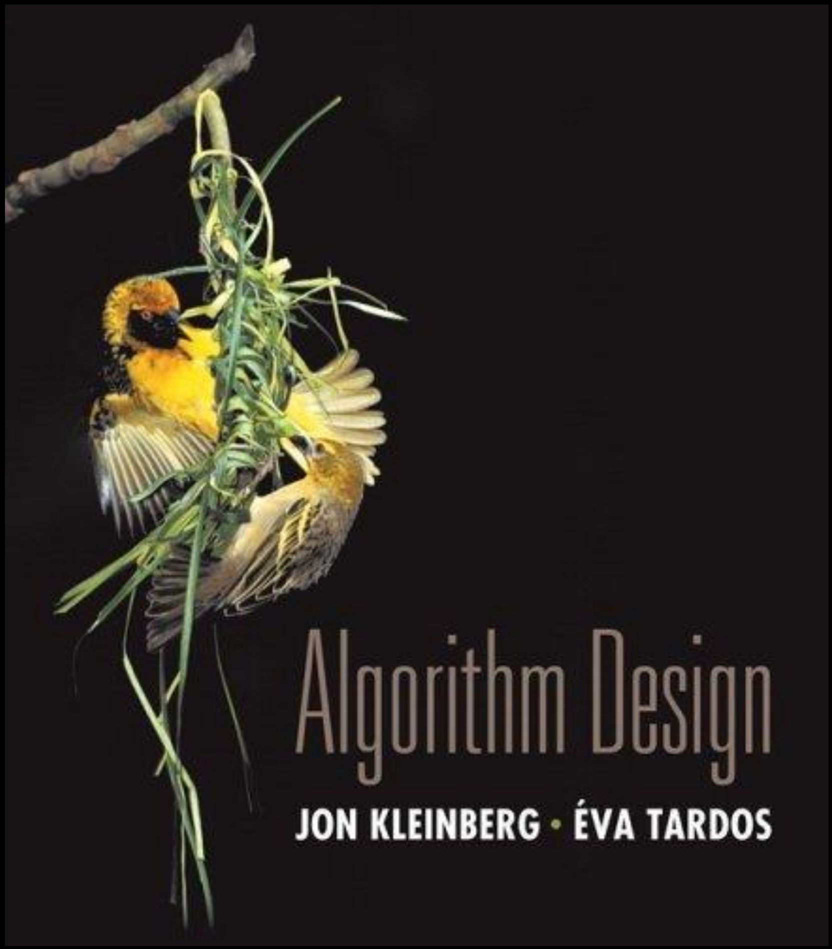


Chapter 1

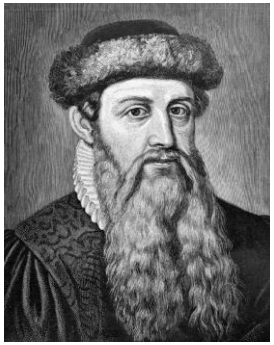
Introduction: Some Representative Problems

CS 350: Winter 2018

0.1 Prologue

Books and Algorithms

1448: Gutenberg – invention of printing press



600: Invention of decimal system (India)
Gives compact notation/ease of computation

9C: Al Khwarizmi
Wrote early, influential text that laid out basic arithmetic methods, including extracting square roots and estimating π, and solving linear and quadratic equations.

These procedures were precise, unambiguous, mechanical, efficient and correct *algorithms*.

An Incomplete Timeline of Early Algorithms

1700-2000 BC: Egyptians develop earliest algorithms for multiplying two numbers.

1600 BC. Babylonians develop earliest known algorithms for factorization and finding square roots.

300 BC: Euclid's Algorithm

200 BC: Sieve of Eratosthenes

263 AD Gaussian Elimination described by Liu Hui

820 Al Khawarizmi treatise on algorithms

Fibonacci

Al Khwarizmi's work would not have gained a foothold in the West were it not for the efforts of Fibonacci (13C).

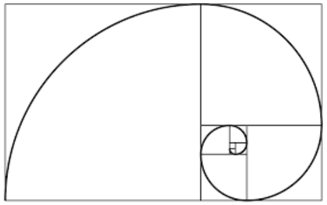
Fibonacci saw the potential of the positional system and worked hard to develop it and further propagandize it.

Fibonacci Sequence:

0,1, 1, 2, 3, 5, 8, 13, 21, 34,...

More formally,

$$F_n = \begin{cases} F_{n-1} + F_{n-2}, & n > 2 \\ 1, & n = 1 \\ 0, & n = 0 \end{cases}$$



No other sequence of numbers has been studied as extensively, or applied to more fields: biology, demography, art, architecture, music, etc.

Fibonacci (cont'd)

The Fibonacci numbers grow almost as fast as the powers of 2: for example, F_{30} is over a million and F_{100} is already 21 digits long! In general: $F_n \approx 2^{0.694n}$

But what is the precise value, of say, F_{100} or F_{200}?

An Exponential Algorithm:

```
Function fib1(n)
 If n=0; return 0
 If n=1; return 1
 Return fib1(n-1)+fib1(n-2)
```

Whenever we have an algorithm, there are (3) fundamental questions we always ask:

(1) Is it correct?
(2) How much time does it take, as a function of n (the input)?
(3) Can we do better?

Fibonacci (cont'd)

Function fib1(n)
 If n=0; return 0
 If n=1; return 1
 Return fib1(n-1)+fib1(n-2)

(1) Is it correct? Yes.

(2) How much time does it take, as a function of n (the input)?

Let T(n) be the number of "computer steps" needed to compute fib1(n).

What can we say about this function? $T(n) \leq 2 \quad for\, n \leq 1$

For n > 1, there are two recursive invocations of fib1, taking time T(n-1) and T(n-2), respectively plus three computer steps (check on the value n and a final addition). Therefore:

$$T(n) = T(n-1) + T(n-2) + 3 \quad for\, n > 1$$

Fibonacci (cont'd)

In summary we have a "correct" algorithm that requires:

$$T(n) = T(n-1) + T(n-2) + 3 \quad for\, n > 1$$

Computer steps in general.

Is this a "good" procedure? No, in fact, it is terrible. Note that:

$$T(n) \geq F_n$$

(why?) Thus T(n) is exponential in n (why?).

More concretely, $T(200) > F_{200} > 2^{138}$.

Note that for a computer that clocks 40 trillion steps per second, fib1(200) would take at least 292 seconds to compute – but by this time our sun will have burnt out.

Lesson learned: the naïve recursive algorithm was hopelessly inefficient.

Fibonacci (cont'd)

Let's try a more sensible scheme:

```
Function fib2(n)
if n=0; return 0
Create array f[0…n]
f[0]=0, f[1] = 1
for i=2:n
 f[i] = f[i-1] + f[i-2]
Return f[n]
```

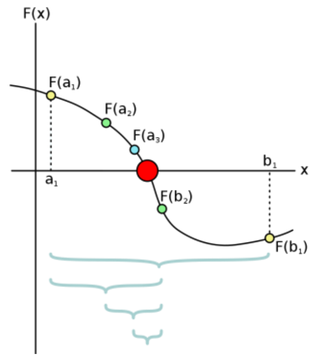
Is this algorithm correct? What is the run-time in terms of n? Are there any additional drawbacks?

Note that using fib2() it is now reasonable to compute F_{200} or even $F_{200,000}$.

Babylonian Cuneiform (c 1700 BC), procedure to approximate root(2).

Root Finding

Suppose that we wish to approximate the root r of a function f(x), continuous on [a,b] where f(r)=0, for r in [a,b]; suppose that f(a)f(b)<0.



Each iteration performs these steps:

Calculate *c*, the midpoint of the interval, *c* = *a* + *b*/2.

Calculate the function value at the midpoint, *f*(*c*).

If convergence is satisfactory (that is, *c* - *a* is sufficiently small, or |*f*(*c*)| is sufficiently small), return *c* and stop iterating.

Examine the sign of *f*(*c*) and replace either (*a*, *f*(*a*)) or (*b*, *f*(*b*)) with (*c*, *f*(*c*)) so that there is a zero crossing within the new interval.

Bisection Method

<u>Root Finding</u>

Suppose that we wish to approximate the root r of a function f(x), continuous on [a,b] where f(r)=0, for r in [a,b]; suppose that f(a)f(b)<0.

Each iteration performs these steps:
Calculate c, the midpoint of the interval, $c = a + b/2$.
Calculate the function value at the midpoint, $f(c)$.
If convergence is satisfactory (that is, $c - a$ is sufficiently small, or $|f(c)|$ is sufficiently small), return c and stop iterating.
Examine the sign of $f(c)$ and replace either $(a, f(a))$ or $(b, f(b))$ with $(c, f(c))$ so that there is a zero crossing within the new interval.

Is the algorithm "correct" (confirm that a solution exists and convergence)? How many iterations are required if the error tolerance is ε?

Bisection Method

Is the algorithm "correct" ?

Yes, by IVT (intermediate value theorem) there exists a solution.

One can show: $$\lim_{n\to\infty}\left|r-c_n\right| \le \lim_{n\to\infty}\frac{\left|b-a\right|}{2^n} = 0$$

Why?

How many iterations are required if the error tolerance is ε?

Since $\left|r-c_n\right| \le \dfrac{\left|b-a\right|}{2^n}$,

Solving for n yields: $n = \log_2\left(\dfrac{b-a}{\varepsilon}\right)$.

How can we use this to approximate root(2), or any root for that matter?
Are there better iterative/approximation methods?

1.1 A First Problem: Stable Matching

Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	1st (favorite)	2nd	3rd (least favorite)
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair m-w is unstable if man m and woman w prefer each other to current partners.
- Unstable pair m-w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

(1st column: favorite; 3rd column: least favorite)

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. Bertha and Xavier will hook up.

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?
A. Yes.

favorite ↓ ... least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ ... least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Formulating the Problem

Let's make the problem as "clean" and clear as possible. Suppose there are n applicants and n companies (analogously: n men and n women).

This simplified version of the stable matching problem preserves the fundamental issues inherent in the problem; in particular, our solution to the simplified version will extend to the more general case as well.

Let sets $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ denote the men and women.
Let MxW denote the set of all possible ordered pairs of the form: (m,w).

A matching S is a set of ordered pairs, each from MxW , with the property that each member of M and each member of W appears in at most one pair in S.

A perfect matching S' is a matching with the property that each member of M and each member of W appears in exactly one pair in S'.

Formulating the Problem

Next, we add the notion of preferences to this setting.

Each man ranks all the women, e.g. m prefers w to w' if m ranks w higher than w'. We refer to the ordered ranking of m as his preference list (no ties allowed). Each woman also ranks all the men.

It is useful to consider the situation: *given a perfect matching S, what can go wrong?*

> There are two pairs (m,w) and (m',w') in S with the property that m prefers w' to w, and w' prefers m to m'. Such a pair: (m,w') is an instability wrt S.

Stable Matching Problem

Formulating the Problem

Our goal, then is a set of marriages with no instabilities.

We say that a set of marriages S is stable if:

(1) It is perfect
(2) There is no instability

(2) Immediate questions:
(*) Does there exist a stable matching for every preference list; if so, is the solution unique?

(*) Given preference lists, can we efficiently construct a stable matching?

Stable Matching Problem

Some examples

(I) Consider the preference list:

m prefers w to w'

m' prefers w' to w

w prefers m to m'

w' prefers m' to m

This list represents complete agreement, and we accordingly have the following *unique* stable matching: (m,w), (m',w').

(II) Consider:

m prefers w to w'

m' prefers w' to w

w prefers m' to m

w' prefers m to m'

Here there are two distinct stable matchings: (m,w), (m', w') (men happy as possible); (m',w), (m,w') (women happy).

Designing an Algorithm

(*) We show that there exists a stable matching for every set of preference lists among men and women.

Motivating the algorithm:

(*) Initially everyone is unmarried; suppose an unmarried man m chooses the woman w who ranks highest on his preference list and *proposes* to her.

(*) Can we declare immediately that (m,w) will be one of the pairs in our final stable matching? Not necessarily (why?). Thus we introduce engagement as a natural intermediate state.

(*) Suppose we are at a state in which some men and women are free (i.e. not engaged); an arbitrary free man m chooses the highest-ranked woman w to whom he has not yet proposed, and he proposes to her.

Designing an Algorithm

Motivating the algorithm:
(*) Suppose we are at a state in which some men and women are free (i.e. not engaged); an arbitrary free man m chooses the highest-ranked woman w to whom he has not yet proposed, and he proposes to her.

If w is also free, then m and w become engaged. Otherwise, w is already engaged to some other man m'. In this case, she determines which of m or m' ranks higher on her preference list; this man becomes engaged to w and the other becomes free.

(*) Finally, the algorithm will terminate when no one is free; at this moment, all engagements are declared find, an the resulting perfect matching is returned.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Bertha.

- Bertha accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Diane.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Diane.

- Diane accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Xavier proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Xavier proposes to Bertha.

- Bertha dumps Victor and accepts Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Amy.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	~~Victor~~	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Victor proposes to Amy.

- Amy accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Amy.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Amy.

- Amy rejects since she prefers Victor.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Diane.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Diane.

- Diane dumps Wyatt and accepts Yancey.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Bertha.

- Bertha rejects since she prefers Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Amy.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Amy.

- Amy rejects since she prefers Victor.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Clare.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Wyatt proposes to Clare.

- Clare accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Bertha.

- Bertha rejects since she prefers Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Diane.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Zeus proposes to Diane.

- Diane rejects Yancey and accepts Zeus.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Clare.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	~~Clare~~	Bertha	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Wyatt~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	~~Yancey~~	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Clare.

- Clare rejects since she prefers Wyatt.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	~~Bertha~~	Amy	Diane	Erika	Clare
Wyatt	~~Diane~~	~~Bertha~~	~~Amy~~	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	~~Amy~~	~~Diane~~	~~Clare~~	Bertha	Erika
Zeus	~~Bertha~~	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	~~Wyatt~~	~~Yancey~~	Xavier
Bertha	Xavier	~~Wyatt~~	Yancey	~~Victor~~	~~Zeus~~
Clare	Wyatt	Xavier	~~Yancey~~	Zeus	Victor
Diane	Victor	Zeus	~~Yancey~~	Xavier	~~Wyatt~~
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Bertha.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Bertha.

- Bertha rejects since she prefers Xavier.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Erika.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

Yancey proposes to Erika.

- Erika accepts since previously unmatched.

Men's Preference Profile

	0th	1st	2nd	3rd	4th
Victor	Bertha	Amy	Diane	Erika	Clare
Wyatt	Diane	Bertha	Amy	Clare	Erika
Xavier	Bertha	Erika	Clare	Diane	Amy
Yancey	Amy	Diane	Clare	Bertha	Erika
Zeus	Bertha	Diane	Amy	Erika	Clare

Women's Preference Profile

	0th	1st	2nd	3rd	4th
Amy	Zeus	Victor	Wyatt	Yancey	Xavier
Bertha	Xavier	Wyatt	Yancey	Victor	Zeus
Clare	Wyatt	Xavier	Yancey	Zeus	Victor
Diane	Victor	Zeus	Yancey	Xavier	Wyatt
Erika	Yancey	Wyatt	Zeus	Xavier	Victor

STOP

- Everyone matched.
- Stable matching!

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

- Case 1: Z never proposed to A.
  - ⇒ Z prefers his GS partner to A. ← men propose in decreasing order of preference
  - ⇒ A-Z is stable.

- Case 2: Z proposed to A.
  - ⇒ A rejected Z (right away or later)
  - ⇒ A prefers her GS partner to Z. ← women only trade up
  - ⇒ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

S*
Amy-Yancey
Bertha-Zeus
. . .

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named 1, …, n.
- Assume women are named 1', …, n'.

Engagements.

- Maintain a list of free men, e.g., in a queue (FIFO)
- Maintain two arrays `wife[m]`, and `husband[w]`.
  - set entry to `0` if unmatched
  - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man `m`.

Efficient Implementation

Women rejecting/accepting.

- Does woman `w` prefer man `m` to man `m'`?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after O(n) preprocessing.

Amy	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1st	2nd	3rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S* is man-optimal.

Pf. (by contradiction)

- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

since this is first rejection by a valid partner

S
Amy-Yancey
Bertha-Zeus
. . .

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, A-Z is an unstable in S. ▪

S
Amy-Yancey
Bertha-Zeus
. . .

Extensions: Matching Residents to Hospitals

Ex: Men $\approx$ hospitals, Women $\approx$ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Def. Matching S unstable if there is a hospital h and resident r such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

Lessons Learned

Powerful ideas learned in course.

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

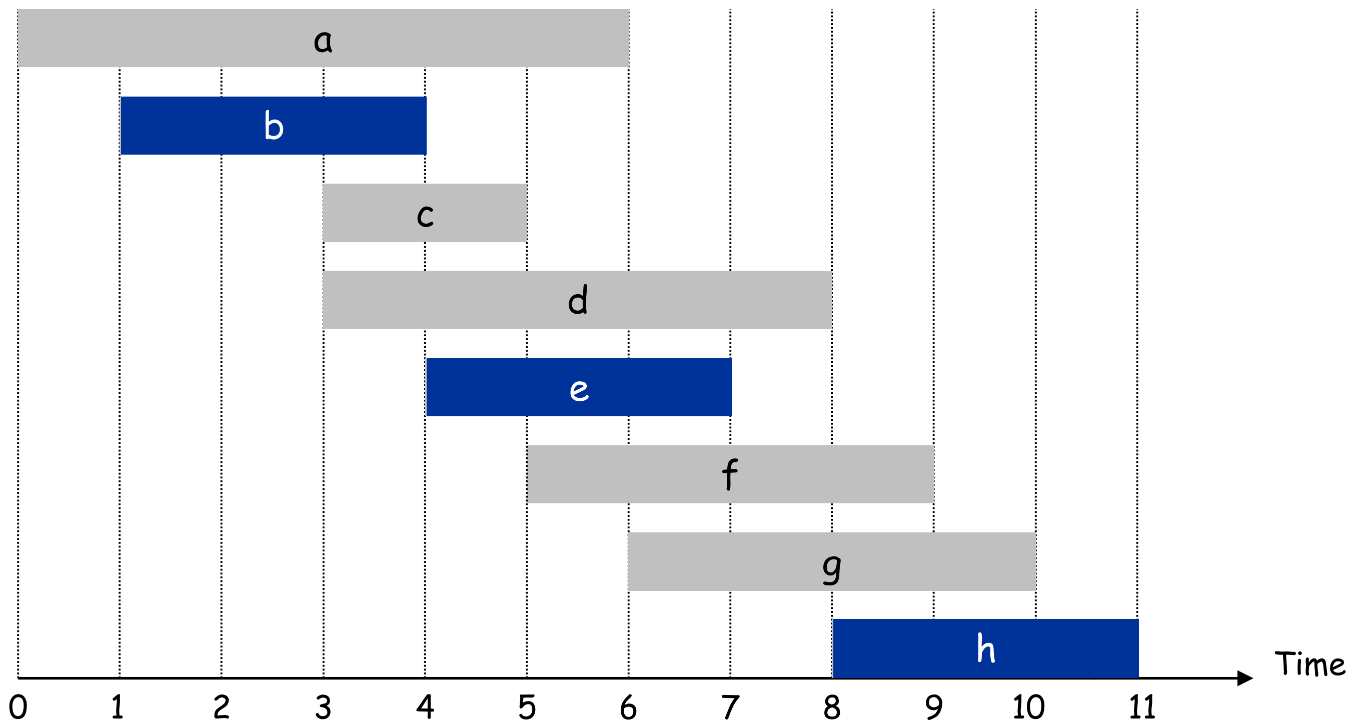
1.2 Five Representative Problems

Interval Scheduling

Input. Set of jobs with start times and finish times.

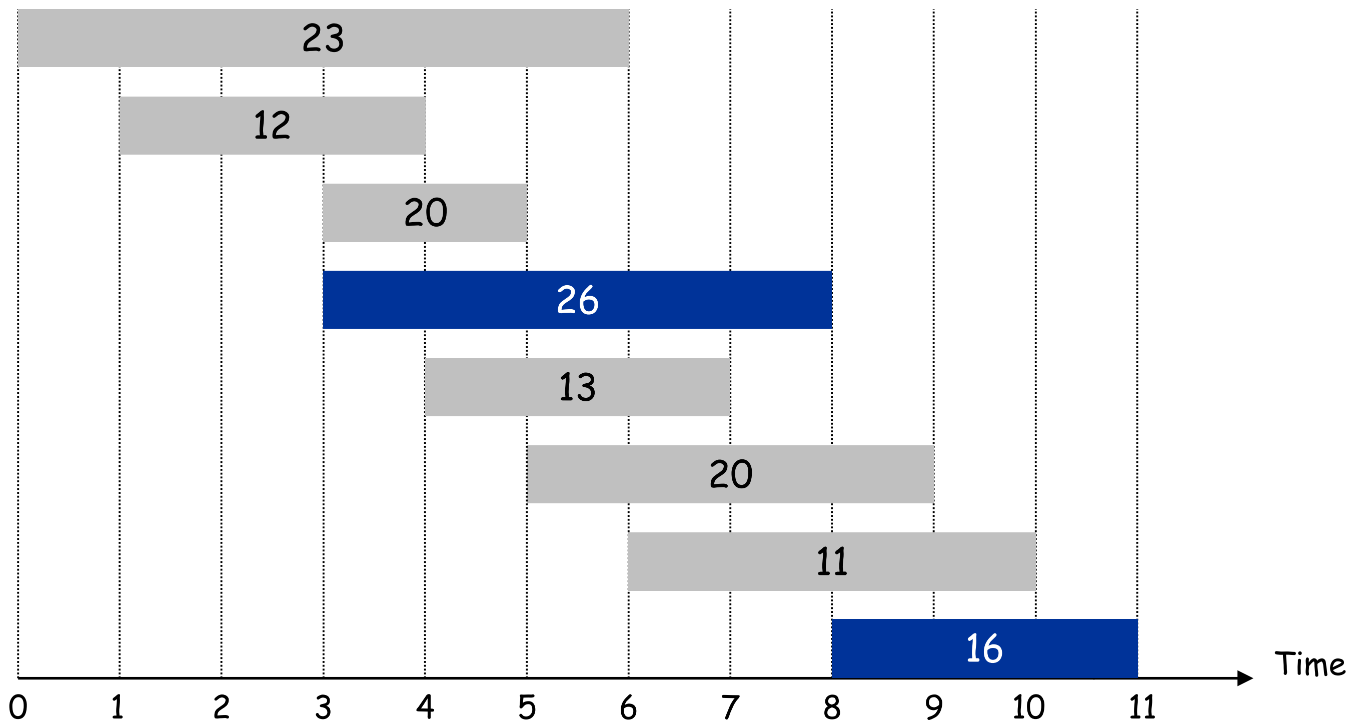
Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



Weighted Interval Scheduling

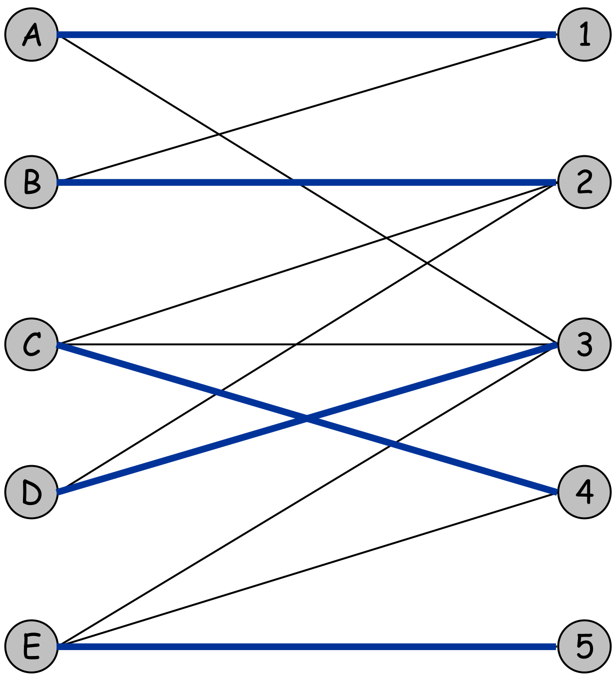
Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.



Bipartite Matching

Input. Bipartite graph.

Goal. Find maximum cardinality matching.

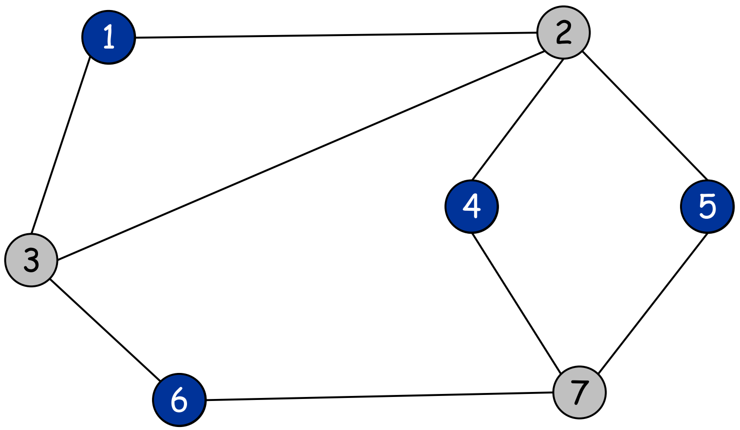


Independent Set

Input. Graph.

Goal. Find maximum cardinality independent set.

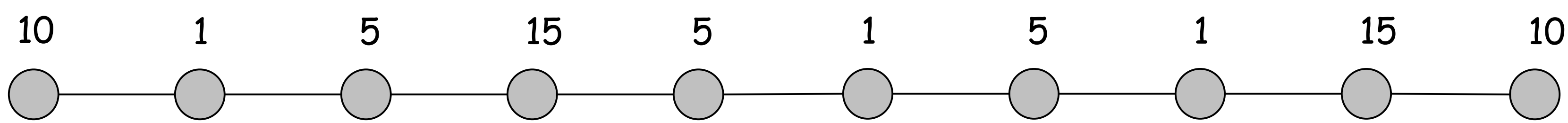
subset of nodes such that no two joined by an edge



Competitive Facility Location

Input. Graph with weight on each node.
Game. Two competing players alternate in selecting nodes.
Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: n log n greedy algorithm.
Weighted interval scheduling: n log n dynamic programming algorithm.
Bipartite matching: n^k max-flow based algorithm.
Independent set: NP-complete.
Competitive facility location: PSPACE-complete.