7.2: Integration by Parts

Summary of important rules, p512.

\[
\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)
\]

\[
f(x)g(x) = \int f'(x)g(x) + f(x)g'(x) \, dx
\]

\[
\int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx
\]

Integrating both sides at step 2. Below is a bit easier notation to remember that also keeps track of the differentials.

\[
\frac{d}{dx} [uv] = (du/dx)v + u(dv/dx)
\]

\[
uv = \int v(du/dx) \, dx + \int u(dv/dx) \, dx
\]

\[
\int u \, dv = uv - \int v \, du
\]

How do we take advantage of this pattern? We choose parts of our integral to identify as \(u, dv\) and then calculate \(v, du\), and plug in the appropriate values to the right-hand side of the equation above.

Two general guidelines for choosing \(u, dv\):
- You want \(u\) to get \textit{simpler} after differentiation.
- \(dv\) needs to be \textit{possible} to integrate.

Though different than the book, it is better ‘housekeeping’ to keep good track of the differentials, like we did with \(u\)-substitution.

\textbf{Two Worked Examples:}

1. \(\int xe^x \, dx\)

   Let \(u = x, dv = e^x \, dx\). I like to use an array to organize my information:

   \[
   \begin{array}{l|l}
   u = x & v = e^x \\
   dv = e^x \, dx & \int dv = \int e^x \, dx
   \end{array}
   \]

   Now we differentiate \(u\) to find \(du\), and we integrate \(dv\) to find \(v\).

   \[
   \begin{array}{l|l}
   u = x & v = e^x \\
   du = (1)dx & \int dv = \int e^x \, dx
   \end{array}
   \]

   Then

   \[
   \int xe^x \, dx = xe^x - \int e^x \, (1)dx
   \]

   \[
   = xe^x - e^x + C
   \]

   Check by differentiating to see that this really is the desired ant-derivative.
2. \( \int x^2 \ln(x) \, dx \)

Let \( u = \ln(x), \, dv = x^2 \, dx \), since at this point we don’t know the anti-derivative of \( \ln(x) \). Again, we differentiate \( u \) to find \( du \), and we integrate \( dv \) to find \( v \).

\[
\begin{array}{c|c}
 u = \ln(x) & v = \frac{x^3}{3} \\
 du = \left( \frac{1}{x} \right) dx & dv = x^2 dx \\
\end{array}
\]

Then

\[
\int x^2 \ln(x) \, dx = \frac{x^3}{3} \ln(x) - \int \frac{x^3}{3} \left( \frac{1}{x} \right) \, dx \\
= \frac{x^3}{3} \ln(x) - \int \frac{x^2}{3} \, dx \\
= \frac{x^3}{3} \ln(x) - \frac{x^3}{9} + C
\]

Check by differentiating to see that this really is the desired ant-derivative.