Applications to Leray–Schauder’s theorem

Consider the BVP

\[
-\Delta u + b(u) + \mu u = f \quad \text{in } \Omega \subset \mathbb{R}^n
\]

\[u|_{\partial \Omega} = 0\]

where \(\Omega\) is a bounded domain with smooth boundary, \(b: \mathbb{R}^n \to \mathbb{R}\) is smooth, Lipschitz function and \(f \in L^2(\Omega)\).

\textbf{Theorem (existence)} If \(\mu > 0\) is sufficiently large then there is a weak solution \(u \in H^1(\Omega) \cap H^2(\Omega)\) to (1) [there is a constant \(\delta > 0\) such that (1) has a weak solution for any \(\mu \geq \delta\)].

\textbf{Proof} Since \(b: \mathbb{R}^n \to \mathbb{R}\) is Lipschitz then

\[|b(p) - b(q)| \leq \mathcal{L}p - 2\|p\|_{\mathbb{R}^n}\]

such that \(b\) satisfies the growth condition

\[|b(p)| \leq \mathcal{L}p + |b(0)| \leq C(\|p\|_{\mathbb{R}^n} + 1), \quad \forall p \in \mathbb{R}^n\]

where \(C = \max\{\mathcal{L}, |b(0)|\}\)
For \( w \in H^1_0(\Omega) \) let \( f : \Omega \to \mathbb{R} \) defined as

\[
    f(x) = -b(\nabla w(x))
\]

then

\[
    |f(x)| \leq C(1 + |\nabla w(x)| + 1)
\]

Thus \( f \in L^2(\Omega) \) and \( \|f\|_{L^2(\Omega)} \leq C \|\nabla w\|_{L^2(\Omega)} + 1 \)

Consider the linear problem

\[
    \begin{cases}
    -\Delta u + \mu u = f + g & \text{in } \Omega \\
    u = 0 & \text{on } \partial \Omega
    \end{cases}
\]

Remark: linear elliptic theory, \( \exists ! u \in H^1_0(\Omega) \cap H^2(\Omega) \)

and such that

\[
    \|u\|_{H^1_0} \leq C(\|f\|_{L^2} + \|g\|_{L^2})
\]

in addition, \( \|u\|_{H^2} \leq C_1(\|f\|_{L^2} + \|g\|_{L^2}) \)

Multiply \((***)\) by \(-\Delta u\) and integrate \(\Omega\) to \(\Omega\):

\[
    \int_\Omega |\Delta u|^2 - \mu \int_\Omega u \Delta u = -\int_\Omega (f + g) \Delta u
\]

\[
    \implies \int_\Omega |\Delta u|^2 + \mu \int_\Omega |\Delta u|^2 \leq C(\|f\|_{L^2} + \|g\|_{L^2}) \|\Delta u\|_{L^2}^2
\]

\[
    \implies \int_\Omega |\Delta u|^2 + \mu \gamma_1 \int_\Omega |u|^2 \leq C(\|f\|_{L^2} + \|g\|_{L^2}) \|u\|_{H^2}^2
\]
\[
\Rightarrow \int u^2 + \int |u|^2 \leq C(\|f\|_{L^2}^2 + \|g\|_{L^2}^2) \|u\|_{H^2}^2
\]

On \( H^2(\Omega) \cap H_0^1(\Omega) \), \( \|u\|_{L^2} + \|\Delta u\|_{L^2} \) defines a norm equivalent with \( \|u\|_{H^2} \)

\[
\text{[since A = -\Delta: } H^2 \cap H_0^1 \subset L^2 \hookrightarrow L^2 \text{ is maximal monotone, thus closed operator, } D(A) \text{ is Banach with the graph norm } \|u\|_A = \|u\|_{L^2} + \|\Delta u\|_{L^2} = \|u\|_{L^2} + \|\Delta u\|_{L^2}]
\]

Then \( \|u\|_{H^2} \leq C(\|f\|_{L^2}^2 + \|g\|_{L^2}^2) \).

Define \( T: H_0^1(\Omega) \rightarrow H_0^1(\Omega) \), \( T(u) = u \)

A fixed point to \( T \), \( T(u) = u \) solves the weak problem

\[
\int_\Omega u \delta v + \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v + \int \delta(v) b(v) u, \text{ for all } v
\]

This is weak solution to (\#).

We show that \( T \) is continuous and compact operator.
We show that in fact $T$ is a Lipschitz operator
Let $w_1, w_2 \in H^1_0(\Omega)$, $u_1 = T(w_1)$, $u_2 = T(w_2)$

Then
\[
\int_\Omega \nabla u_1 \cdot \nabla \psi + \mu \int_\Omega u_1 \psi = -\int_\Omega b(\sigma w_1) \psi + \int_\Omega g \psi
\]
\[
\int_\Omega \nabla u_2 \cdot \nabla \psi + \mu \int_\Omega u_2 \psi = -\int_\Omega b(\sigma w_2) \psi + \int_\Omega g \psi
\]

\[
\Rightarrow \int_\Omega \nabla (u_1 - u_2) \cdot \nabla \psi + \mu \int_\Omega (u_1 - u_2) \psi = -\int_\Omega \left( b(\sigma w_1) - b(\sigma w_2) \right) \psi
\]

Let $\psi = u_1 - u_2$

\[
\int_\Omega |\nabla (u_1 - u_2)|^2 + \mu \int_\Omega (u_1 - u_2)^2 \leq L \int_\Omega |\sigma (w_1 - w_2) - (u_1 - u_2)|
\]

\[
\Rightarrow |u_1 - u_2|_{L^1} \leq \frac{L}{\lambda_1} |w_1 - w_2|_{L^\infty}
\]

Thus $T$ is Lipschitz and therefore continuous.
Show that $T$ is compact operator:

If $\{w_k\}_{k=1}^\infty$ denotes a bounded sequence in $H_0^1$, then $\{u_k\}_{k=1}^\infty$, $u_k = T(w_k)$ has a convergent subsequence in $H_0^1$.

Let $\{w_k\}_{k=1}^\infty$, bounded in $H_0^1$ then $\{u_k\}_{k=1}^\infty$, is bounded in $H^2(\Omega)$.

$H_0^1(\Omega) \cap H^2(\Omega) \hookrightarrow H_0^1(\Omega)$ compact implies $u_k \to u$ in $H_0^1$ (subsequence).

Thus $T$ is compact.

Next we show that there is a constant $r > 0$ such that if $\mu > r$ then

$\Delta = \exists u \in H_0^1 : u = \lambda T(u)$ for some $0 \leq \lambda \leq 1$ is bounded in $H_0^1(\Omega)$.

Let $u \in \Delta$, thus $u = \lambda T(u)$ for some $0 \leq \lambda \leq 1$.

Assume $\lambda > 0$ (else $u \equiv 0$).

Then $T(u) = \frac{u}{\lambda}$ such that
\[ \int \sigma \nabla \cdot \nu + \mu \int \nabla \cdot \nu = -2 \int \lambda (\sigma \nabla u) \cdot \nu + 2 \int g \nu \]

for \( \nu = u \)

\[ \Rightarrow \int \lambda \nabla u \cdot \nu + \mu \int \nabla u \cdot \nu = -2 \int \lambda \nabla u \cdot u + 2 \int g \nu \]

Since \( 0 \leq \lambda \leq 1 \)

\[ \Rightarrow \int \lambda \nabla u \cdot \nu + \mu \int \nabla u \cdot \nu \leq \int \lambda \nabla u \cdot \lambda u + \int \nabla u \cdot \nabla u \]

Since \( b \) Lipschitz \( \leq \int \lambda \nabla u \cdot \lambda u + \int \nabla u \cdot \nabla u \]

\[ \leq \int \left( \frac{1}{2} |\nabla u|^2 + \frac{c^2}{2} |u|^2 + \frac{|\nabla u|^2 + c^2}{2} \right) + \]

\[ + \frac{1}{2} \nabla u \cdot \nabla u + \frac{1}{2} \nabla u \cdot \nabla u \]

\[ \Rightarrow \frac{1}{2} \int \lambda \nabla u \cdot \lambda u + \left( \mu - \frac{c^2 + 1}{2} \right) \nabla u \cdot \nabla u \leq C + \frac{1}{2} \nabla u \cdot \nabla u \]

\[ \text{positive for } \mu \geq \frac{c^2 + 1}{2} \]

\[ \text{independent on } \lambda \]

\[ \Rightarrow \text{ for large } \mu, \quad |\nabla u| \text{ is bounded by a constant, independent on } \lambda \]

Thus \( T \) satisfies the Leray-Schauder theorem, there is a fixed point \( T(u) = u \).
Steady-state Navier-Stokes equations

\[ u_t + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p = \mu \Delta u + f \] in \( \mathbb{R}^n \), \( n = 2, 3 \)

\[ u = 0 \] on \( \partial \Omega \)

\( u = (u_1, u_2, u_3) \) velocity field

\( p = \) pressure

\( \rho = \) density \( \Rightarrow \rho_t + \nabla \cdot (\rho u) = 0 \Rightarrow \rho_t + \rho \nabla \cdot u + u \cdot \nabla \rho = 0 \)

Incompressible fluid \( \Rightarrow \rho = \) constant

\[ \nabla \cdot u = 0 \] in \( \mathbb{R}^n \) \( (\nabla \cdot u = 0) \)

Assume \( \rho = 1 \) and steady-state flow \( \Rightarrow u_t = 0 \)

\[ (u \cdot \nabla) u + \nabla p = \mu \Delta u + f \] \( \Rightarrow \) PDE system

\[ (u \cdot \nabla) u = (u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{\partial u}{\partial x} u \]

where \( \frac{\partial u}{\partial x} \) denotes the Jacobian matrix \( [\frac{\partial u_i}{\partial x_j}] = \frac{\partial u_i}{\partial x_j} \)

Component-wise:

\[ \begin{cases} 
    u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} + \frac{\partial p}{\partial x_i} = \mu \Delta u_i + f_i, & i = 1:3 \\
    u_i |_{\partial \Omega} = 0, & i = 1:3 
\end{cases} \]
Weak formulation: Let \( H = H_0^1(\Omega; \mathbb{R}^n) \)
with the inner product \( \langle u, v \rangle_1 = \sum_{i=1}^n \langle u_i, v_i \rangle \),
for \( v \in H_0^1(\Omega; \mathbb{R}^n) \)
\[
\int_\Omega (u \cdot \nabla) u \cdot v + \int_\Omega \nabla p \cdot v = \mu \int_\Omega \Delta u \cdot v + \int_\Omega f \cdot v
\]
Remark
\[
\int_\Omega \nabla p \cdot v = \int_\Omega \nabla (p v) - p \nabla v = \int_\Omega p v \cdot \nabla v - \int_\Omega p \nabla v
\]
\[
\text{if } v = 0 \text{ on } \partial \Omega \quad \Rightarrow \quad \int_\Omega \nabla v = 0 \quad \text{in } \Omega.
\]
Define \( \tilde{H}_0^1 = \{ v \in H_0^1(\Omega; \mathbb{R}^n) : \nabla v = 0 \text{ in } \Omega \} \)
Remark: \( \tilde{H}_0^1 \subset H_0^1 \) closed subspace (prove it!)
thus \( (\tilde{H}_0^1, \langle \cdot, \cdot \rangle_1) \) is Hilbert space.
Notice that \( \tilde{H}_0^1 = F^{-1}(0) \)
where
\[
F: H_0^1(\Omega; \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n), \quad F(v) = \nabla v
\]
is continuous function.
\[
\int_\Omega |\nabla v_k - \nabla v|^2 \leq \int_\Omega \left[ \sum_{i=1}^n \left( \frac{\partial v_{k,i}}{\partial x_i} - \frac{\partial v_{i}}{\partial x_i} \right) \right]^2
\]
\[
\leq n \int_\Omega \sum_{i=1}^n \left( \frac{\partial v_{k,i}}{\partial x_i} - \frac{\partial v_{i}}{\partial x_i} \right)^2 = n \sum_{i=1}^n \|v_{k,i} - v_i\|^2 \to 0
\]
if \( v_k \to v \) in \( H_0^1(\Omega, \mathbb{R}^n) \).
Weak solution: Find \( u \in \tilde{H}_0^1(\Omega; \mathbb{R}^n) \) such that

\[
\int_\Omega (u \cdot \nabla) u \cdot v = \mu \sum \int_{\Omega} \frac{\partial u_i}{\partial x_i} v_i + \int_\Gamma f \cdot v, \quad \forall v \in \tilde{H}_0^1(\Omega; \mathbb{R}^n)
\]

equivalent to

\[
\mu \sum \int_{\Omega} \frac{\partial u_i}{\partial x_i} v_i = \int_\Gamma f \cdot v - \int_\Omega (u \cdot \nabla) u \cdot v, \quad \forall v \in \tilde{H}_0^1(\Omega; \mathbb{R}^n)
\]

For \( w \in \tilde{H}_0^1(\Omega; \mathbb{R}^n) \) arbitrary fixed, define

\[
L : \tilde{H}_0^1 \rightarrow \mathbb{R}, \quad L(v) = \int_\Omega f \cdot v - \int_\Omega (w \cdot \nabla) w \cdot v
\]

\( L \) is linear functional, well-defined on \( \tilde{H}_0^1 \), and continuous.

Remark: \( (w \cdot \nabla) w = \left[ \frac{\partial w}{\partial x} \right] \cdot w \) and \( \|A\|_F \leq \|A\|_F \) \( R^n \rightarrow \mathbb{R}^m \)

Thus pointwise,

\[
|(w \cdot \nabla) w \cdot v|_R \leq \|w \cdot \nabla\|_F \|w\|_R \|v\|_R \leq \|w\|_R \|\frac{\partial w}{\partial x}\|_F \|v\|_R
\]

\( w_i \in H_0^1(\Omega; \mathbb{R}^m) = \begin{cases} w_i \in L^6(\Omega) & \text{if } n = 3 \\ w_i \in L^p(\Omega), \ p > 1 & \text{if } n = 2 \\ w_i \in L^4(\Omega) & \text{if } n = 4 \end{cases} \)

Hölder inequality: \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \), \( f_i \in L^{p_i} \) then

\[
f_1 f_2 f_3 \in L^\frac{3}{2}, \quad \|f_1 f_2 f_3\|_{L^\frac{3}{2}} \leq \frac{3}{2} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}
\]
Then \( \| w \|_{L^2} \in L^4(\mathbb{R}) \) \[
\begin{aligned}
\| \frac{\partial w}{\partial x} \|_{L^1} &\in L^2(\mathbb{R}) \\
\| w \|_{L^4(\mathbb{R})} &\in L^4(\mathbb{R})
\end{aligned}
\]
\[
\therefore \text{(w. o) w. v} \in L^4(\mathbb{R}),
\]
\[
\sum_{i=1}^{n} \| v_i \|_{L^4(\mathbb{R})} \leq \sum_{i=1}^{n} \| w_i \|_{L^4(\mathbb{R})} \leq \sum_{i=1}^{n} \| w_i \|_{L^2(\mathbb{R})} \leq \sum_{i=1}^{n} \| w_i \|_{L^4(\mathbb{R})},
\]
\[
\text{such that } \nu \rightarrow \sum_{i=1}^{n} (w_i, v_i) \text{ w. v is continuous from } \tilde{H}^1 \text{ to } \mathbb{R}.
\]

Riesz theorem: Given \( w \in \tilde{H}^0 \), \( \exists ! u \in \tilde{H}^1 \) solution to
\[
\mu \sum_{\alpha \in \tilde{A}} \omega \nu_{\alpha} u : = \sum_{\alpha \in \tilde{A}} f_{\alpha} \nu - \sum_{\alpha \in \tilde{A}} (w, e_{\alpha}) \nu, \forall v \in \tilde{H}^0
\]

Therefore, we may consider the operator
\[
T : \tilde{H}^1(\mathbb{R}^2, \mathbb{R}^2) \rightarrow \tilde{H}^1(\mathbb{R}^2, \mathbb{R}^2)
\]
\[
T(w) = u
\]

We show that \( T \) satisfies the hypothesis of the Leray-Schauder theorem, such that there is a fixed point to \( T \).
Let $u = T(w)$, $\hat{u} = T(\hat{w})$.

\[ \Rightarrow \mu \sum_{i=1}^{n} \nabla (\hat{u}_i - u_i) \cdot \nabla v_i = - \int_{\Omega} [ (\hat{w} \cdot \sigma) \hat{w} - (w \cdot \sigma) w ] \cdot v \, \mathrm{d}V. \]

Let $v = \hat{u} - u$

\[ \Rightarrow \mu \sum_{i=1}^{n} | \nabla (\hat{u}_i - u_i) |^2 \leq \int_{\Omega} | [(\hat{w} \cdot \sigma) \hat{w} - (w \cdot \sigma) w ] \cdot (\hat{u} - u) | \, \mathrm{d}V. \]

\[ \Rightarrow \mu | \hat{u} - u |^2 \leq \int_{\Omega} | (\hat{w} - w) \cdot \frac{\partial \hat{w}}{\partial x} \cdot (\hat{u} - u) | + \int_{\Omega} | w \cdot \frac{\partial (\hat{w} - w)}{\partial x} \cdot (\hat{u} - u) | \, \mathrm{d}V. \]

\[ \leq C | \hat{w} - w |_1 \cdot [ | \hat{w} |_1 + | w |_1 ] \cdot | \hat{u} - u |_1 \]

\[ \Rightarrow | \hat{u} - u |_1 \leq C | \hat{w} - w |_1 ( | \hat{w} |_1 + | w |_1 ) \]

Thus $T$ is continuous.

Show that $T$ is compact

Let $\{ w^{(m)} \}_{m \geq 1}$, bounded sequence in $\tilde{H}_0^1$, 

$\| w^{(m)} \| \leq M$. Show that $u^{(m)} = T(w^{(m)})$ has a convergent subsequence in $\tilde{H}_0^1$. 
since \( \tilde{H}_0' \subset L^q \) compact, we may assume that, after passing to a subsequence, \( 3w^{(m)} \) is Cauchy (thus convergent) in \( L^q(\Omega; \mathbb{R}^n) \).

We show that the corresponding subsequence \( u^{(m)} = T(3w^{(m)}) \) is Cauchy in \( \tilde{H}_0' \).

\[
\mu \sum_{i=1}^{n} \nabla (w^{(m)}_i - w^{(k)}_i) \cdot \nabla v_i = - \int_{\Omega} (w^{(m)} \cdot \nabla) w^{(m)} - (w^{(k)} \cdot \nabla) w^{(k)}_i v_i.
\]

Use the following identity:

\[ \forall u, v, w \in \tilde{H}_0' : \int_{\Omega} (u \cdot \nabla) v \cdot w = - \int_{\Omega} (u \cdot \nabla) w \cdot v \]

In particular, \( \int_{\Omega} (u \cdot \nabla) u \cdot u = 0, \quad u \in \tilde{H}_0'(\Omega; \mathbb{R}^n) \)

\[
\Rightarrow \mu \sum_{i=1}^{n} \nabla (w^{(m)}_i - w^{(k)}_i) \cdot \nabla v_i = \left( (w^m \cdot \nabla) v \cdot w^m - (w^k \cdot \nabla) v \cdot w^k \right)
\]

\[
= \left[ \left( w^m - w^k \right) \cdot \nabla \right] v \cdot w^m + \left( w^k \cdot \nabla \right) v \cdot \left( w^m - w^k \right)
\]

\[
\leq \left\| \left( w^m - w^k \right) \cdot \nabla \right\|_2 \left\| v \right\|_1 \left\| w^m \right\|_2 + \left\| w^k \right\|_2 \left\| \left( w^m - w^k \right) \right\|_2
\]

\[
\leq C \left\| w^m - w^k \right\|_2 \left\| v \right\|_1 \left( \left\| w^m \right\|_2 + \left\| w^k \right\|_2 \right) \leq C \left\| w^m - w^k \right\|_2 \left\| v \right\|_1 \text{ bounded by } C M
\]
Let $v = u^m - u^k$:

\[
\|u^m - u^k\|_2 \leq C \|w^m - w^k\|_2 \to 0 \text{ since } \\tilde{w}^m \text{ is Cauchy in } L^2.
\]

Thus $\tilde{u}^m$ is Cauchy in $H_0^1$, thus convergent.

Show that $\Lambda = \{ u : u = \tilde{T}(u), 0 \leq \alpha \leq 1 \}$ is bounded.

Let $\alpha > 0$, $u = \tilde{T}(u)$ : $\frac{1}{\alpha} u = \tilde{T}(u)$.

Then

\[
\frac{\mu}{2} \int \sum_{i=1}^{n} \nabla u_i \cdot \nabla \tilde{v}_i = \int f \cdot \tilde{v} - \int (u \cdot \nabla u) \cdot \nabla \tilde{v}
\]

\[
= \mu \int \sum_{i=1}^{n} \nabla u_i \cdot \nabla \tilde{v}_i = 2 \int f \cdot \tilde{v} - 2 \int (u \cdot \nabla u) \cdot \nabla \tilde{v}
\]

Let $v = u$

\[
\mu \int \sum_{i=1}^{n} 1 |\nabla u_i|^2 = 2 \int f \cdot u - 2 \int (u \cdot \nabla u) \cdot \nabla u
\]

\[
= 0 \text{ for any } \tilde{v} \\
\Rightarrow \mu |u|^2 = 2 \int f \cdot u \leq C \|f\|_L^2 |u|_2
\]

\[
\Rightarrow |u|_2 \leq \frac{C_2}{\mu} \|f\|_L^2 \leq \frac{c}{\mu} \|f\|_L^2, \quad c > 0
\]

Thus $\Lambda$ is bounded.