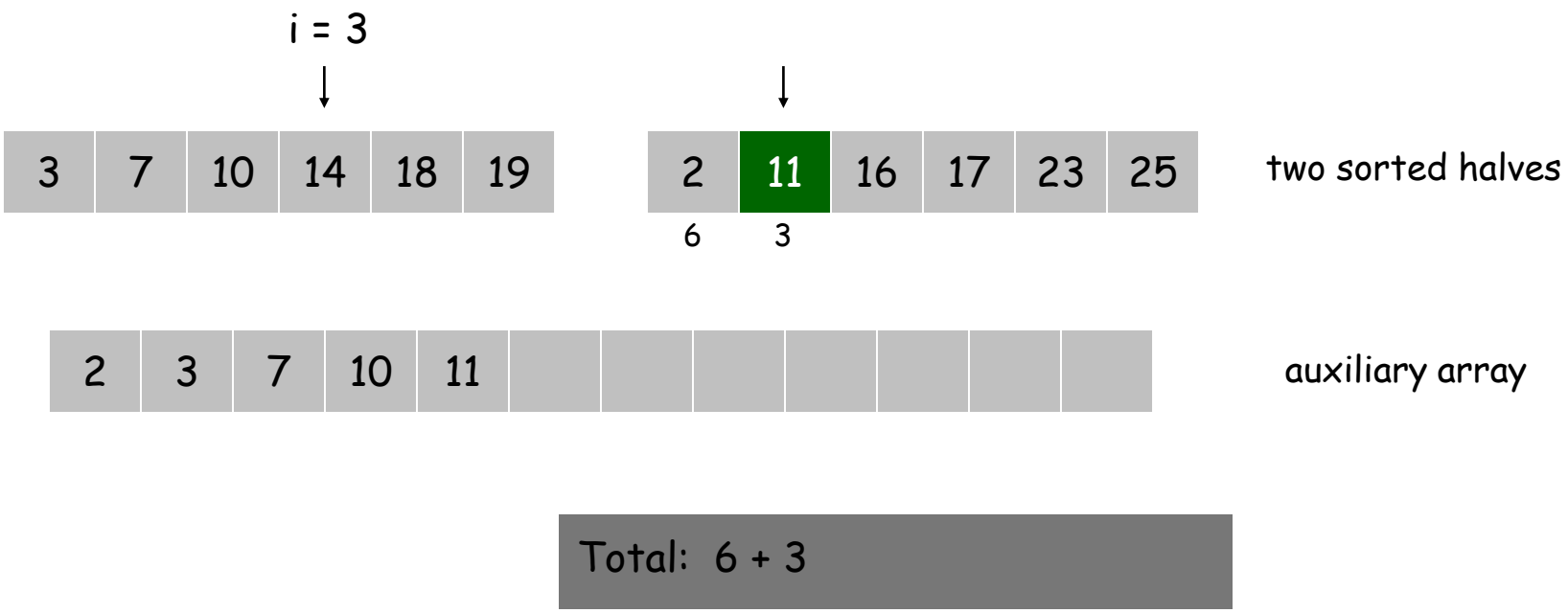

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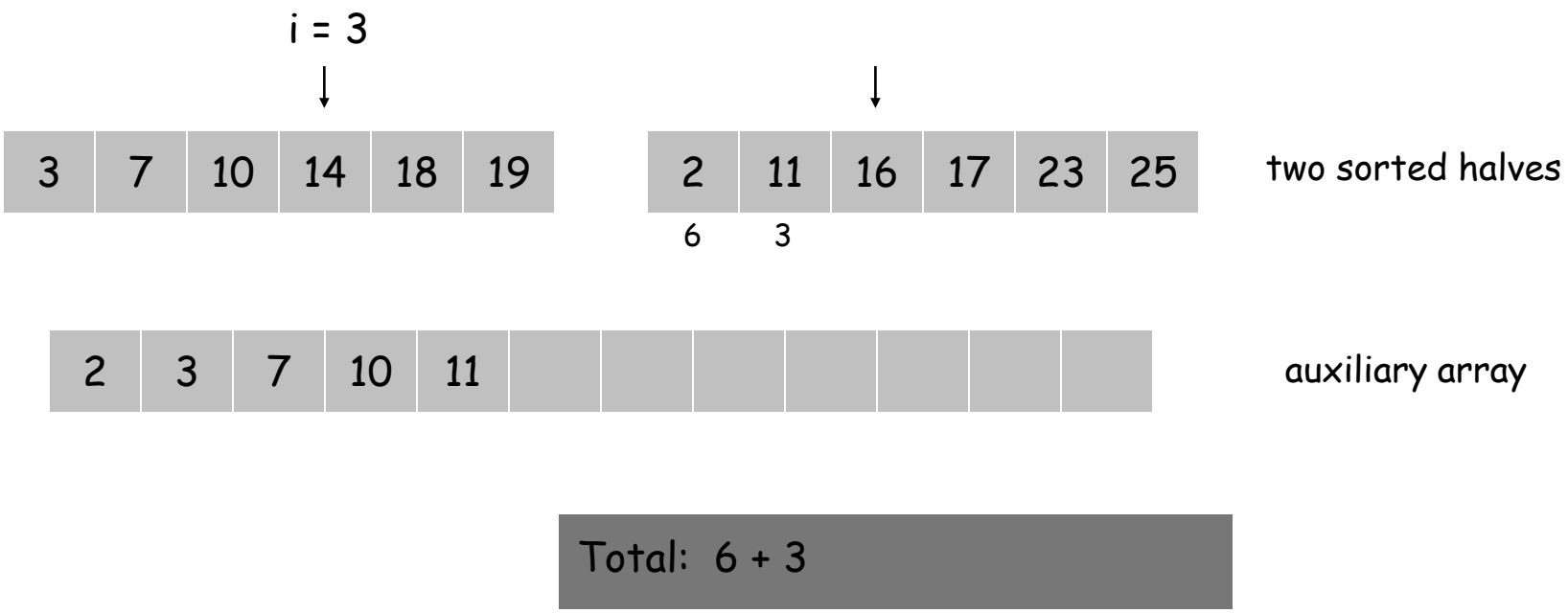

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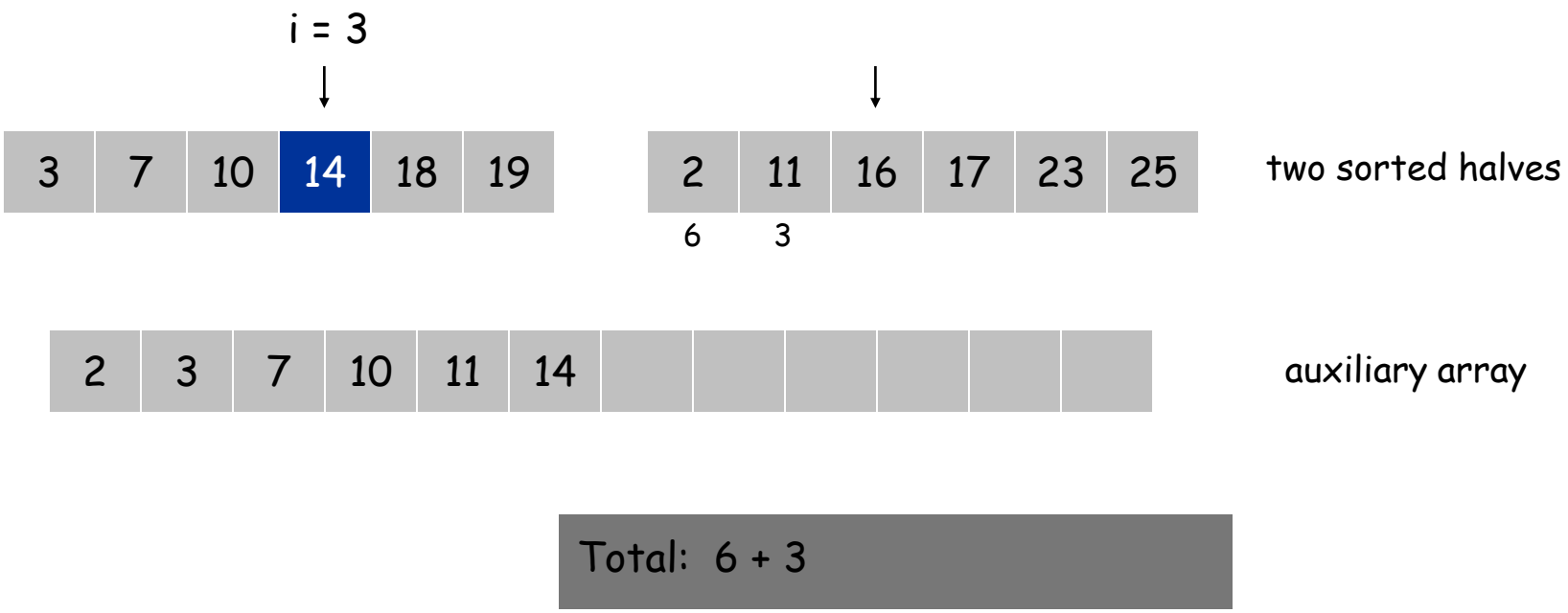

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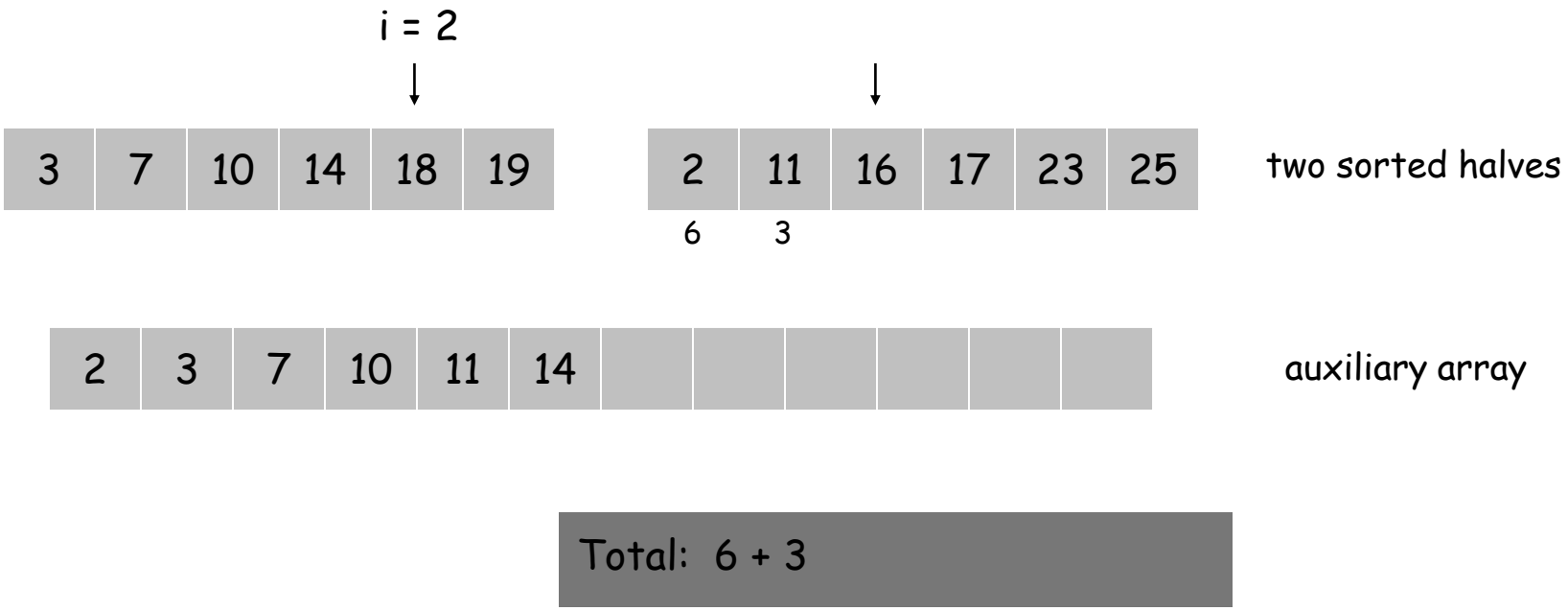

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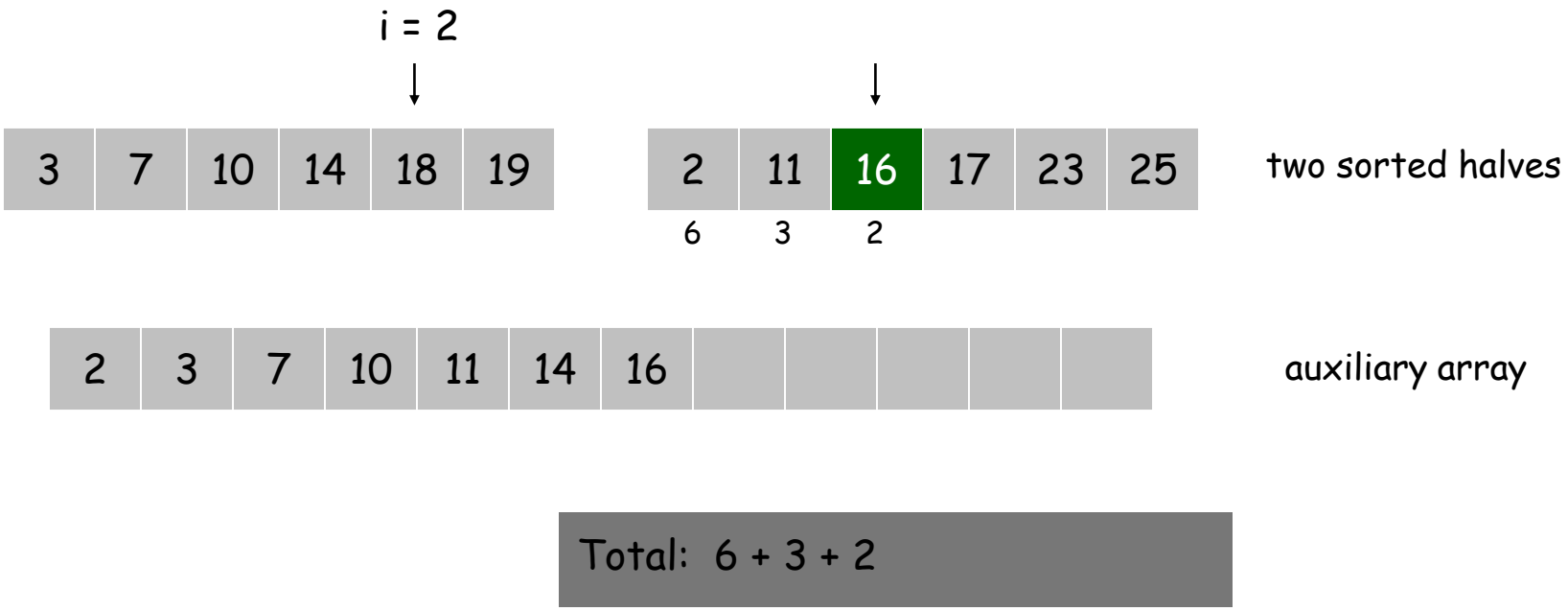

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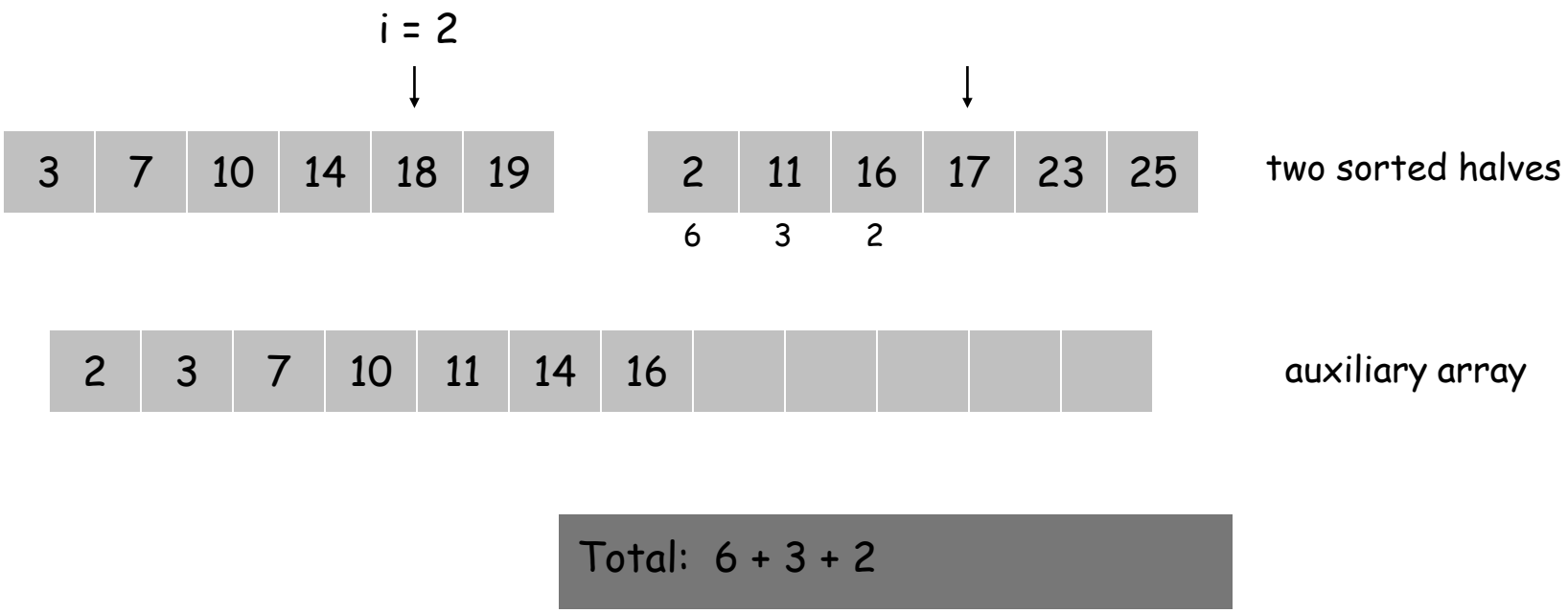

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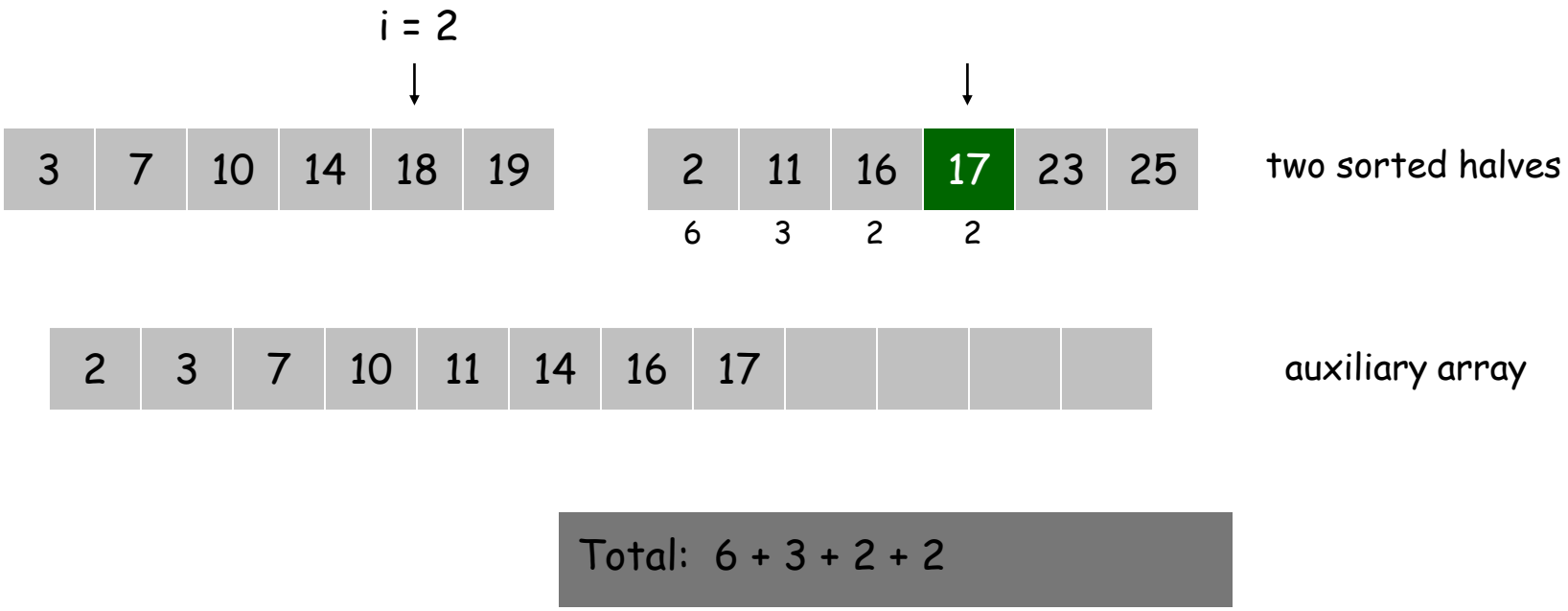

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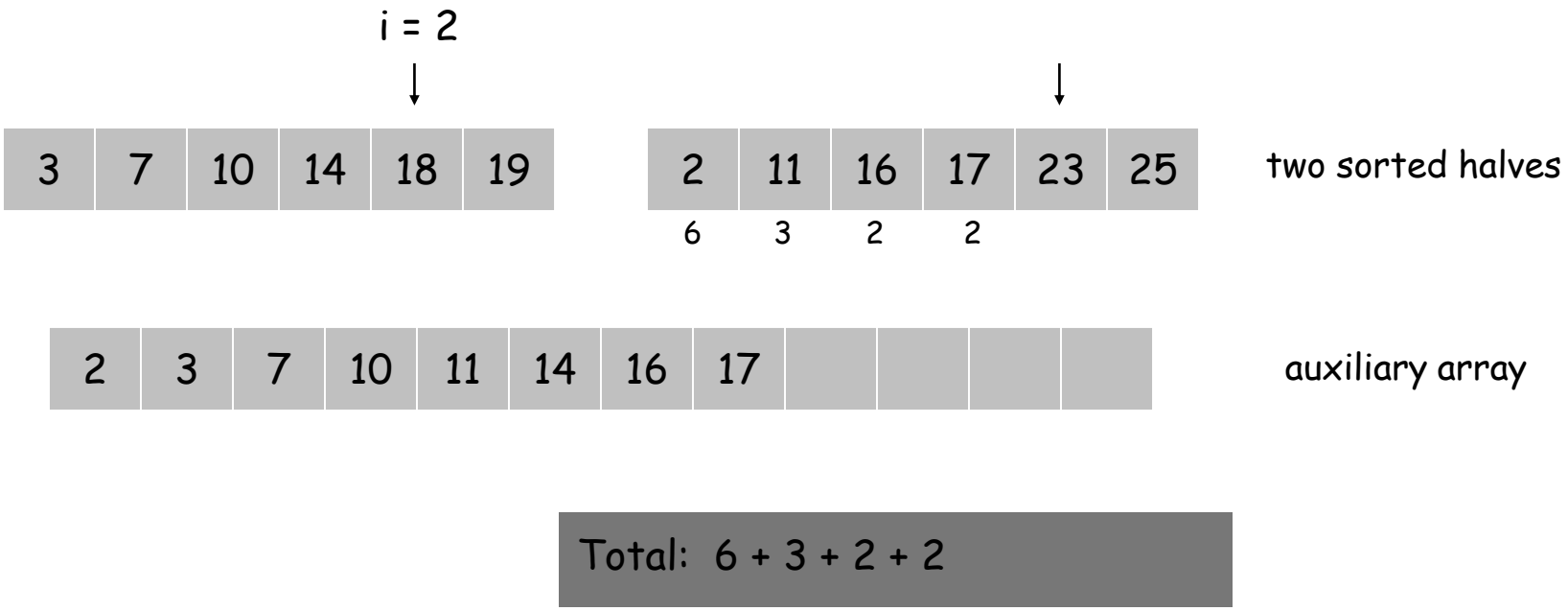

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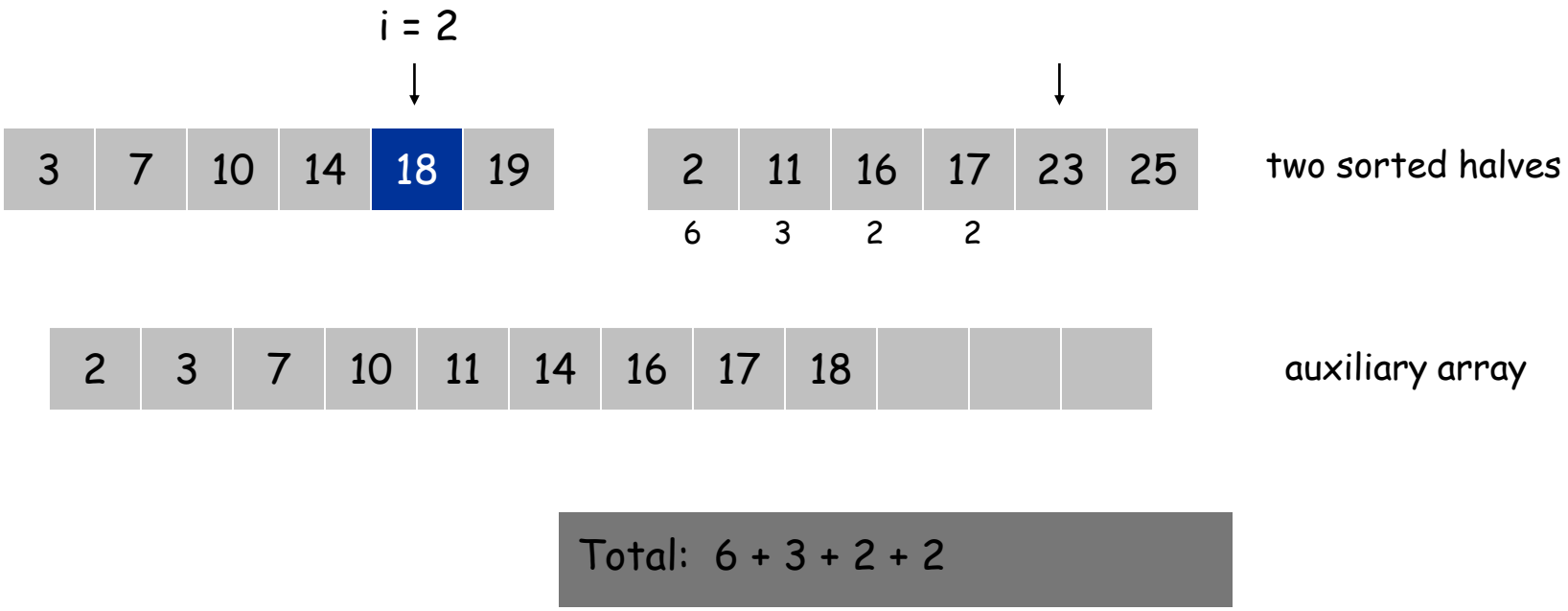

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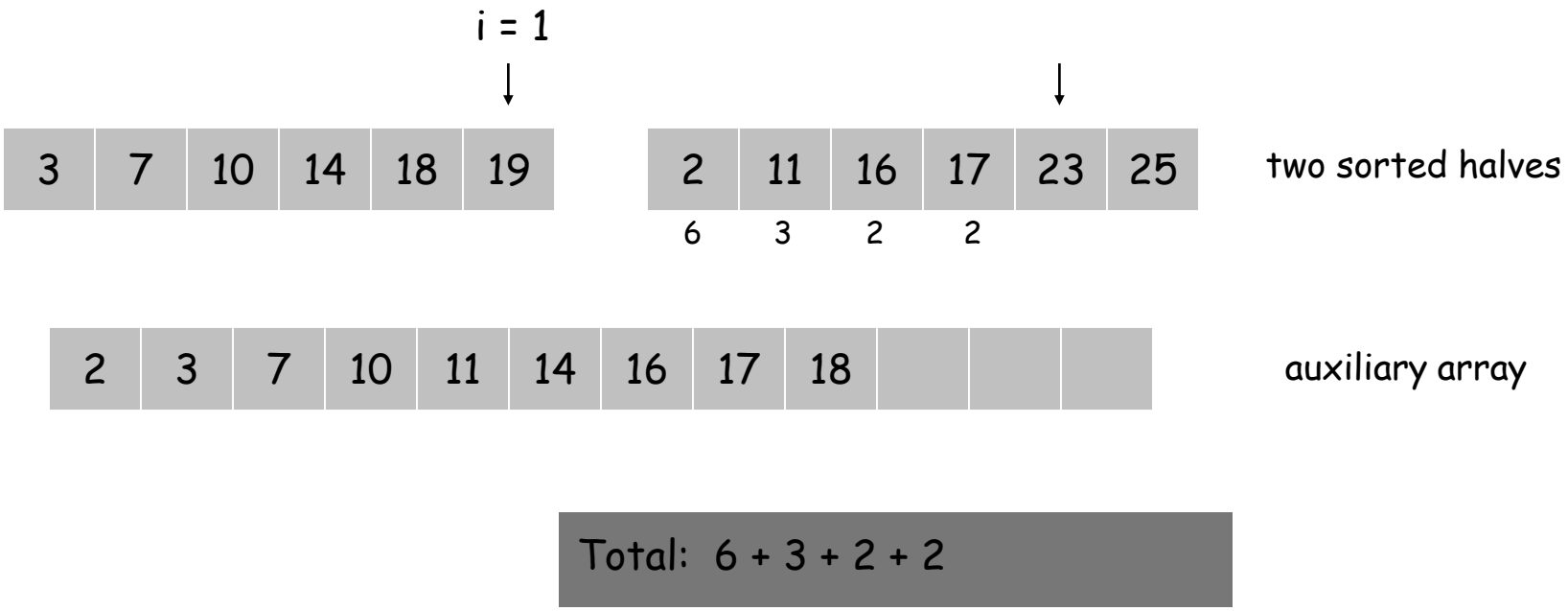

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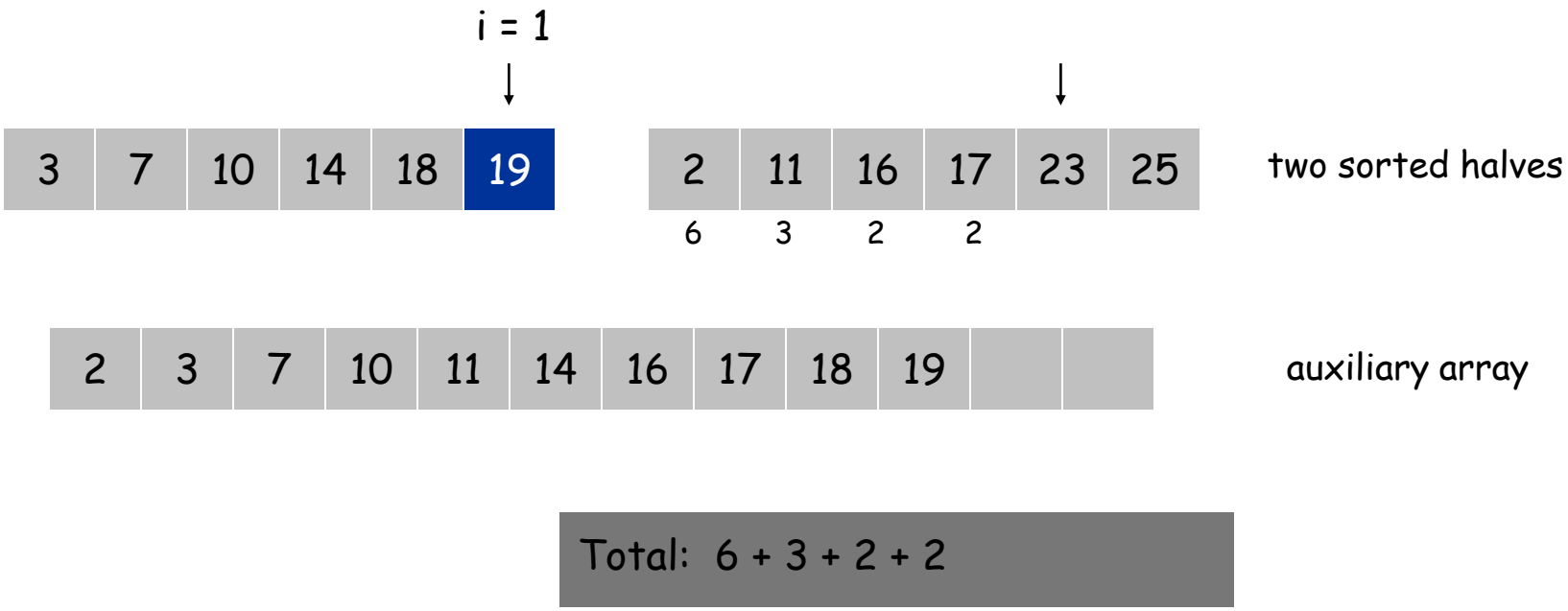

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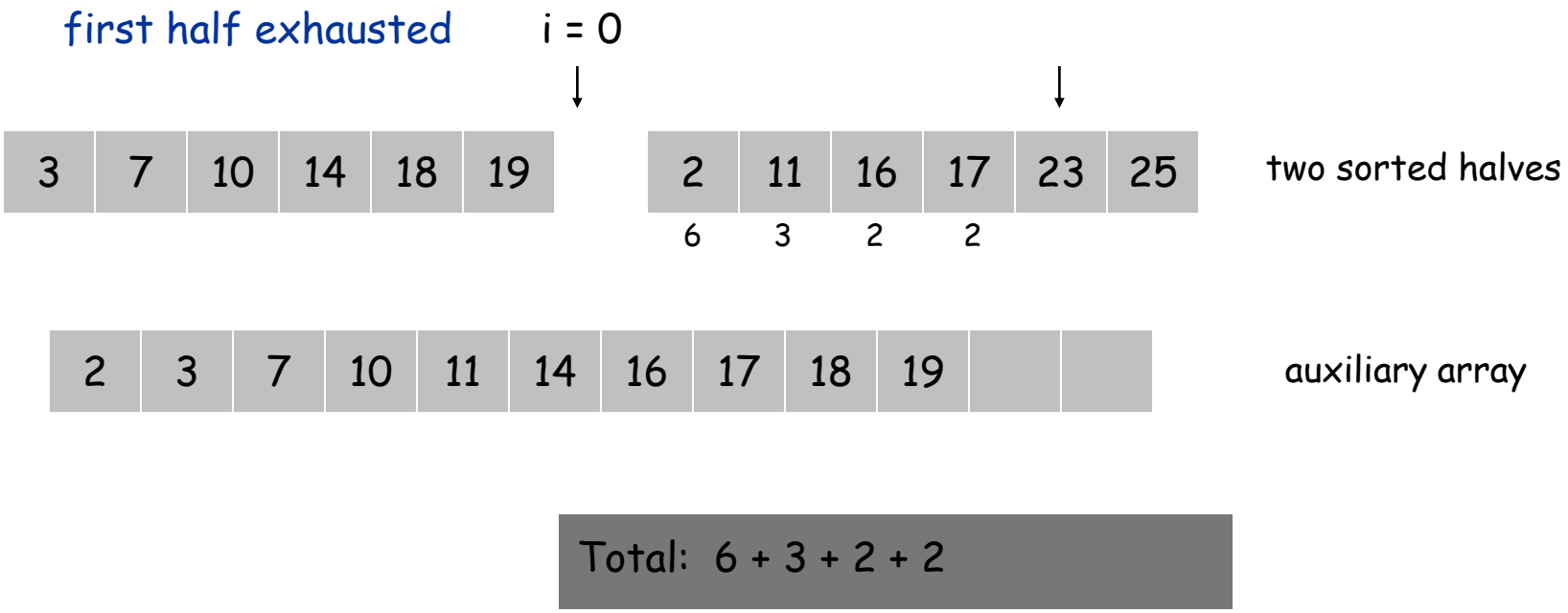

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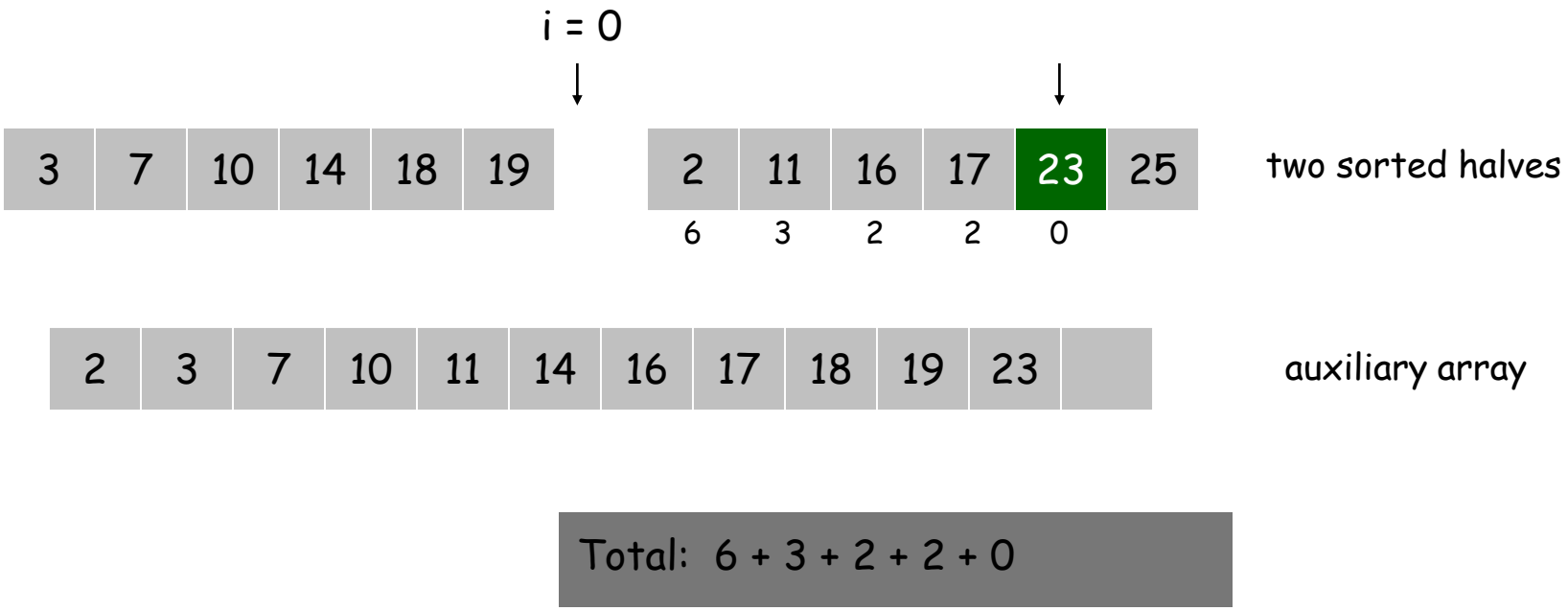

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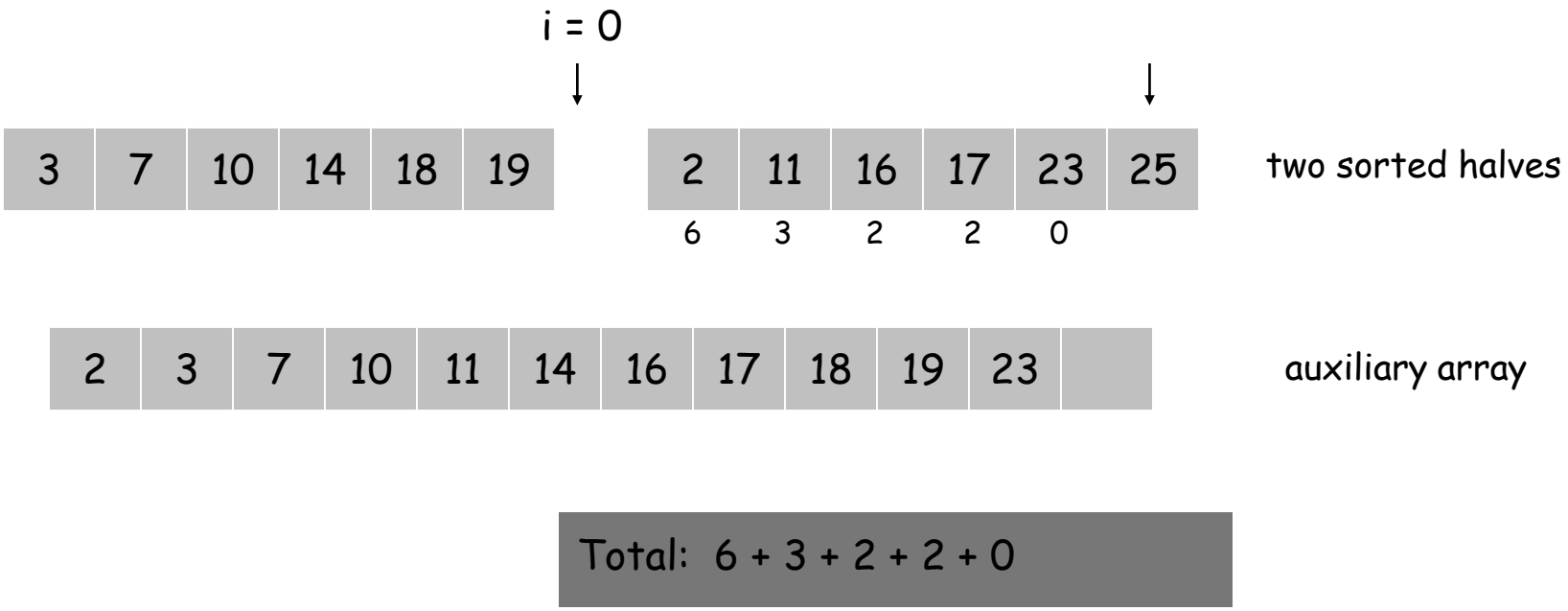

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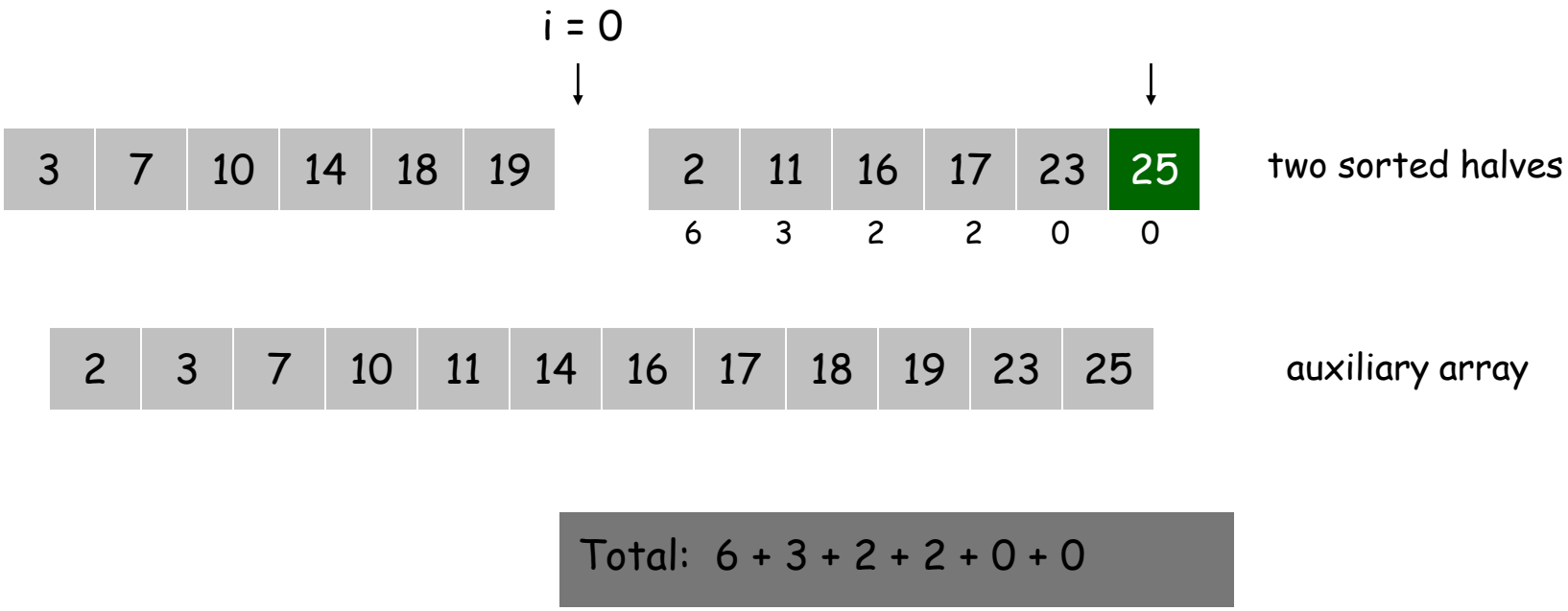

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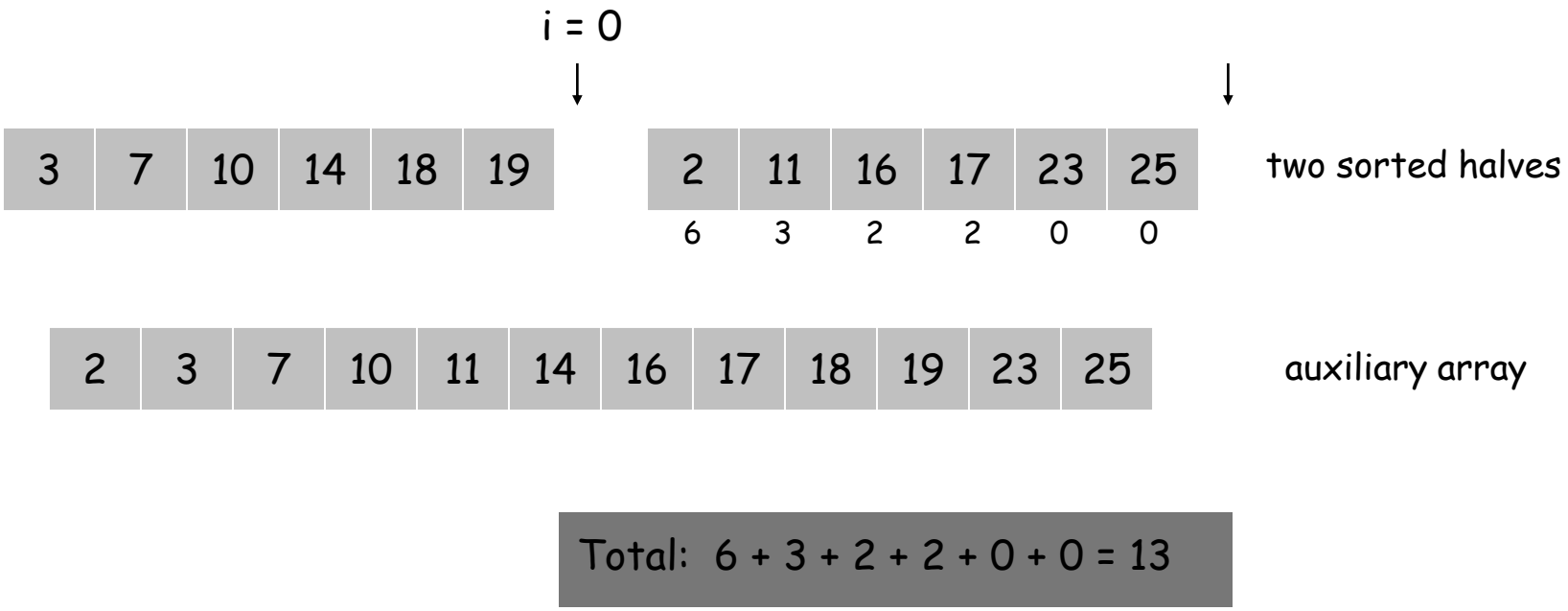

HW#3, Kleinberg #5.1

You are interested in analyzing some hard-to-obtain data from two separate databases. Each database contains n numerical values – so there are 2n values total – and you may assume that no two values are the same. You'd like to determine the median of this set of 2n values, which we will define here to be the nth smallest value.

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However, the only way you can access these values is through queries to the databases. In a single query, you can specify a value k to one of the two databases, and the chosen database will return the kth smallest value that it contains. Since queries are expensive, you would like to compute the median using as few queries as possible.

Given an algorithm that finds the median value using at most $O(n \log n)$ queries.

HW#3, Kleinberg #5.1

Given an algorithm that finds the median value using at most $O(n \log n)$ queries.

Begin by querying $A(k)$ and $B(k)$ – these are the medians of the two databases, respectively.

HW#3, Kleinberg #5.1

Given an algorithm that finds the median value using at most O(n log n) queries.

Begin by querying A(k) and B(k) – these are the medians of the two databases, respectively.

Suppose, WLOG, that A(k)<B(k).

Now, B(k) is: (1) larger than the first k elements of A and, (2) naturally, larger than the first k-1 elements of B. Thus, B(k) is at least the $2k^{th}$ element in the combined database.

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This implies that B(k) is greater than the overall median. So let's eliminate the second half of the B dataset; let B' = the first k elements in B.

HW#3, Kleinberg #5.1

Begin by querying A(k) and B(k) – these are the medians of the two databases, respectively (let k = ceiling(1/2n)).

Suppose, WLOG, that A(k)<B(k).

Now, B(k) is: (1) larger than the first k elements of A and, (2) naturally, larger than the first k-1 elements of B. Thus, B(k) is at least the $2k^{th}$ element in the combined database.

This implies that B(k) is greater than the overall median. So let's eliminate the second half of the B dataset; let B' = the first k elements in B.

Now show that the elements in the first half of A (i.e. the first floor(1/2n) elements) are also less than B(k) and can be discarded.

Divide and conquer…

HW#3, Kleinberg #5.2

Recall the problem of finding the number of inversions. As in the text, we are given a sequence of n numbers $a_1,\ldots,a_n$, which we assume are all distinct, and we define an inversion to be a pair $i<j$ such that $a_i>a_j$.

We motivated the problem of counting inversions as a good measure of how different two orderings are. However, one might feel that this measure is too sensitive. Let's call a pair a significant inversion if $i<j$ and $a_i>2a_j$.

Give an O(n log n) algorithm to count the number of significant inversions between two orderings.

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Give an O(n log n) algorithm to count the number of significant inversions between two orderings.

Idea: Let k=floor(n/2); call algorithm (ALG) on each (sorted) half:

ALG($a_1,\ldots,a_k$) -> return N1, number of significant inversions and sorted list.

ALG(a_{k+1}, a_n) -> return N2, number of significant inversions and sorted list.

HW#3, Kleinberg #5.2

Let's call a pair a significant inversion if $i<j$ and $a_i>2a_j$.

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ALG($a_1,\ldots,a_k$) -> return N1, number of significant inversions and sorted list.

ALG(a_{k+1}, a_n) -> return N2, number of significant inversions and sorted list.

Lastly, we need N3, the count for number of significant inversions where left endpoint is in the first set, right endpoint in the second set.

Last point: How to merge in O(n) time for significant inversion counts?
Hint: Merge list 1 and list two times list 2.

HW#3, Kleinberg #5.6

Consider an n-node complete binary tree T, where $n=2^d-1$ for some d. Each node v of T is labeled with a real number xv. You may assume the real numbers labeling the nodes are all distinct. A node v of T is a local minimum if the label x_v is less than the label x_w for all nodes w that are joined to v by an edge.

You are given such a complete binary tree T, but the labeling is only specified in the following implicit way: for each node v, you can determine the value x_v, by probing the node v. Show how to find a local minimum of T using only O(log n) probes to the nodes of T.

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One idea: Recursive step – begin at root, if it is smaller than children we are done. What next?

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One idea: Recursive step – begin at root, if it is smaller than children we are done.

Next, choose a smaller child and iterate. We still must prove the run-time is O(log n) and correctness.

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

↑ fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

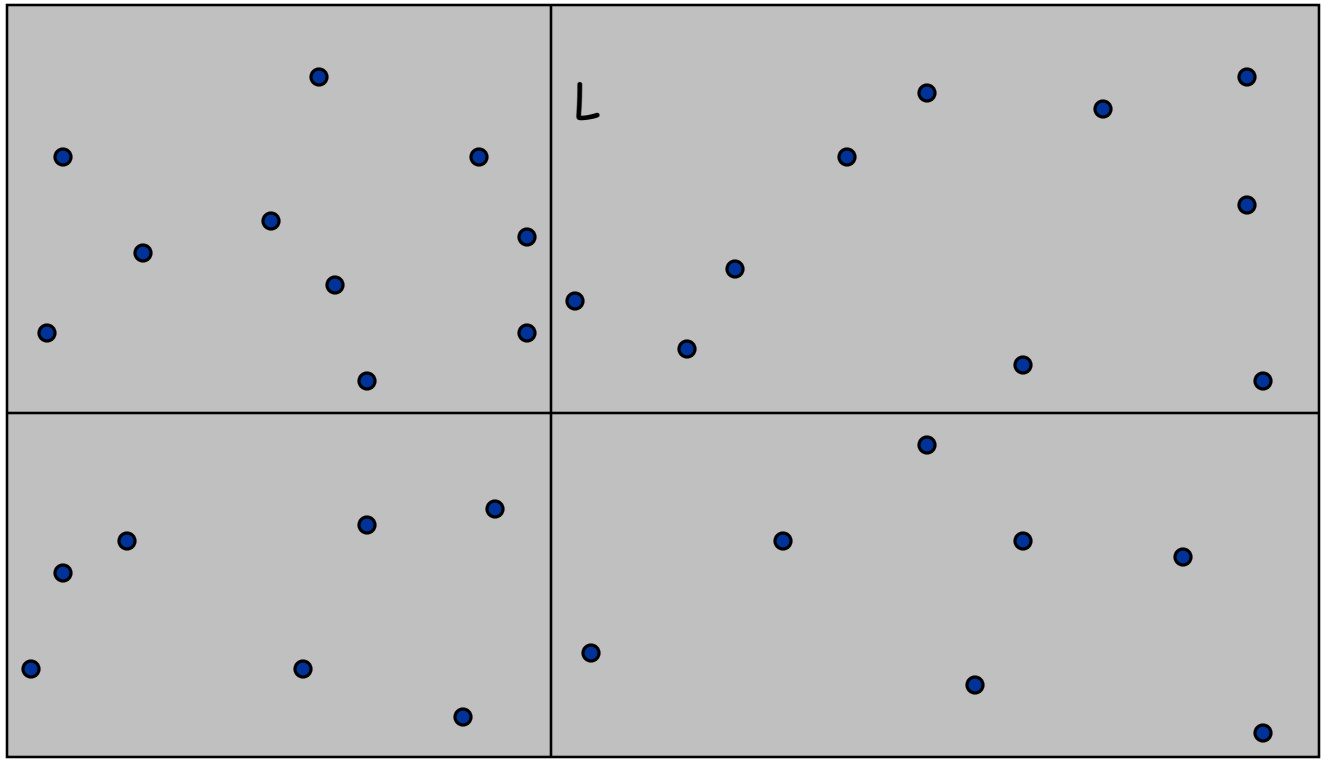
1-D version. $O(n \log n)$ easy if points are on a line.

Assumption. No two points have same x coordinate.

↑ to make presentation cleaner

Closest Pair of Points: First Attempt

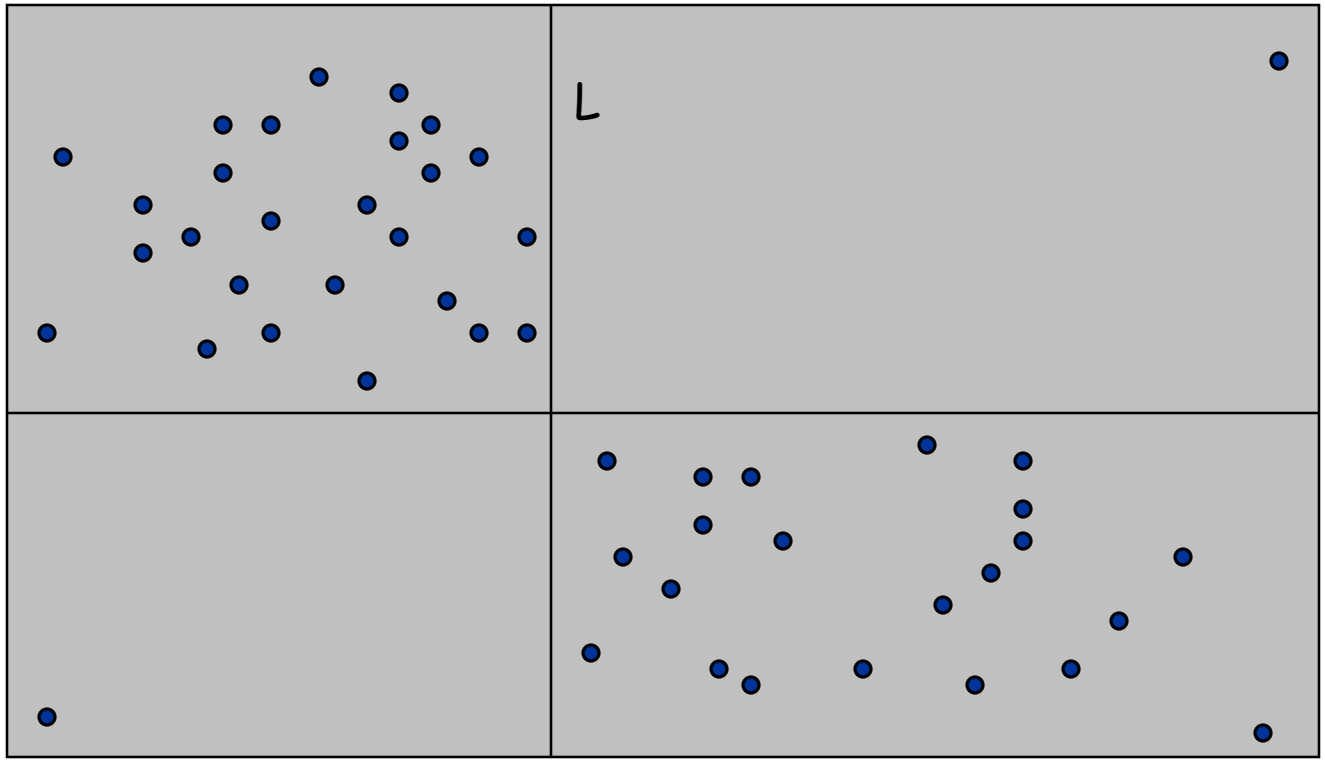
Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.

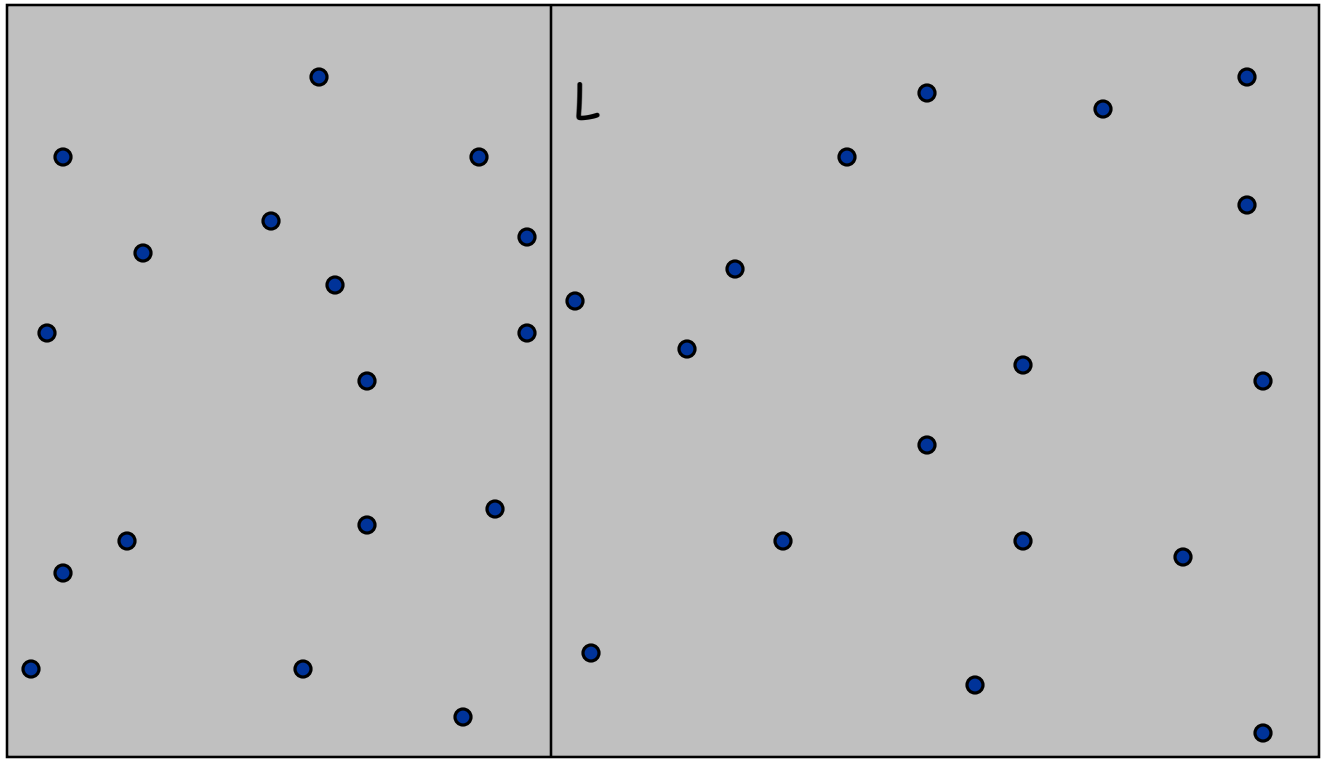
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



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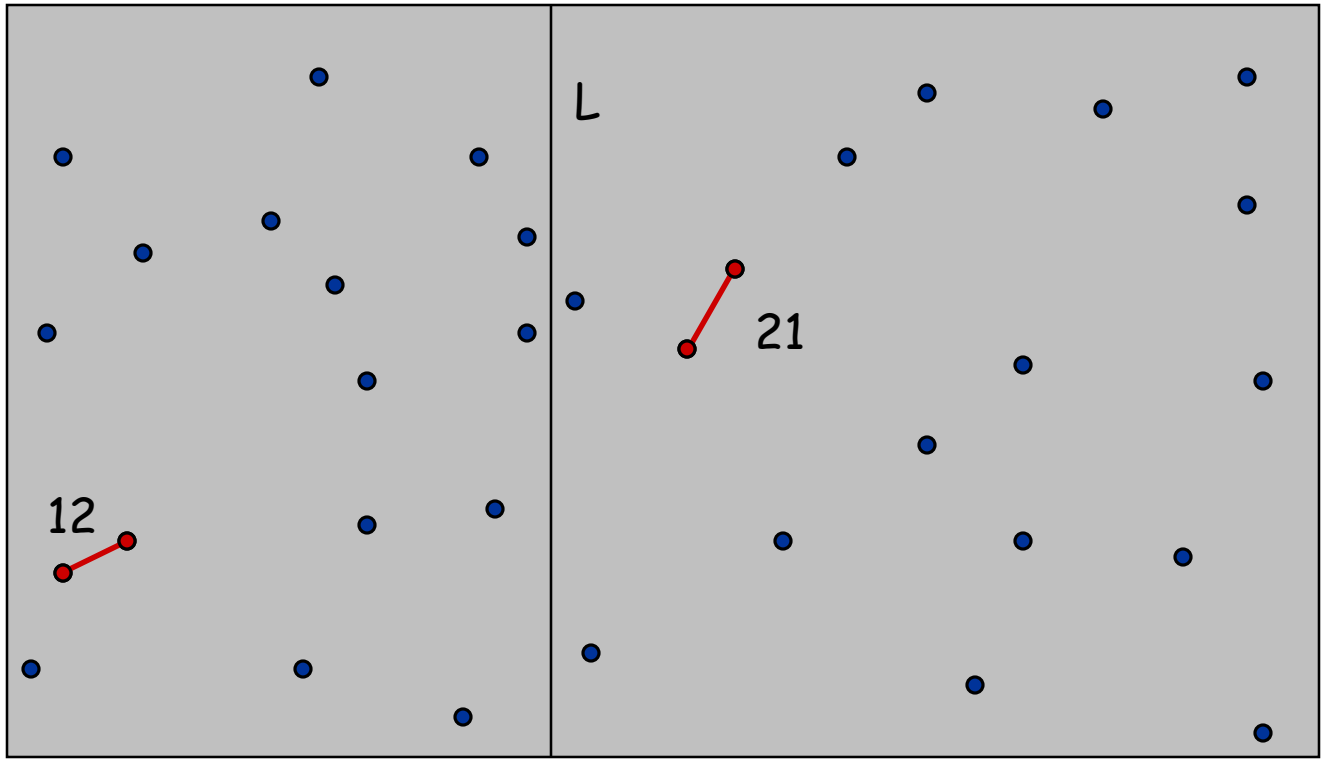
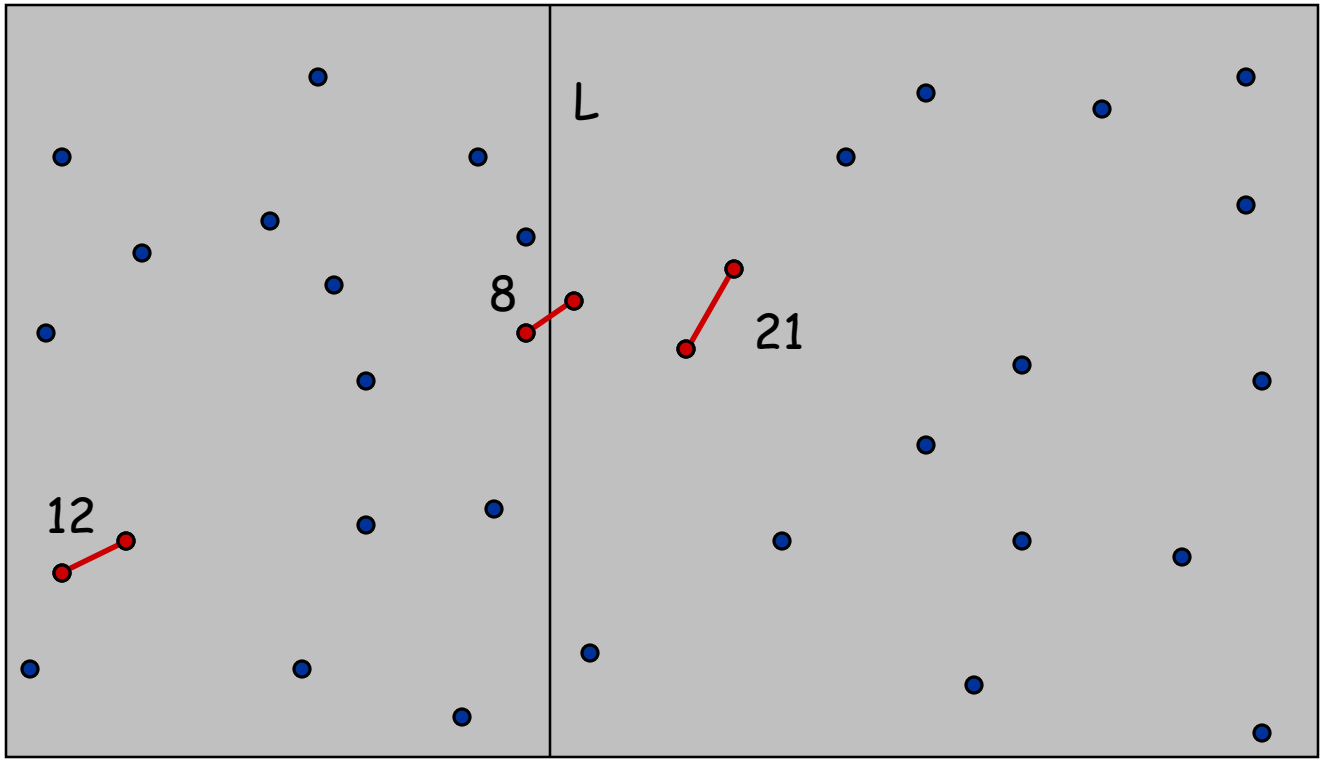
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

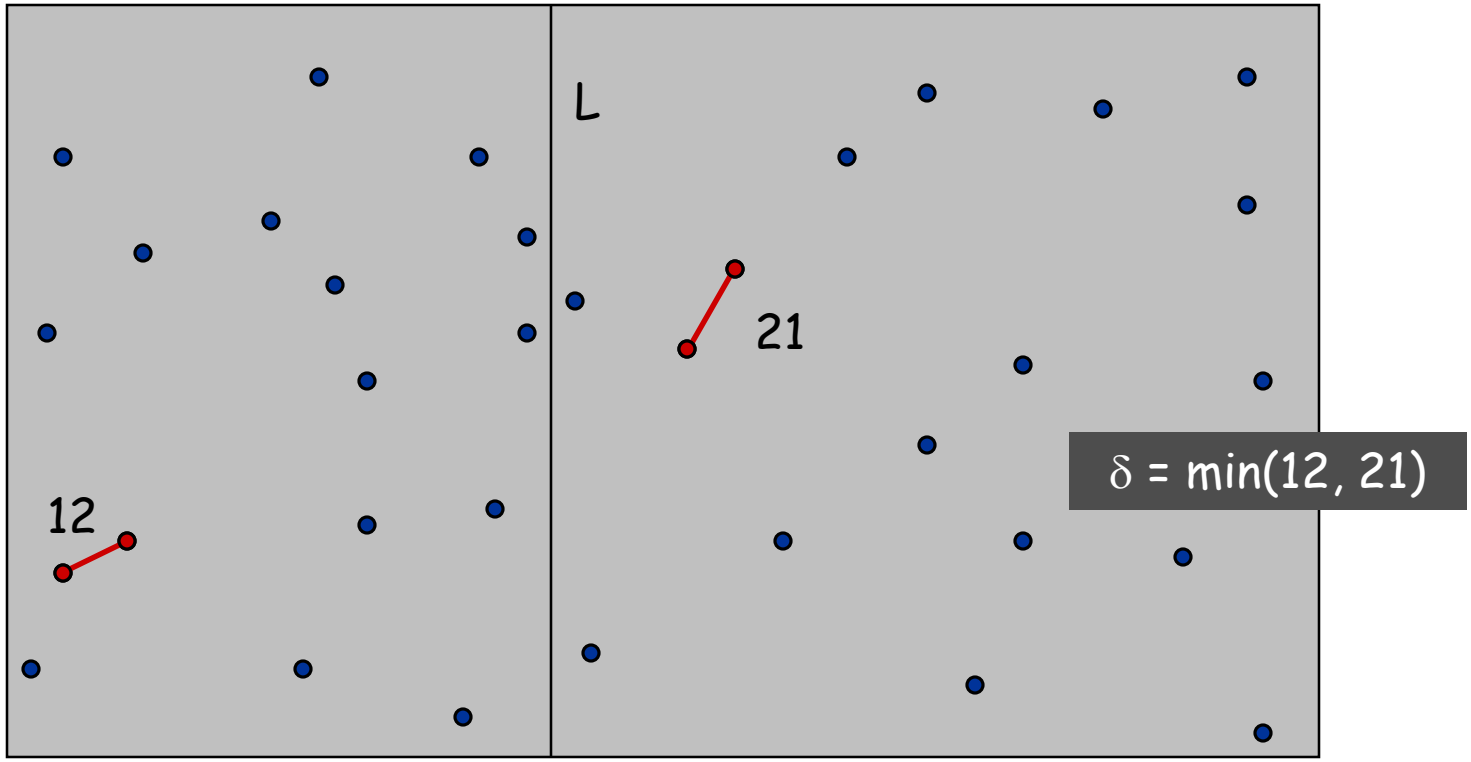
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

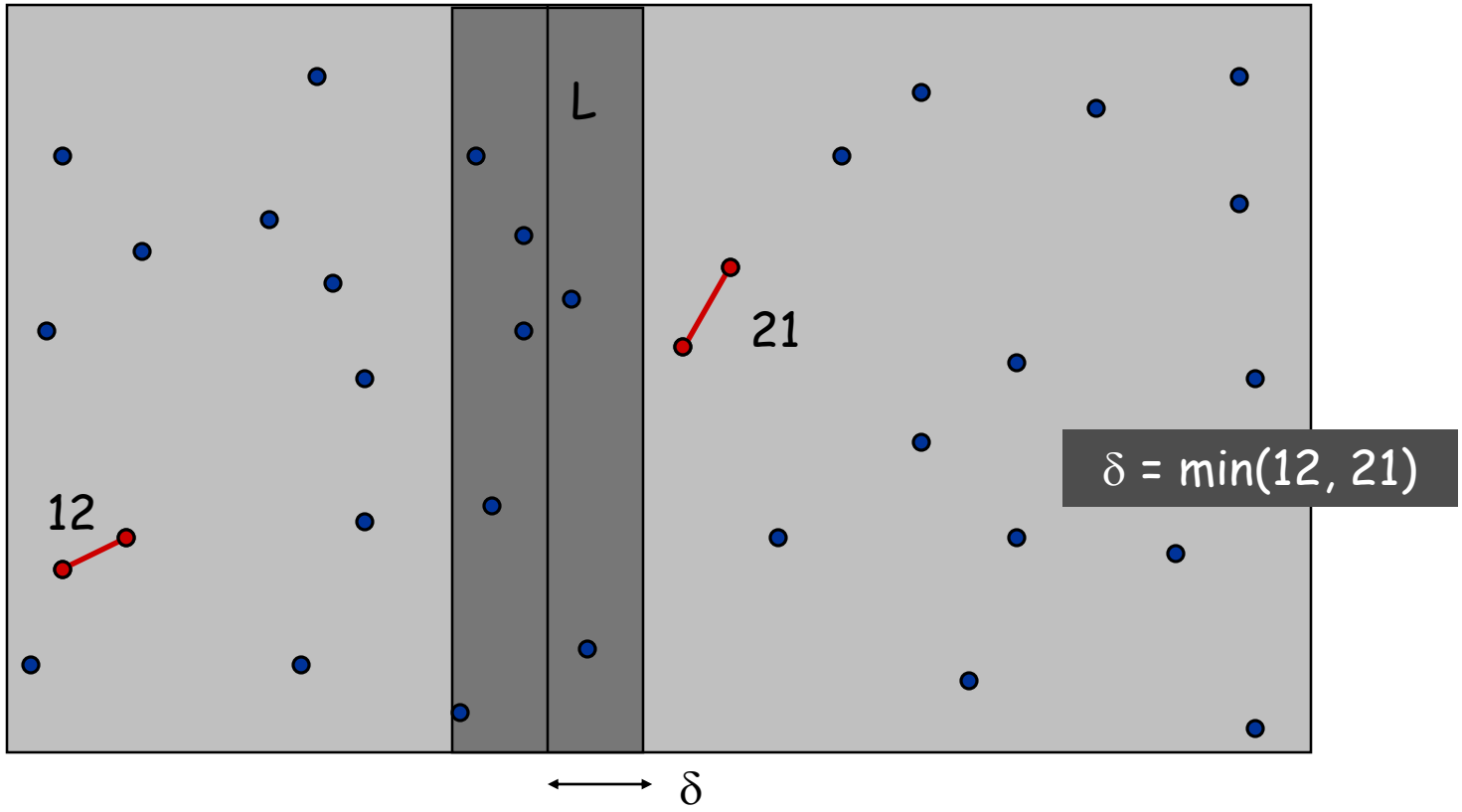
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

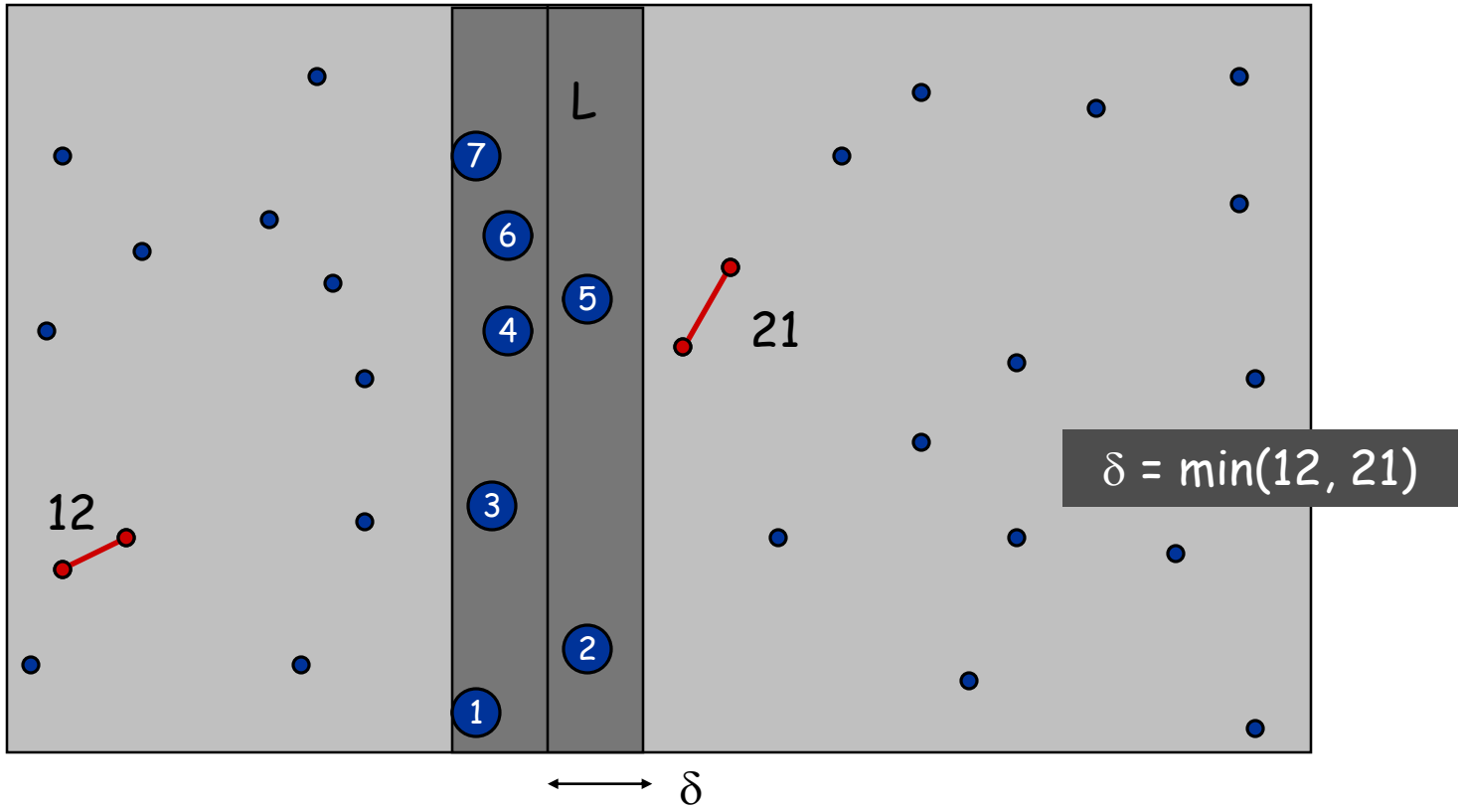
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

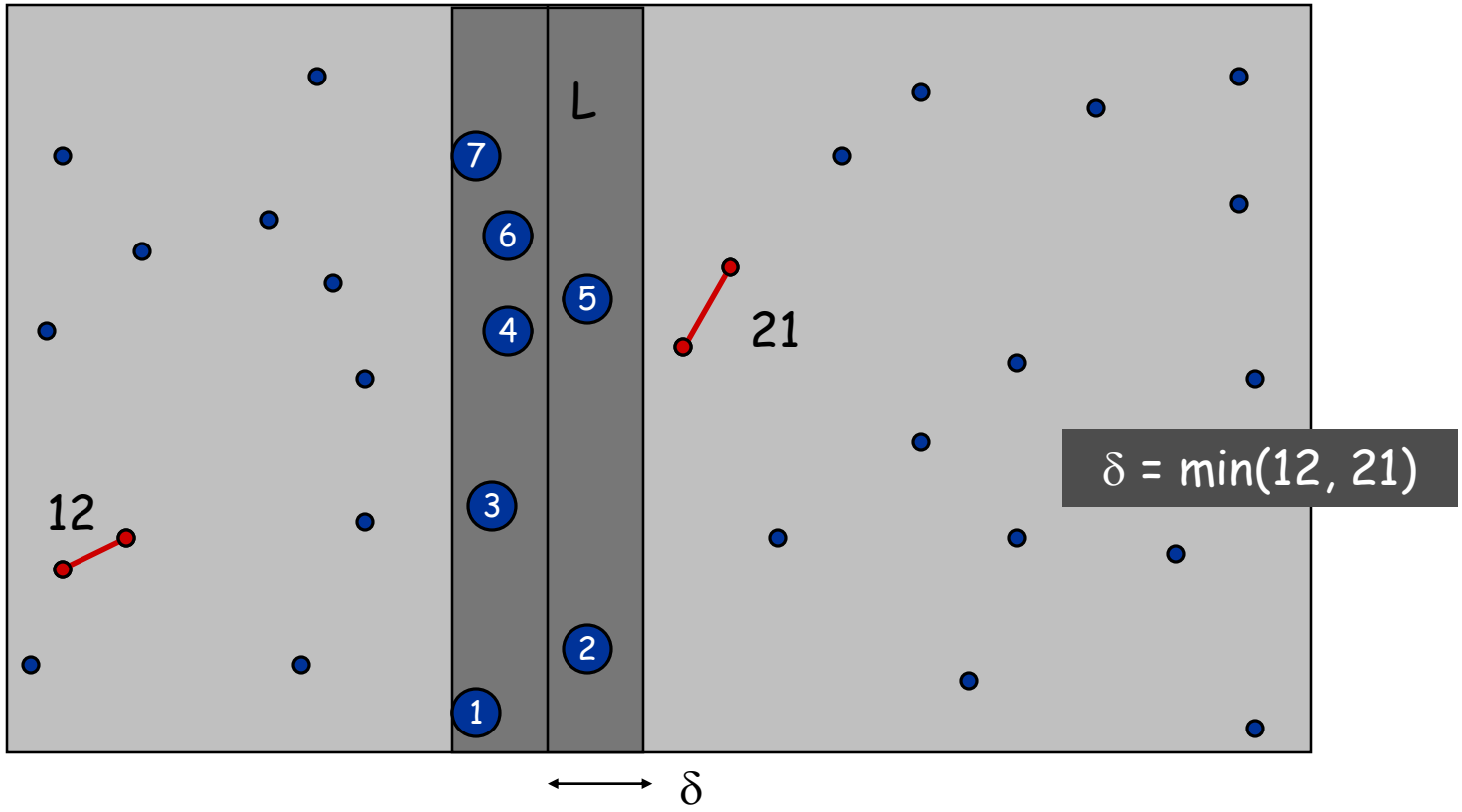
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



Closest Pair of Points

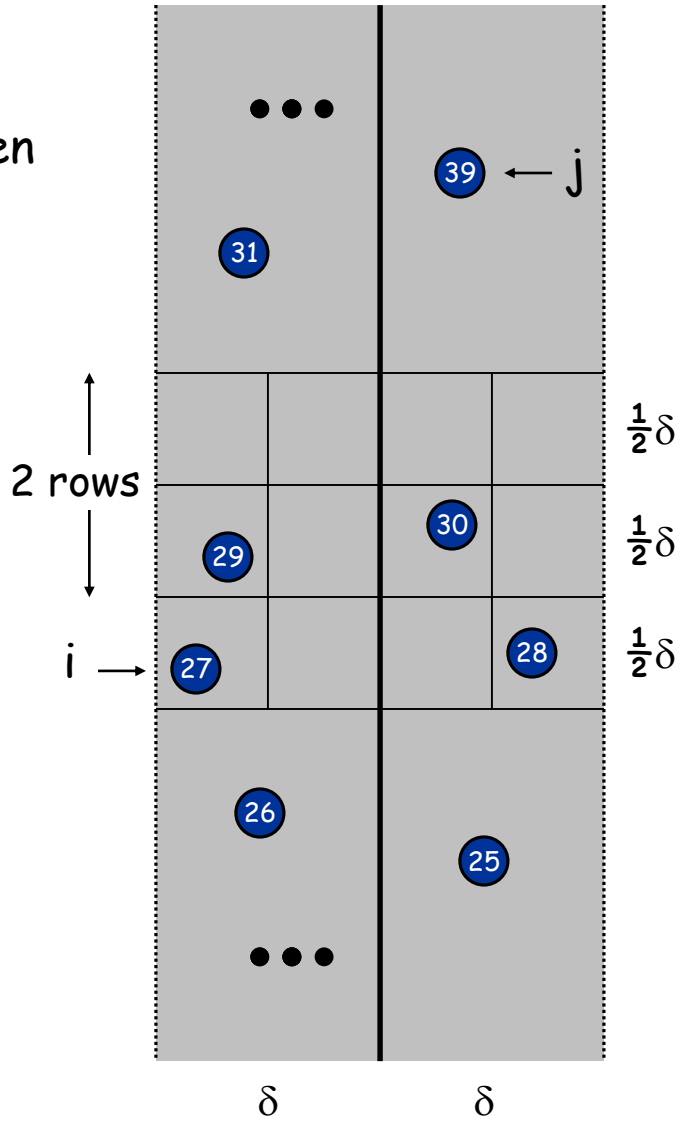
Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.



Closest Pair Algorithm

```
Closest-Pair(p1, …, pn) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

O(n log n)

2T(n / 2)

O(n)

O(n log n)

O(n)

Closest Pair of Points: Analysis

Running time.

$$\mathrm{T}(n) \leq 2T(n/2) + O(n \log n) \Rightarrow \mathrm{T}(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \leq 2T(n/2) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

HW #3 / Exercise #6

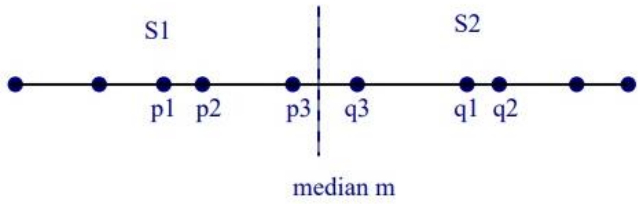
In this exercise we consider the task of finding the closest pair of points in 1-D (i.e. points on a line).

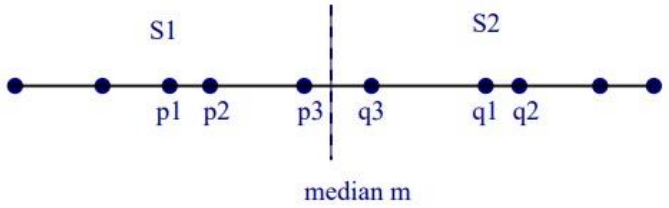
(iii) Using explicit divide and conquer techniques, devise an algorithm (different from part (ii)) that solves the problem in $\Theta(n \log n)$.

HW #3 / Exercise #6

In this exercise we consider the task of finding the closest pair of points in 1-D (i.e. points on a line).

(iii) Using explicit divide and conquer techniques, devise an algorithm (different from part (ii)) that solves the problem in Θ(n log n).





- Let δ be the smallest separation found so far:

$$\delta = \min(|p_2 - p_1|,\ |q_2 - q_1|)$$

Matrix Multiplication

Dot Product

Dot product. Given two length n vectors a and b, compute $c = a \cdot b$.

Grade-school. $\Theta(n)$ arithmetic operations.

$$a \cdot b = \sum_{i=1}^{n} a_i b_i$$

$$a = [\,.70 \quad .20 \quad .10\,]$$

$$b = [\,.30 \quad .40 \quad .30\,]$$

$$a \cdot b = (.70 \times .30) + (.20 \times .40) + (.10 \times .30) = .32$$

Remark. Grade-school dot product algorithm is optimal.

Matrix Multiplication

Matrix multiplication. Given two n-by-n matrices A and B, compute $C = AB$.

Grade-school. $\Theta(n^3)$ arithmetic operations.

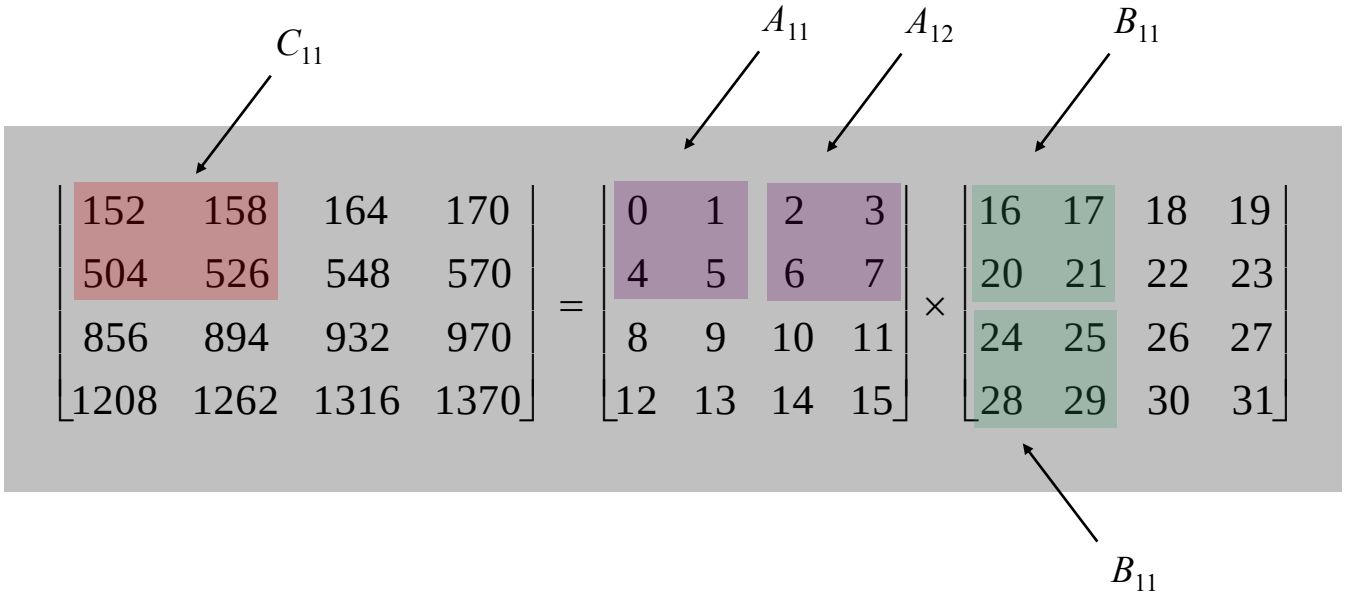
$$c_{ij} = \underbrace{\sum_{k=1}^{n} a_{ik} b_{kj}}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$\begin{bmatrix} .59 & .32 & .41 \\ .31 & .36 & .25 \\ .45 & .31 & .42 \end{bmatrix} = \begin{bmatrix} .70 & .20 & .10 \\ .30 & .60 & .10 \\ .50 & .10 & .40 \end{bmatrix} \times \begin{bmatrix} .80 & .30 & .50 \\ .10 & .40 & .10 \\ .10 & .30 & .40 \end{bmatrix}$$

Q. Is grade-school matrix multiplication algorithm optimal?

Block Matrix Multiplication



$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

Matrix Multiplication: Warmup

To multiply two n-by-n matrices A and B:

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Conquer: multiply 8 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: add appropriate products using 4 matrix additions.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$T(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow T(n) = \Theta(n^3)$$

Fast Matrix Multiplication

Key idea. multiply 2-by-2 blocks with only 7 multiplications.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

- 7 multiplications.
- $18 = 8 + 10$ additions and subtractions.

Fast Matrix Multiplication

To multiply two n-by-n matrices A and B: [Strassen 1969]

- Divide: partition A and B into $\frac{1}{2}n$-by-$\frac{1}{2}n$ blocks.
- Compute: 14 $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices via 10 matrix additions.
- Conquer: multiply 7 pairs of $\frac{1}{2}n$-by-$\frac{1}{2}n$ matrices, recursively.
- Combine: 7 products into 4 terms using 8 matrix additions.

Analysis.

- Assume n is a power of 2.
- $T(n)$ = # arithmetic operations.

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Fast Matrix Multiplication: Practice

Implementation issues.

- Sparsity.
- Caching effects.
- Numerical stability.
- Odd matrix dimensions.
- Crossover to classical algorithm around $n = 128$.

Common misperception. *"Strassen is only a theoretical curiosity."*

- Apple reports 8x speedup on G4 Velocity Engine when $n \approx 2{,}500$.
- Range of instances where it's useful is a subject of controversy.

Remark. Can "Strassenize" $Ax = b$, determinant, eigenvalues, SVD, ….

Fast Matrix Multiplication: Theory

Q. Multiply two 2-by-2 matrices with 7 scalar multiplications?

A. Yes! [Strassen 1969] $\Theta(n^{\log_2 7}) = O(n^{2.807})$

Q. Multiply two 2-by-2 matrices with 6 scalar multiplications?

A. Impossible. [Hopcroft and Kerr 1971] $\Theta(n^{\log_2 6}) = O(n^{2.59})$

Q. Two 3-by-3 matrices with 21 scalar multiplications?

A. Also impossible. $\Theta(n^{\log_3 21}) = O(n^{2.77})$

Begun, the decimal wars have. [Pan, Bini et al, Schönhage, ...]

- Two 20-by-20 matrices with 4,460 scalar multiplications. $O(n^{2.805})$
- Two 48-by-48 matrices with 47,217 scalar multiplications. $O(n^{2.7801})$
- A year later. $O(n^{2.7799})$
- December, 1979. $O(n^{2.521813})$
- January, 1980. $O(n^{2.521801})$

Fast Matrix Multiplication: Theory

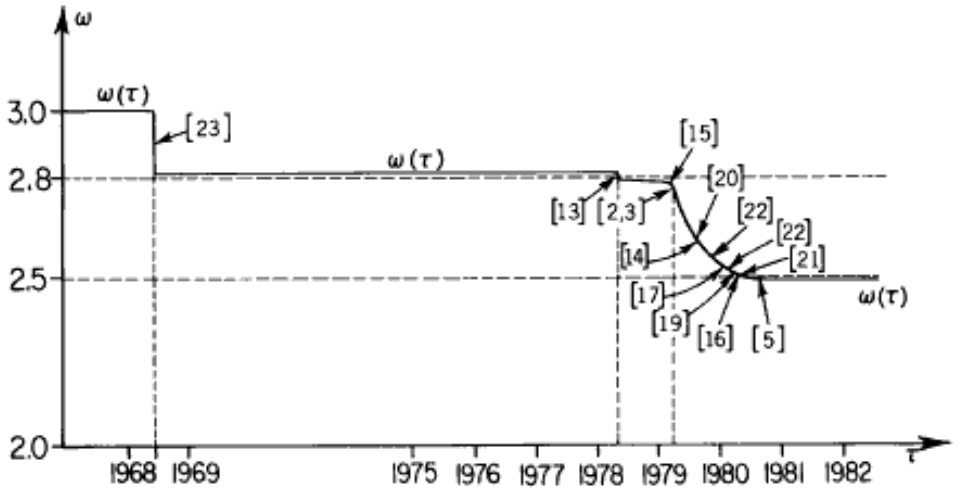


FIG. 1. *ω(t) is the best exponent announced by time τ.*

Best known. $O(n^{2.376})$ [Coppersmith-Winograd, 1987]

Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Caveat. Theoretical improvements to Strassen are progressively less practical.

5.6 Convolution and FFT

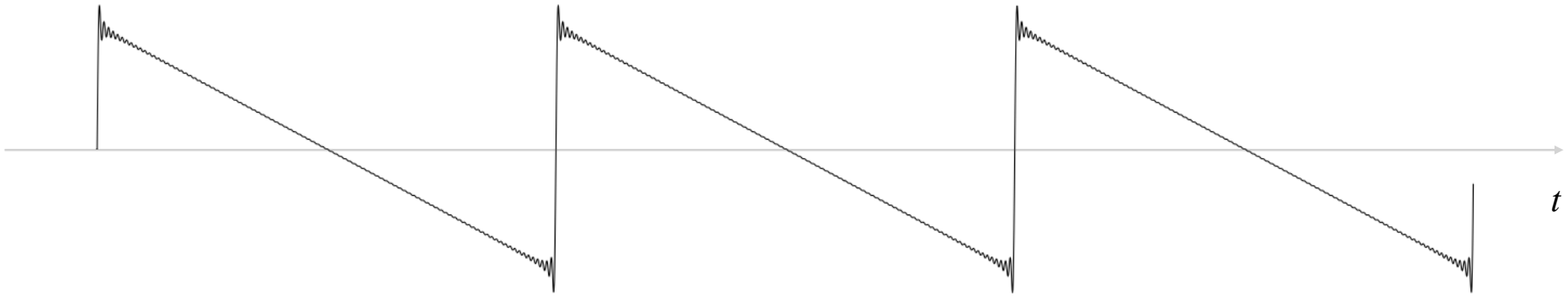
The FFT is the most important algorithm of the 20th century.

-- Gilbert Strang

Fourier Analysis

Fourier theorem. [Fourier, Dirichlet, Riemann] Any periodic function can be expressed as the sum of a series of sinusoids.

sufficiently smooth



$$y(t) = \frac{2}{\pi} \sum_{k=1}^{N} \frac{\sin kt}{k} \qquad N = 100$$

Euler's Identity

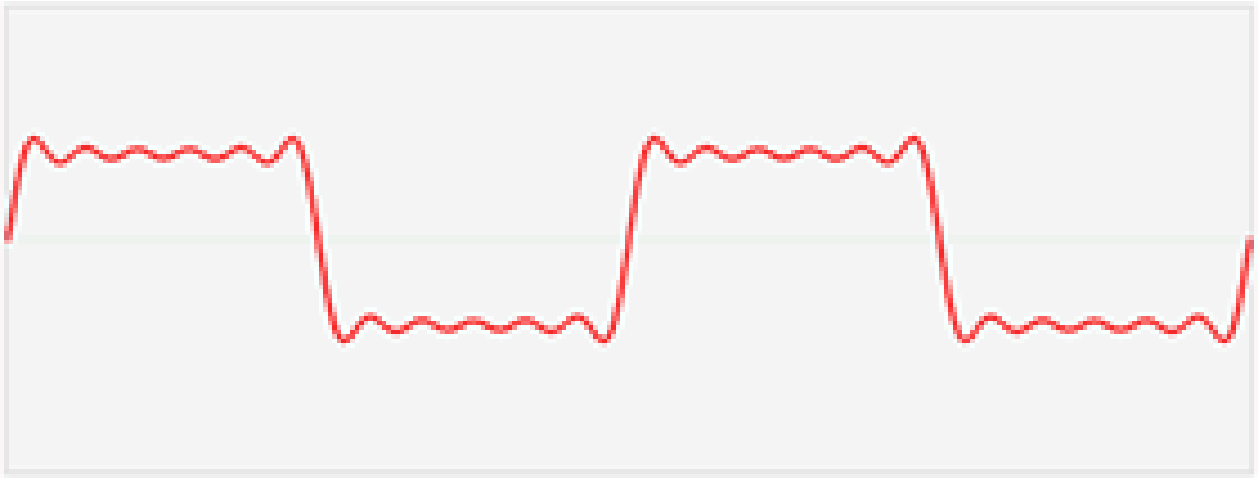
Sinusoids. Sum of sine an cosines.

$$e^{ix} = \cos x + i \sin x$$

Euler's identity

Sinusoids. Sum of complex exponentials.

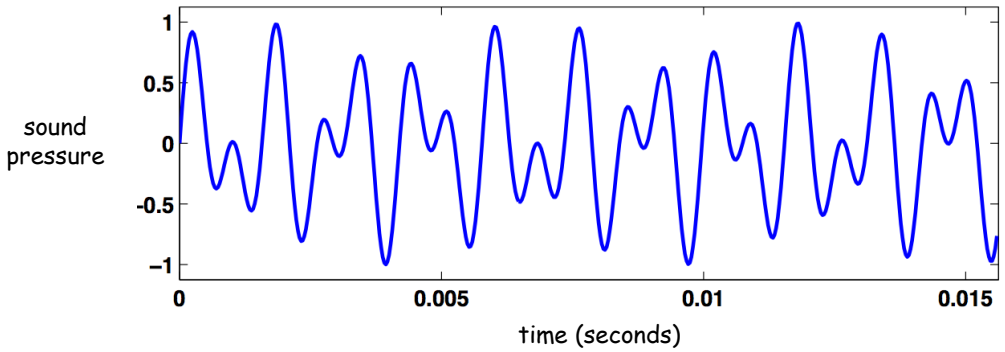
Time Domain vs. Frequency Domain



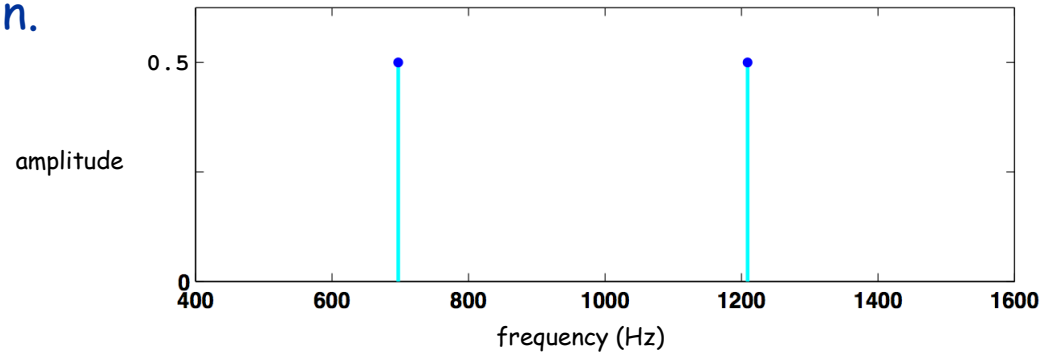
Time Domain vs. Frequency Domain

Signal. [touch tone button 1] $y(t) = \frac{1}{2}\sin(2\pi \cdot 697\, t) + \frac{1}{2}\sin(2\pi \cdot 1209\, t)$

Time domain.



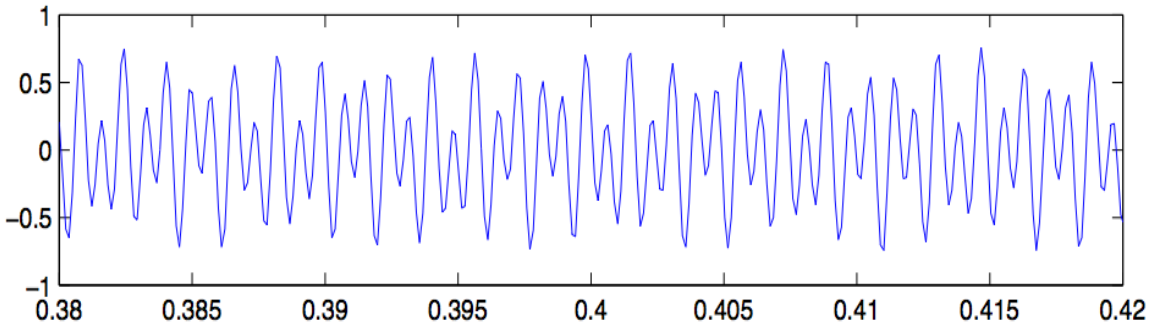
Frequency domain.



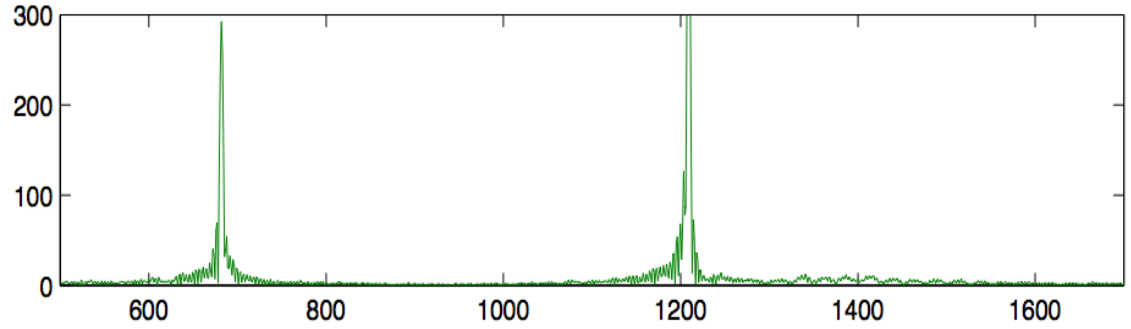
Reference: Cleve Moler, Numerical Computing with MATLAB

Time Domain vs. Frequency Domain

Signal. [recording, 8192 samples per second]



Magnitude of discrete Fourier transform.



Reference: Cleve Moler, Numerical Computing with MATLAB

Fast Fourier Transform

FFT. Fast way to convert between time-domain and frequency-domain.

Alternate viewpoint. Fast way to multiply and evaluate polynomials.

we take this approach

> If you speed up any nontrivial algorithm by a factor of a million or so the world will beat a path towards finding useful applications for it. -Numerical Recipes

Fast Fourier Transform: Applications

Applications.

- Optics, acoustics, quantum physics, telecommunications, radar, control systems, signal processing, speech recognition, data compression, image processing, seismology, mass spectrometry…
- Digital media. [DVD, JPEG, MP3, H.264]
- Medical diagnostics. [MRI, CT, PET scans, ultrasound]
- Numerical solutions to Poisson's equation.
- Shor's quantum factoring algorithm.
- …

> The FFT is one of the truly great computational developments of [the 20th] century. It has changed the face of science and engineering so much that it is not an exaggeration to say that life as we know it would be very different without the FFT. -Charles van Loan

Fast Fourier Transform: Brief History

Gauss (1805, 1866). Analyzed periodic motion of asteroid Ceres.

Runge-König (1924). Laid theoretical groundwork.

Danielson-Lanczos (1942). Efficient algorithm, x-ray crystallography.

Cooley-Tukey (1965). Monitoring nuclear tests in Soviet Union and tracking submarines. Rediscovered and popularized FFT.

Importance not fully realized until advent of digital computers.

Polynomials: Coefficient Representation

Polynomial. [coefficient representation]

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$B(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1} x^{n-1}$$

Add. $O(n)$ arithmetic operations.

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$$

Evaluate. $O(n)$ using Horner's method.

$$A(x) = a_0 + (x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1}))\cdots))$$

Multiply (convolve). $O(n^2)$ using brute force.

$$A(x) \times B(x) = \sum_{i=0}^{2n-2} c_i x^i, \text{ where } c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

A Modest PhD Dissertation Title

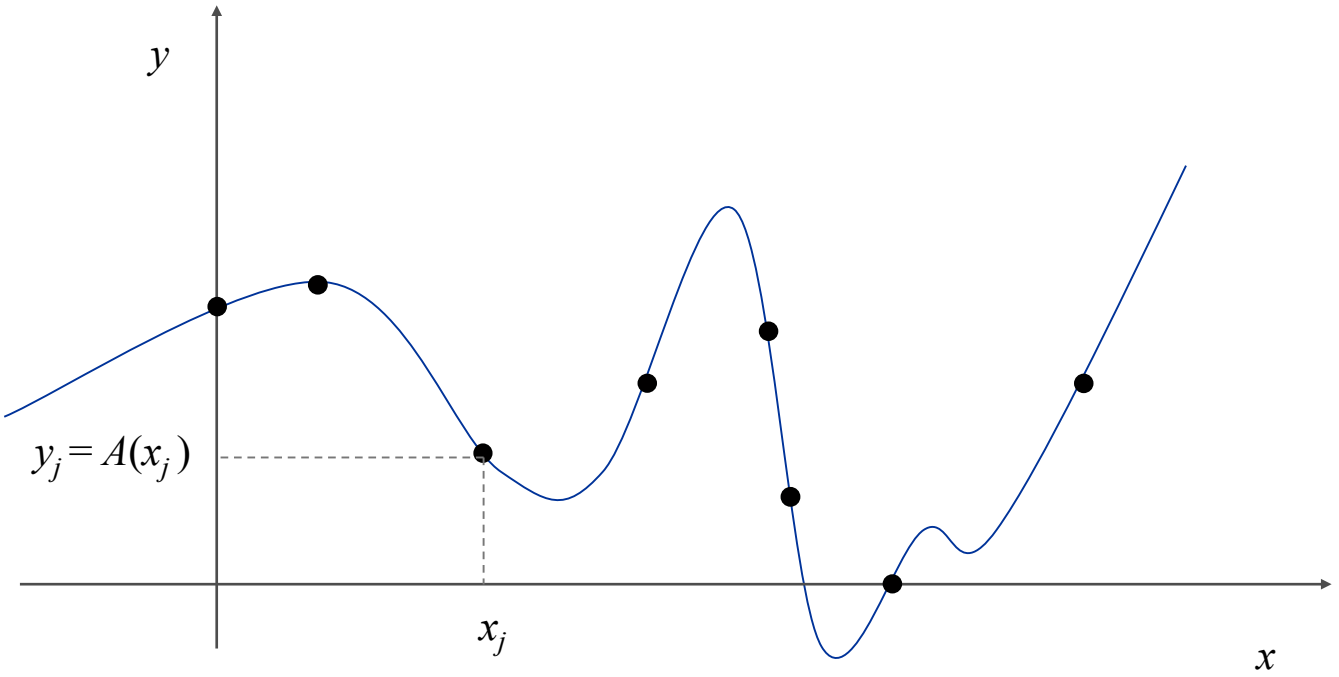
"New Proof of the Theorem That Every Algebraic Rational Integral Function In One Variable can be Resolved into Real Factors of the First or the Second Degree."

- PhD dissertation, 1799 the University of Helmstedt

Polynomials: Point-Value Representation

Fundamental theorem of algebra. (FTA) [Gauss, PhD thesis] A degree n polynomial with complex coefficients has exactly n complex roots.

Corollary. A degree n-1 polynomial $A(x)$ is uniquely specified by its evaluation at n distinct values of x.



Polynomials: Point-Value Representation

Polynomial. [point-value representation]

$$A(x): (x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$$
$$B(x): (x_0, z_0), \ldots, (x_{n-1}, z_{n-1})$$

Add. $O(n)$ arithmetic operations.

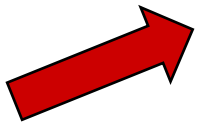
$$A(x)+B(x): (x_0, y_0+z_0), \ldots, (x_{n-1}, y_{n-1}+z_{n-1})$$

Multiply (convolve). $O(n)$, but need $2n$ points.

$$A(x) \times B(x): (x_0, y_0 \times z_0), \ldots, (x_{2n-1}, y_{2n-1} \times z_{2n-1})$$

Evaluate. $O(n^2)$ using Lagrange's formula.

$$A(x) = \sum_{k=0}^{n-1} y_k \frac{\prod_{j \neq k} (x - x_j)}{\prod_{j \neq k} (x_k - x_j)}$$



Commonly used for polynomial interpolation.

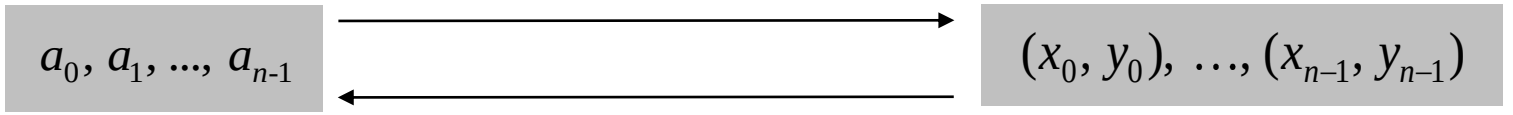
Converting Between Two Polynomial Representations

Tradeoff. Fast evaluation or fast multiplication. We want both!

representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

Goal. Efficient conversion between two representations $\Rightarrow$ all ops fast.

FFT: Given coefficient representation of polynomials, (1) convert to point-value, (2) multiply, (3) then convert back to coefficient representation.



coefficient representation

point-value representation

Converting Between Two Representations: Brute Force

Coefficient $\Rightarrow$ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Vandermonde matrix

FFT: Given coefficient representation of polynomials, (1) convert to point-value, (2) multiply, (3) then convert back to coefficient representation.

Running time. $O(n^2)$ for matrix-vector multiply (or n Horner's).

- Main Idea: Multiplication by the Vandermonde matrix renders conversion from coefficient polynomial representation to point-value representation (step (1)).

Converting Between Two Representations: Brute Force

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

FFT: Given coefficient representation of polynomials, (1) convert to point-value, (2) multiply, (3) then convert back to coefficient representation.

Vandermonde matrix is invertible iff x_i distinct

Running time. $O(n^3)$ for Gaussian elimination (i.e. find inverse of Vandermonde matrix).

or $O(n^{2.376})$ via fast matrix multiplication

Main Idea: Multiplication by the inverse of the Vandermonde matrix renders conversion from point-value representation to coefficient representation (step (3)). $\mathbf{y} = V\mathbf{a} \rightarrow V^{-1}\mathbf{y} = \mathbf{a}$

Why do we need to multiply polynomials?

Convolutions: In sampling theory, a convolution operator (*) computes a weighted average of an input signal (x) and a filter (h).

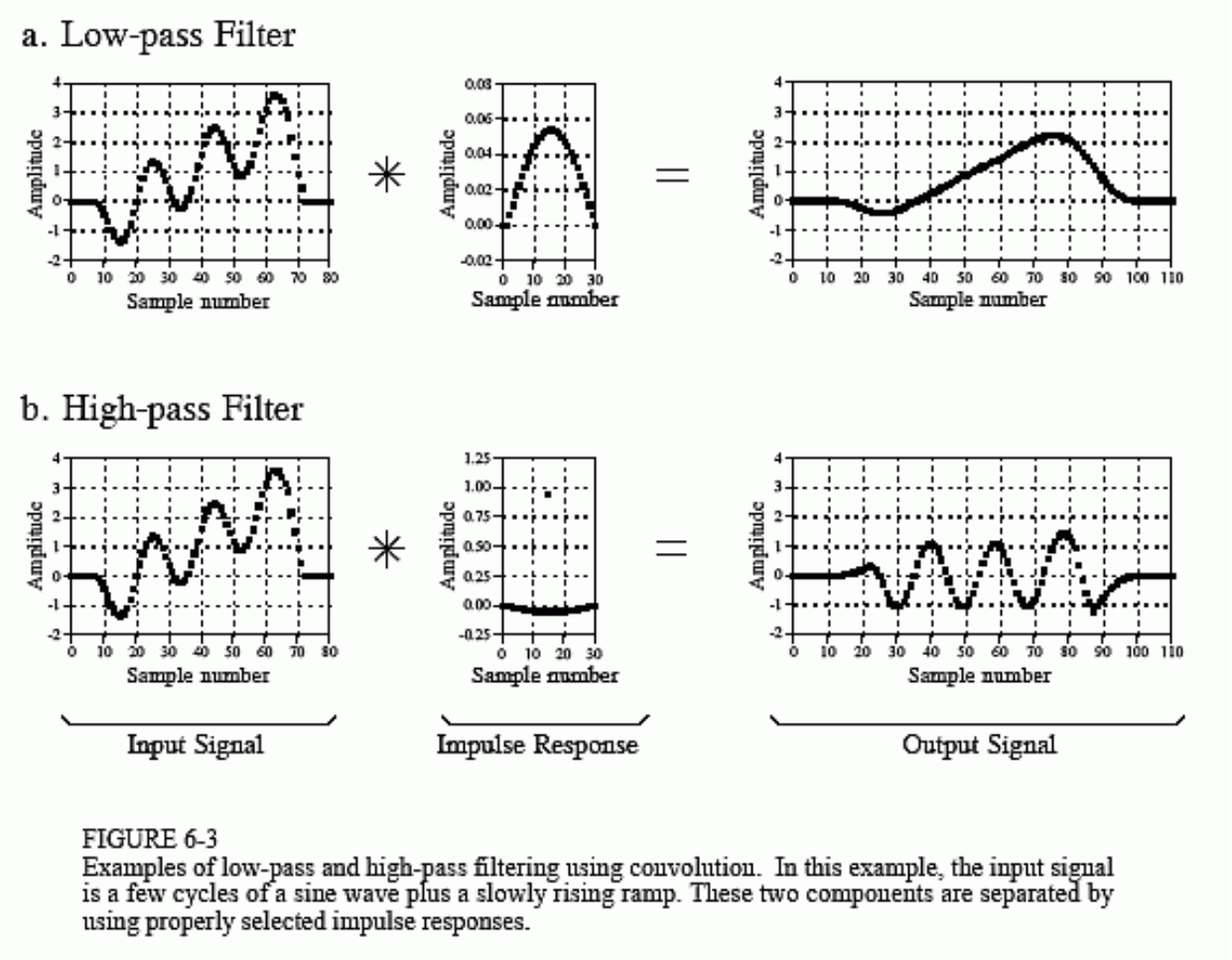
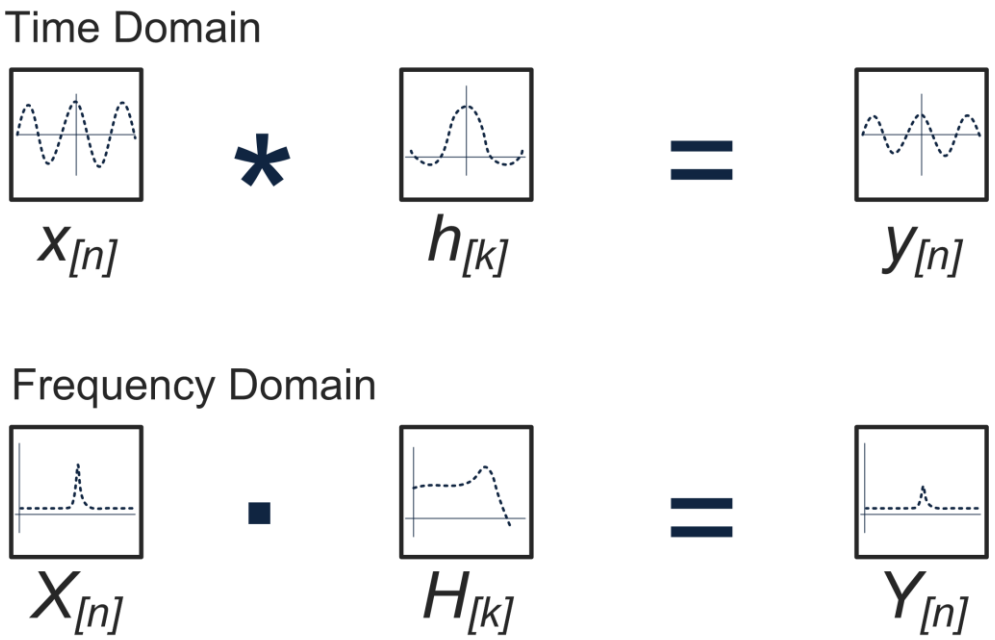


FIGURE 6-3
Examples of low-pass and high-pass filtering using convolution. In this example, the input signal is a few cycles of a sine wave plus a slowly rising ramp. These two components are separated by using properly selected impulse responses.

Why do we need to multiply polynomials?

The convolution theorem states that convolution of two signals in the time domain is equivalent to the multiplication of their corresponding Fourier transforms.

Essentially, we will obtain the same result if we multiply the Fourier transforms of our signals as we would if we convolved the signals directly.



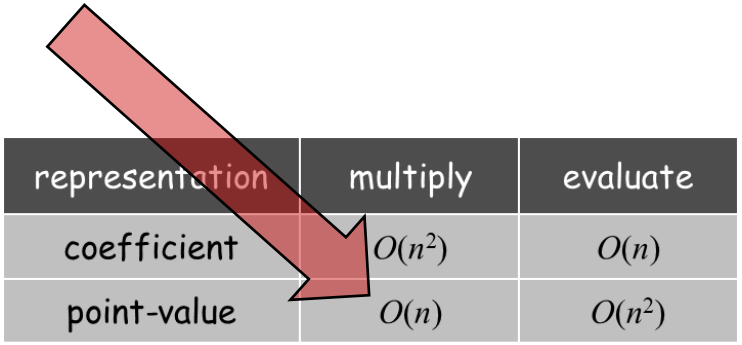
(*) NB: This is why we want to multiply polynomials!

Why do we need to multiply polynomials?

Summary of ideas so far for FFT:

(*) By the convolution theorem, multiplication of polynomials in the frequency domain is equivalent to "convolving polynomials" (i.e. performing discrete sampling).

Thus, we need an efficient procedure to convert from the (conventional) coefficient representation of polynomials to the point-value representation (as multiplication then costs O(n)).



representation	multiply	evaluate
coefficient	$O(n^2)$	$O(n)$
point-value	$O(n)$	$O(n^2)$

Why do we need to multiply polynomials?

Summary of ideas so far for FFT:

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(*) When we multiply by V, the Vandermonde matrix, this converts the polynomial representation from coefficient to point-value; multiplying by V^{-1} reverses this transformation from: coefficient -> point-value.

Are we done?

Why do we need to multiply polynomials?

Summary of ideas so far for FFT:

(*) By the convolution theorem, multiplication of polynomials in the frequency domain is equivalent to "convolving polynomials" (i.e. performing discrete sampling).

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(*) When we multiply by V, the Vandermonde matrix, this converts the polynomial representation from coefficient to point-value; multiplying by V^{-1} reverses this transformation from: coefficient -> point-value.

Are we done? Not quite. Multiplying naively by V requires $O(n^2)$ time. Why? Also, while inverting a matrix in general requires $O(n^3)$ time, the Vandermonde structure allows inversion in $O(n^2)$.

Issue still remains: $O(n^2)$ bound.

FFT

Issue still remains: $O(n^2)$ bound.

How do we remedy this? Divide and conquer!

In summary: We will transform, recursively, the problem of multiplying the n coefficients by a form of the Vandermonde matrix – by rendering the size-n problem as two size n/2 problems.

Divide and conquer for FFT will consequently yield a recursion:

$$T(n)=2T(n/2)+O(n)$$

What is the natural big-O upper bound?

FFT

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Divide and conquer for FFT will consequently yield a recursion:

$$T(n)=2T(n/2)+O(n)$$

What is the natural big-O upper bound? $O(n \log n)$ (recall the previous Mergesort discussion and solution).

Geometry of FFT

(*) FYI: The columns of the Vandermonde matrix are orthogonal.

Meaning that they are pairwise orthogonal, i.e. perpendicular.

(*) This means that they form an alternative coordinate system, which is often called the Fourier basis.

The effect of multiplying a vector by V is, geometrically, the effect of rotating the vector from the standard basis to the Fourier basis (defined by the columns of V). The inverse (V^{-1}) is the opposite rotation.

Divide-and-Conquer

Decimation in frequency. Break up polynomial into low and high powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{low}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$
- $A_{high}(x) = a_4 + a_5 x + a_6 x^2 + a_7 x^3.$
- $A(x) = A_{low}(x) + x^4 A_{high}(x).$

Decimation in time. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x A_{odd}(x^2).$

Coefficient to Point-Value Representation: Intuition

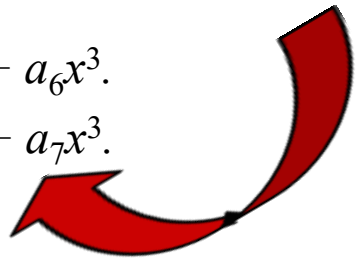
Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

we get to choose which ones!

Divide. Break polynomial up into even and odd coefficients.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$



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Divide. Break polynomial up into even and odd coefficients.

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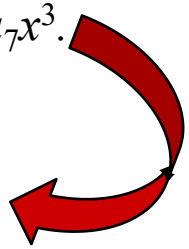
- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3.$
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$



- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Coefficient to Point-Value Representation: Intuition

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

we get to choose which ones!

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
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- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose two points to be ± 1.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1).$
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$

Can evaluate polynomial of degree $\leq n$ at 2 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 1 point.

Coefficient to Point-Value Representation: Intuition

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

we get to choose which ones!

Divide. Break polynomial up into even and odd powers.

- $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7.$
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- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3.$
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2).$
- $A(-x) = A_{even}(x^2) - x\, A_{odd}(x^2).$

Intuition. Choose four complex points to be $\pm 1, \pm i$.

- $A(1) = A_{even}(1) + 1\, A_{odd}(1).$
- $A(-1) = A_{even}(1) - 1\, A_{odd}(1).$
- $A(i) = A_{even}(-1) + i\, A_{odd}(-1).$
- $A(-i) = A_{even}(-1) - i\, A_{odd}(-1).$

Can evaluate polynomial of degree $\leq n$ at 4 points by evaluating two polynomials of degree $\leq \frac{1}{2}n$ at 2 points.

Discrete Fourier Transform

Coefficient ⇒ point-value. Given a polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, evaluate it at n distinct points $x_0, \ldots, x_{n-1}$.

Key idea. Choose $x_k = \omega^k$ where ω is principal n^{th} root of unity.

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

DFT

Fourier matrix F_n

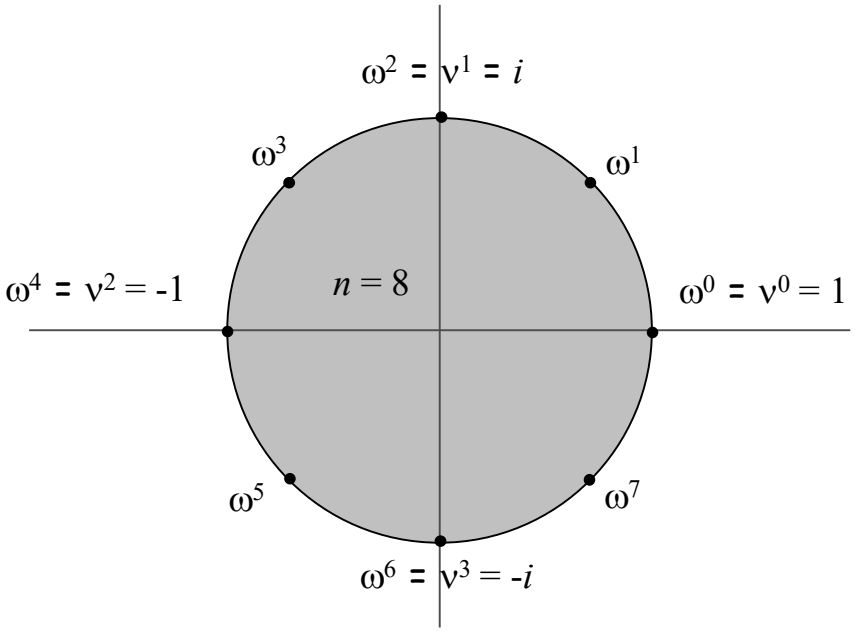
Roots of Unity

Def. An n^{th} root of unity is a complex number x such that $x^n = 1$.

Fact. The n^{th} roots of unity are: $\omega^0, \omega^1, \ldots, \omega^{n-1}$ where $\omega = e^{2\pi i / n}$.

Pf. $(\omega^k)^n = (e^{2\pi i k / n})^n = (e^{\pi i})^{2k} = (-1)^{2k} = 1$.

Fact. The $\frac{1}{2}n^{th}$ roots of unity are: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$ where $\nu = \omega^2 = e^{4\pi i / n}$.



Fast Fourier Transform

Goal. Evaluate a degree n-1 polynomial $A(x) = a_0 + \ldots + a_{n-1}\, x^{n-1}$ at its n^{th} roots of unity: $\omega^0, \omega^1, \ldots, \omega^{n-1}$.

Divide. Break up polynomial into even and odd powers.

- $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + \ldots + a_{n-2}\, x^{n/2\,-\,1}$.
- $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + \ldots + a_{n-1}\, x^{n/2\,-\,1}$.
- $A(x) = A_{even}(x^2) + x\, A_{odd}(x^2)$.

Conquer. Evaluate $A_{even}(x)$ and $A_{odd}(x)$ at the $\frac{1}{2}n^{th}$ roots of unity: $\nu^0, \nu^1, \ldots, \nu^{n/2-1}$.

Combine.

$\nu^k = (\omega^k)^2$

- $A(\omega^k) = A_{even}(\nu^k) + \omega^k\, A_{odd}(\nu^k), \quad 0 \le k < n/2$
- $A(\omega^{k+\,½n}) = A_{even}(\nu^k) - \omega^k\, A_{odd}(\nu^k), \quad 0 \le k < n/2$

$\nu^k = (\omega^{k\,+\,½n})^2$ $\qquad$ $\omega^{k+\,½n} = -\omega^k$

FFT Algorithm

```
fft(n, a_0,a_1,…,a_n-1) {
    if (n == 1) return a_0

    (e_0,e_1,…,e_n/2-1) ← FFT(n/2, a_0,a_2,a_4,…,a_n-2)
    (d_0,d_1,…,d_n/2-1) ← FFT(n/2, a_1,a_3,a_5,…,a_n-1)

    for k = 0 to n/2 - 1 {
        ω^k ← e^2πik/n
        y_k     ← e_k + ω^k d_k
        y_k+n/2 ← e_k - ω^k d_k
    }

    return (y_0,y_1,…,y_n-1)
}
```

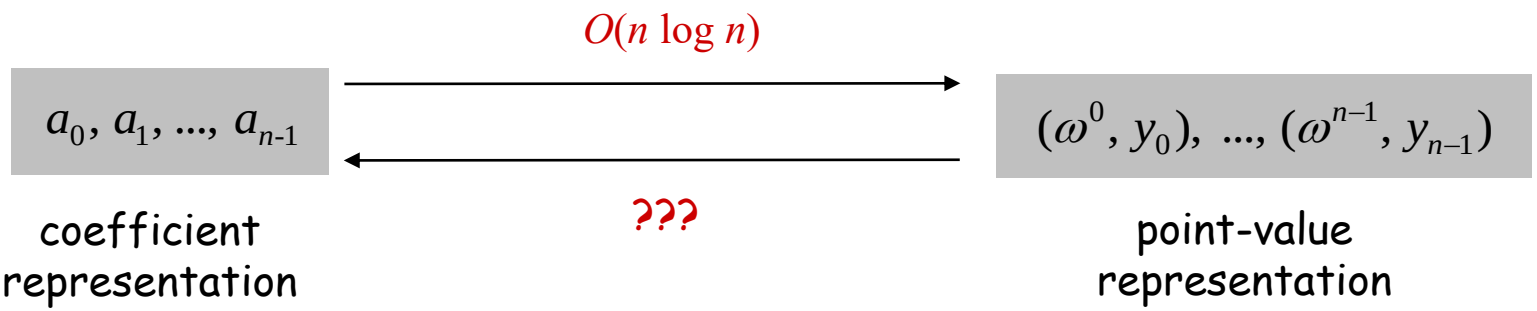
FFT Summary

Theorem. FFT algorithm evaluates a degree n-1 polynomial at each of the n^{th} roots of unity in $O(n \log n)$ steps.

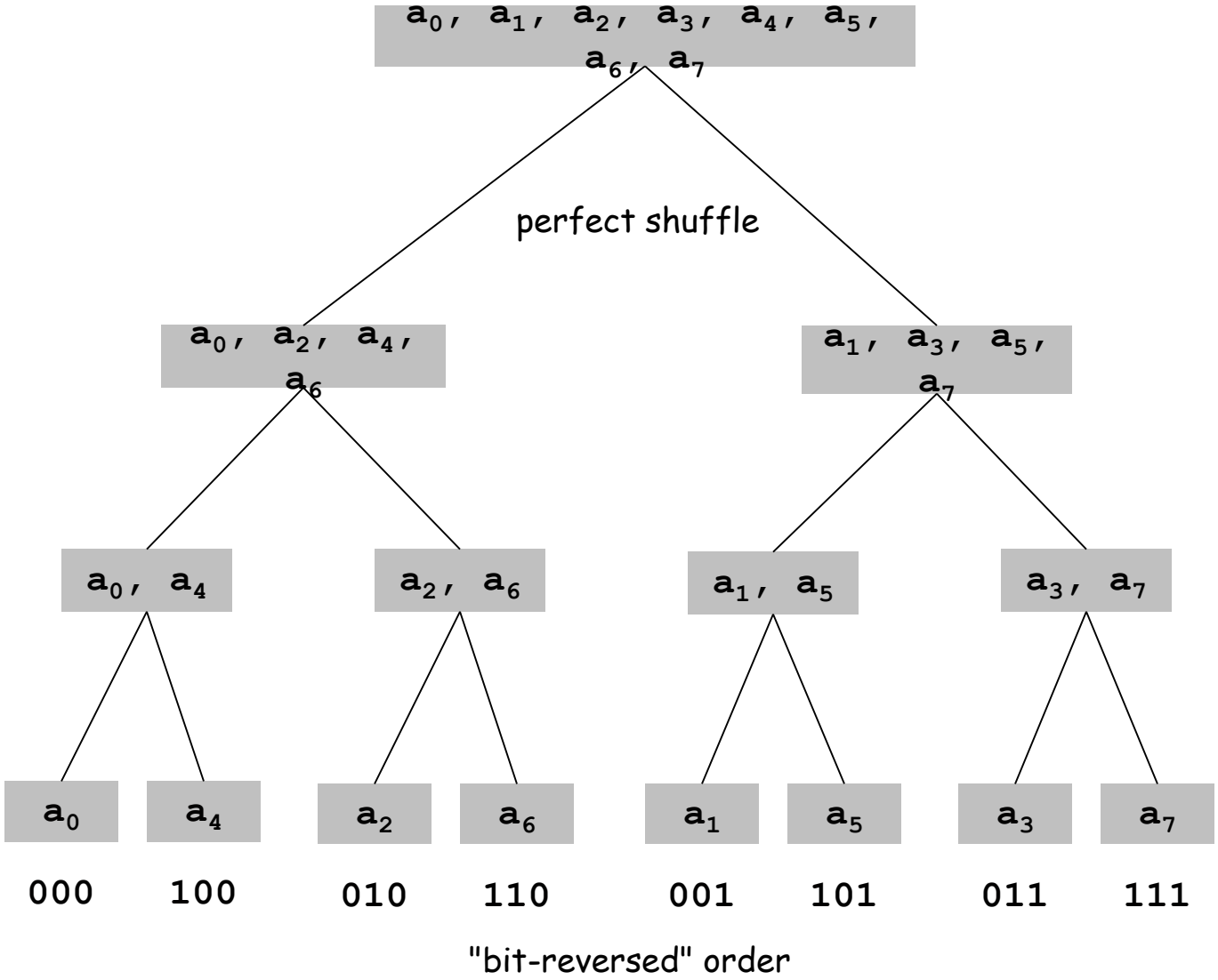
assumes n is a power of 2

Running time.

$$T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n)$$



Recursion Tree



Inverse Discrete Fourier Transform

Point-value $\Rightarrow$ coefficient. Given n distinct points $x_0, \ldots, x_{n-1}$ and values $y_0, \ldots, y_{n-1}$, find unique polynomial $a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$, that has given values at given points.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(n-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Inverse DFT

Fourier matrix inverse $(F_n)^{-1}$

Inverse DFT

Claim. Inverse of Fourier matrix F_n is given by following formula.

$$G_n = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \omega^{-3} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \omega^{-6} & \cdots & \omega^{-2(n-1)} \\ 1 & \omega^{-3} & \omega^{-6} & \omega^{-9} & \cdots & \omega^{-3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \omega^{-3(n-1)} & \cdots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

$\frac{1}{\sqrt{n}} F_n$ is unitary

Consequence. To compute inverse FFT, apply same algorithm but use $\omega^{-1} = e^{-2\pi i / n}$ as principal n^{th} root of unity (and divide by n).

Inverse FFT: Proof of Correctness

Claim. F_n and G_n are inverses.

Pf.

$$\left(F_n\, G_n\right)_{k\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{k\,j}\, \omega^{-j\,k'} = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{(k-k')\,j} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{otherwise} \end{cases}$$

summation lemma

Summation lemma. Let ω be a principal n^{th} root of unity. Then

$$\sum_{j=0}^{n-1} \omega^{k\,j} = \begin{cases} n & \text{if } k \equiv 0 \bmod n \\ 0 & \text{otherwise} \end{cases}$$

Pf.

- If k is a multiple of n then $\omega^k = 1 \;\Rightarrow$ series sums to n.
- Each n^{th} root of unity ω^k is a root of $x^n - 1 = (x - 1)\,(1 + x + x^2 + \ldots + x^{n-1})$.
- if $\omega^k \neq 1$ we have: $1 + \omega^k + \omega^{k(2)} + \ldots + \omega^{k(n-1)} = 0 \;\Rightarrow$ series sums to 0. ▪

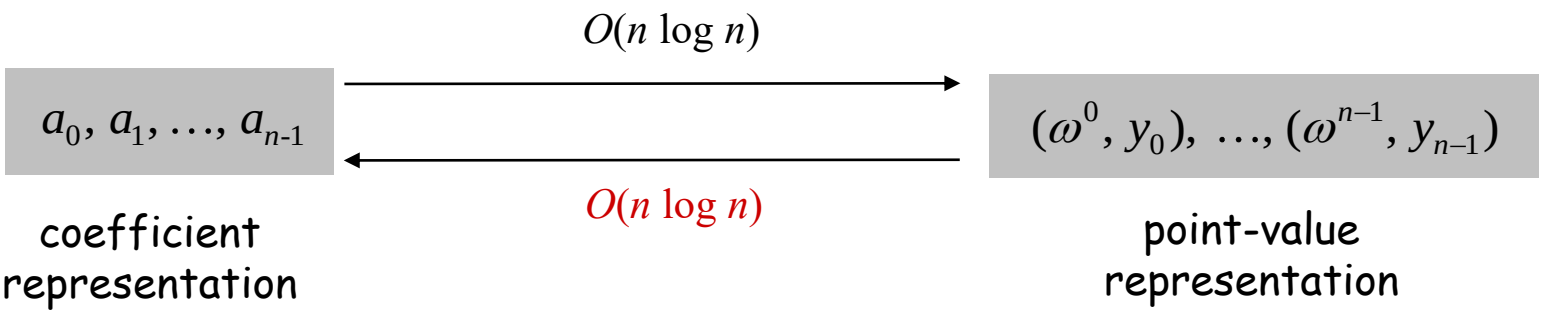
Inverse FFT: Algorithm

```
ifft(n, a_0,a_1,…,a_n-1) {
    if (n == 1) return a_0

    (e_0,e_1,…,e_n/2-1) ← FFT(n/2, a_0,a_2,a_4,…,a_n-2)
    (d_0,d_1,…,d_n/2-1) ← FFT(n/2, a_1,a_3,a_5,…,a_n-1)

    for k = 0 to n/2 - 1 {
        ω^k ← e^-2πik/n
        y_k+n/2 ← (e_k + ω^k d_k) / n
        y_k+n/2 ← (e_k - ω^k d_k) / n
    }

    return (y_0,y_1,…,y_n-1)
}
```

Inverse FFT Summary

Theorem. Inverse FFT algorithm interpolates a degree n-1 polynomial given values at each of the n^{th} roots of unity in $O(n \log n)$ steps.

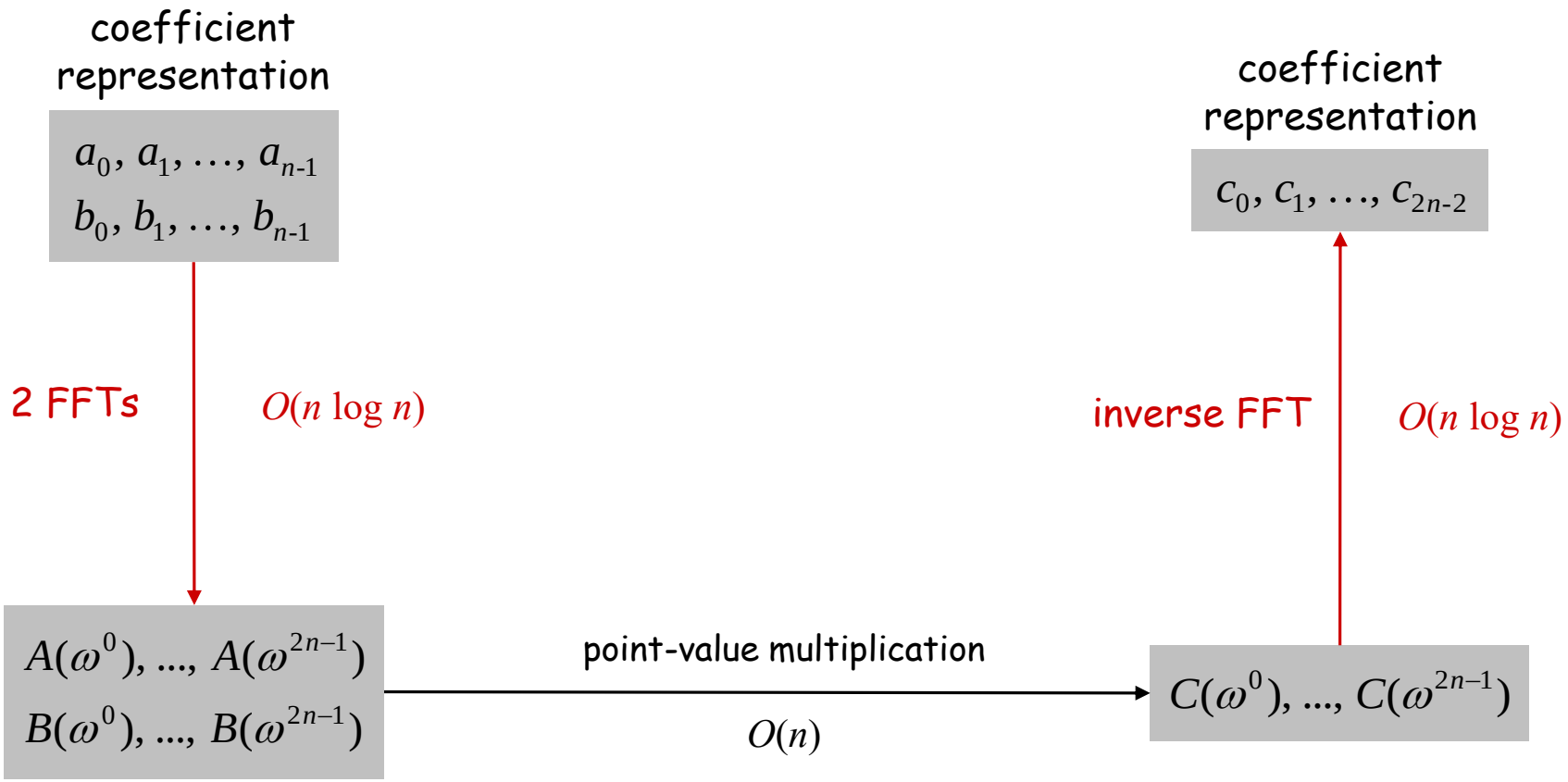
assumes n is a power of 2



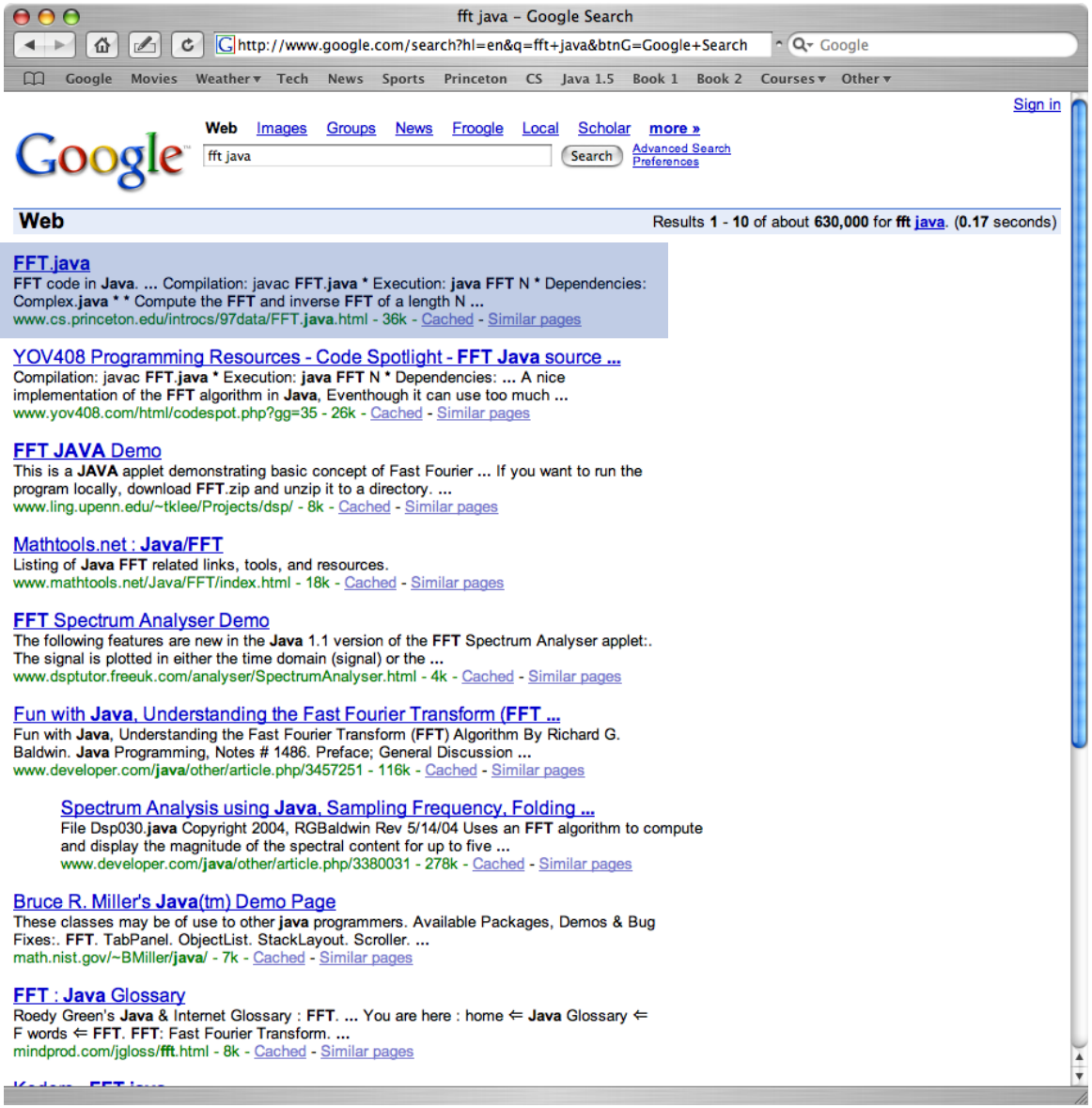
Polynomial Multiplication

Theorem. Can multiply two degree n-1 polynomials in $O(n \log n)$ steps.

pad with 0s to make n a power of 2



FFT in Practice ?



FFT in Practice

Fastest Fourier transform in the West. [Frigo and Johnson]

- Optimized C library.
- Features: DFT, DCT, real, complex, any size, any dimension.
- Won 1999 Wilkinson Prize for Numerical Software.
- Portable, competitive with vendor-tuned code.

Implementation details.

- Instead of executing predetermined algorithm, it evaluates your hardware and uses a special-purpose compiler to generate an optimized algorithm catered to "shape" of the problem.
- Core algorithm is nonrecursive version of Cooley-Tukey.
- $O(n \log n)$, even for prime sizes.

Reference: http://www.fftw.org