Transformations

Let \( X \) and \( Y \) have joint pdf \( f(x,y) = \frac{1}{4} e^{-\frac{1}{2}(x+y)}, \) \( 0 < x < \infty \)
\( 0 < y < \infty \)

Find the pdf of \( U = \frac{1}{2}(X-Y) \)

Let \( V = Y \)

Solve for \( X \) and \( Y \):
\( X = 2u + v \)
\( Y = v \)

Jacobian:
\[ J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2 \]

\( g(u,v) = f(x,y) |J| \)
\[ g(u,v) = \frac{1}{4} e^{-\frac{1}{2}(2u+v+v)} \cdot 2 \]
\[ = \frac{1}{2} e^{-(u+v)} \]

\[ g(u,v) = \begin{cases} \frac{1}{2} e^{-(u+v)} & 0 < 2u+v < \infty \\ 0 < v < \infty \end{cases} \]

\[ \begin{cases} \frac{1}{2} e^{-(v+2u)} & v > -2u \end{cases} \]
\[ g_u(u) = \begin{cases} 
\frac{1}{z} e^{-(u+u)} & \text{if } u < 0 \\
\int_{0}^{\infty} \frac{1}{z} e^{-v} \, dv & \text{if } u \geq 0
\end{cases} \]

\text{Case 1:} \quad \frac{1}{z} e^{-u} \left[ -e^{-v} \right]_{v=-2u}^{\infty} = \frac{1}{z} e^{-u} \left[ 0 - (-e^{2u}) \right]

\text{Case 2:} \quad \frac{1}{z} e^{-u} \left[ -e^{-v} \right]_{v=0}^{\infty} = \frac{1}{z} e^{-u} \left[ 0 - (0) \right] = \frac{1}{z} e^{-u}

\[ g_u(u) = \begin{cases} 
\frac{1}{z} e^{u} & u < 0 \\
\frac{1}{2} e^{-u} & u \geq 0 
\end{cases} \]

\[ = \frac{1}{2} e^{-|u|} \quad -\infty < u < \infty \]

(Laplace distribution)

New example: Let \( X \sim \mathcal{N}(0,1) \) be independent standard normal r.v.'s

Let \( U = \frac{X}{Y} \). Find the pdf of \( U \).
Let \( v = y \), \( x = uv \)

\[ y = v \]

\[ J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v \]

\[ g(u,v) = f(x,v) |J| \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} |v| \]

\[ = \frac{1}{2\pi} e^{-\frac{1}{2}(u^2v^2+v^2)} |v| \]

\[ = \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} |v| \]

\[ -\infty < uv < \infty \\
-\infty < v < \infty \]

\[ g_u(u) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} |v| \, dv \]

\[ = 2 \int_0^{\infty} \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1)} v \, dv \]

(since we are integrating an even function)
\[ \text{let } S = \frac{1}{2} v^2 (u^2+1) \]
\[ \frac{ds}{dv} = \frac{1}{2} 2u (u^2+1) \]
\[ = \int_0^\infty \frac{1}{\pi} e^{-s} \frac{ds}{u^2+1} \]
\[ = \frac{1}{\pi} \frac{1}{u^2+1} \left[-e^{-s}\right]_0^\infty \]
\[ g_u(u) = \frac{1}{\pi} \frac{1}{u^2+1} \quad -\infty < u < \infty \]

This is the Cauchy distribution.

Recall: \( \text{Cov}(x,y) = \sigma_{xy} = E[(x-\mu_x)(y-\mu_y)] \)
\[ = E[xy] - \mu_x \mu_y \]
\[ \text{Corr}(x,y) = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} \]

Example: \( f(x,y) = x+y \quad 0 < x < 1 \)
\[ \quad \quad \quad \quad \quad \quad 0 < y < 1 \]

Find \( \rho_{xy} \)
\[ E(xy) = \int_0^1 \int_0^1 xy(x+y) \, dx \, dy \]
\begin{align*}
&\int_0^1 \int_0^{\frac{x}{y}} (x^2 + y^2) \, dy \, dx \\
&= \int_0^1 \left[ \frac{x^3}{3} + \frac{x^2}{2} \right]_{x/2}^{x} \, dx \\
&= \int_0^1 \frac{x^3}{3} + \frac{x^2}{2} \, dx = \left[ \frac{x^4}{12} + \frac{x^3}{6} \right]_0^1 \\
&= \frac{1}{3} \\
&f_x(x) = \int_0^1 x + y \, dy = (x + \frac{y^2}{2}) \bigg|_0^1 \\
&= x + \frac{1}{2}, \quad 0 < x < 1
\end{align*}

\begin{align*}
E(X) &= \int_0^1 x \left( x + \frac{1}{2} \right) \, dx \\
&= \left[ \frac{x^3}{3} + \frac{x^2}{4} \right]_0^1 \\
&= \frac{5}{12} \\
E(X^2) &= \int_0^1 x^2 \left( x + \frac{1}{2} \right) \, dx \\
&= \left[ \frac{x^4}{4} + \frac{x^3}{3} \right]_0^1 \\
&= \frac{5}{12}
\end{align*}

Since the joint density had the property that $X$ and $Y$ are interchangeable, the density of $Y$ is identical to the density of $X$. 
\[ \mu_x = \frac{7}{12}, \quad \sigma_x = \sqrt{\frac{5}{12} - \frac{44}{144}} = \frac{\sqrt{11}}{12} \]  

\[ \mu_y = \frac{7}{12}, \quad \sigma_y = \frac{\sqrt{11}}{12} \]

\[ \sigma_{xy} = \frac{1}{3} - \frac{7 \cdot 7}{12} = -\frac{1}{144} \]

\[ \rho_{xy} = \frac{-\frac{1}{144}}{\frac{\sqrt{11}}{12} \cdot \frac{\sqrt{11}}{12}} = -\frac{1}{11} \]

What does \( \rho_{xy} \) actually measure?

Suppose \( E[y|X] \) is a linear function of \( X \),
that is, assume \( E[y|X] = a + bX \)

Then \( E[y] = E[E[y|X]] = E[a + bX] \)

\[ = a + bE[X] \]

Also \( X E[y|X] = aX + bX^2 \)

and \( E[X E[y|X]] = aE[X] + bE[X^2] \)
\[ M_{xy} = \mathbb{E}[X \mathbb{E}[Y|X]] = \int_0^\infty \int_0^\infty \mathbb{E}[Y|X=x] f_X(x) \, dx \, dy \]

\[ = \int_0^\infty \left[ \int_0^\infty f(Y|X=x) \, dy \right] \frac{f_X(x)}{f_X(x)} \, dx \]

\[ = \int_0^\infty \int_0^\infty xy \, f(x,y) \, dy \, dx \]

\[ = \mathbb{E}[XY] \]

(14)

\[ \sigma_{xy} = \mathbb{E}[XY] - \mu_x \mu_y \]

\[ = a \mu_x + b \mathbb{E}[X^2] - \mu_x (a + b \mu_x) \]

\[ = b (\sigma_x^2 + \mu_x^2) - b \mu_x^2 \]

\[ = b \sigma_x^2 \]

\[ \rho_{xy} = \frac{b \sigma_x^2}{\sigma_x \sigma_y} = b \frac{\sigma_x}{\sigma_y} \]

HW #1 due 1/14 p.192 #4, 11, 16, 17, 20
**Theorem 4.7.9 (Covariance Inequality)** Let $X$ be any random variable and $g(x)$ and $h(x)$ any functions such that $Eg(X)$, $Eh(X)$, and $E(g(X)h(X))$ exist.

(a) If $g(x)$ is a nondecreasing function and $h(x)$ is a nonincreasing function, then
\[ E(g(X)h(X)) \leq (Eg(X))(Eh(X)). \]

(b) If $g(x)$ and $h(x)$ are either both nondecreasing or both nonincreasing, then
\[ E(g(X)h(X)) \geq (Eg(X))(Eh(X)). \]

The intuition behind the inequality is easy. In case (a) there is negative correlation between $g$ and $h$, while in case (b) there is positive correlation. The inequalities merely reflect this fact. The usefulness of the Covariance Inequality is that it allows us to bound an expectation without using higher-order moments.

**4.8 Exercises**

4.1 A random point $(X, Y)$ is distributed uniformly on the square with vertices $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. That is, the joint pdf is $f(x, y) = \frac{1}{4}$ on the square. Determine the probabilities of the following events.

(a) $X^2 + Y^2 < 1$
(b) $2X - Y > 0$
(c) $|X + Y| < 2$

4.2 Prove the following properties of bivariate expectations (the bivariate analog to Theorem 2.2.5). For random variables $X$ and $Y$, functions $g_1(x,y)$ and $g_2(x,y)$, and constants $a$, $b$, and $c$:

(a) $E(a g_1(X,Y) + b g_2(X,Y) + c) = aE(g_1(X,Y)) + bE(g_2(X,Y)) + c.$
(b) If $g_1(x,y) \geq 0$, then $E(g_1(X,Y)) \geq 0.$
(c) If $g_1(x,y) \geq g_2(x,y)$, then $E(g_1(X,Y)) \geq E(g_2(X,Y)).$
(d) If $a \leq g_1(x,y) \leq b$, then $a \leq E(g_1(X,Y)) \leq b.$

4.3 Using Definition 4.1.1, show that the random vector $(X, Y)$ defined at the end of Example 4.1.5 has the pmf given in that example.

4.4 A pdf is defined by
\[ f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases} \]

(a) Find the value of $C$.
(b) Find the marginal distribution of $X$.
(c) Find the joint cdf of $X$ and $Y$.
(d) Find the pdf of the random variable $Z = 9/(X+1)^2$.

4.5 (a) Find $P(X > \sqrt{Y})$ if $X$ and $Y$ are jointly distributed with pdf
\[ f(x,y) = x + y, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1. \]

(b) Find $P(X^2 < Y < X)$ if $X$ and $Y$ are jointly distributed with pdf
\[ f(x,y) = 2x, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1. \]
4.6 A and B agree to meet at a certain place between 1 PM and 2 PM. Suppose they arrive at the meeting place independently and randomly during the hour. Find the distribution of the length of time that A waits for B. (If B arrives before A, define A’s waiting time as 0.)

4.7 A woman leaves for work between 8 AM and 8:30 AM and takes between 40 and 50 minutes to get there. Let the random variable \( X \) denote her time of departure, and the random variable \( Y \) the travel time. Assuming that these variables are independent and uniformly distributed, find the probability that the woman arrives at work before 9 AM.

4.8 Referring to Miscellanea 4.9.1.
(a) Show that \( P(X = m|X = m) = P(X = 2m|X = m) = 1/2 \), and verify the expressions for \( P(M = x|X = x) \) and \( P(M = x/2|X = x) \).
(b) Verify that one should trade only if \( \pi(x/2) < 2\pi(x) \), and if \( \pi \) is the exponential(\( \lambda \)) density, show that it is optimal to trade if \( x < 2\log 2/\lambda \).
(c) For the classical approach, show that \( P(Y = 2x|X = m) = 1 \) and \( P(Y = x/2|X = 2m) = 1 \) and that your expected winning if you trade or keep your envelope is \( E(Y) = 3m/2 \).

4.9 Prove that if the joint cdf of \( X \) and \( Y \) satisfies
\[
F_{X,Y}(x, y) = F_X(x)F_Y(y),
\]
then for any pair of intervals \((a, b)\) and \((c, d)\),
\[
P(a \leq X \leq b, c \leq Y \leq d) = P(a \leq X \leq b)P(c \leq Y \leq d).
\]

4.10 The random pair \((X, Y)\) has the distribution

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(a) Show that \( X \) and \( Y \) are dependent.
(b) Give a probability table for random variables \( U \) and \( V \) that have the same marginals as \( X \) and \( Y \) but are independent.

4.11 Let \( U \) the number of trials needed to get the first head and \( V \) the number of trials needed to get two heads in repeated tosses of a fair coin. Are \( U \) and \( V \) independent random variables?

4.12 If a stick is broken at random into three pieces, what is the probability that the pieces can be put together in a triangle? (See Gardner 1961 for a complete discussion of this problem.)

4.13 Let \( X \) and \( Y \) be random variables with finite means.
(a) Show that
\[
\min_{g(x)} \mathbb{E} (Y - g(X))^2 = \mathbb{E} (Y - \mathbb{E}(Y|X))^2,
\]
where \( g(x) \) ranges over all functions. ( \( \mathbb{E}(Y|X) \) is sometimes called the regression of \( Y \) on \( X \), the “best” predictor of \( Y \) conditional on \( X \).)
(b) Show that equation (2.2.4) can be derived as a special case of part (a).
4.14 Suppose $X$ and $Y$ are independent $n(0, 1)$ random variables.
(a) Find $P(X^2 + Y^2 < 1)$.
(b) Find $P(X^2 < 1)$, after verifying that $X^2$ is distributed $\chi^2$.

4.15 Let $X \sim \text{Poisson}(\theta)$, $Y \sim \text{Poisson}(\lambda)$, independent. It was shown in Theorem 4.3.2 that the distribution of $X + Y$ is Poisson$(\theta + \lambda)$. Show that the distribution of $X|X + Y$ is binomial with success probability $\theta/(\theta + \lambda)$. What is the distribution of $Y|X + Y$?

4.16 Let $X$ and $Y$ be independent random variables with the same geometric distribution.
(a) Show that $U$ and $V$ are independent, where $U$ and $V$ are defined by

$$U = \min(X, Y) \quad \text{and} \quad V = X - Y.$$ 

(b) Find the distribution of $Z = X/(X + Y)$, where we define $Z = 0$ if $X + Y = 0$.
(c) Find the joint pdf of $X$ and $X + Y$.

4.17 Let $X$ be an exponential(1) random variable, and define $Y$ to be the integer part of $X + 1$, that is

$$Y = i + 1 \quad \text{if and only if} \quad i \leq X < i + 1, \quad i = 0, 1, 2, \ldots.$$ 

(a) Find the distribution of $Y$. What well-known distribution does $Y$ have?
(b) Find the conditional distribution of $X - 4$ given $Y \geq 5$.

4.18 Given that $g(x) \geq 0$ has the property that

$$\int_0^\infty g(x) \, dx = 1,$$

show that

$$f(x, y) = \frac{2g(\sqrt{x^2 + y^2})}{\pi \sqrt{x^2 + y^2}}, \quad x, y > 0,$$

is a pdf.

4.19 (a) Let $X_1$ and $X_2$ be independent $n(0, 1)$ random variables. Find the pdf of $(X_1 - X_2)^2/2$.
(b) If $X_i, i = 1, 2$, are independent gamma$(\alpha_i, 1)$ random variables, find the marginal distributions of $X_1/(X_1 + X_2)$ and $X_2/(X_1 + X_2)$.

4.20 $X_1$ and $X_2$ are independent $n(0, \sigma^2)$ random variables.
(a) Find the joint distribution of $Y_1$ and $Y_2$, where

$$Y_1 = X_1^2 + X_2^2 \quad \text{and} \quad Y_2 = \frac{X_1}{|Y_1|}.$$ 

(b) Show that $Y_1$ and $Y_2$ are independent, and interpret this result geometrically.

4.21 A point is generated at random in the plane according to the following polar scheme. A radius $R$ is chosen, where the distribution of $R^2$ is $\chi^2$ with 2 degrees of freedom. Independently, an angle $\theta$ is chosen, where $\theta \sim \text{uniform}(0, 2\pi)$. Find the joint distribution of $X = R\cos \theta$ and $Y = R\sin \theta$. 