Nonhomogeneous boundary conditions may be considered by formulating the minimization problem over a closed and convex set.

As a general theoretical framework use:

**Theorem** Let $X$ Banach space, reflexive. Let $K \subseteq X$ convex and closed subset.

Let $F : K \rightarrow \mathbb{R}$ continuous, convex, and coercive

\[
\lim_{\|x\| \to \infty} F(x) = +\infty
\]

Then $F$ reaches its minimum on $K$:

There is $x^* \in K : F(x^*) \leq F(x), \; \forall x \in K$

**Remark**: Coercivity is not needed if $K$ is bounded

**Proof**: Let $\{x_k\} \subseteq K : F(x_k) \xrightarrow{k} \inf_{x \in K} F(x)$

Then coercivity implies $\{x_k\}$ is bounded, thus $x_k \xrightarrow{} x^*$ since $X$ is reflexive.

$K$ closed and convex $\Rightarrow$ $K$ is weakly closed, $x^* \in K$

$F$ continuous and convex $\Rightarrow F$ w.l.s.c. $\Rightarrow \inf_{x \in K} F(x^*) \leq \inf_{x \in K} F(x)$
Model problem \( \begin{cases} -\Delta u + g(u) = f \text{ in } \Omega \\ u |_{\partial \Omega} = u_0 \end{cases} \)

Consider \( K = \{ v \in H^1(\Omega) : v |_{\partial \Omega} = u_0 \} \)

Then \( K \) is closed subset of \( H^1(\Omega) \) since trace operator \( \text{Tr}(v) = v |_{\partial \Omega} \) is continuous and \( K \) is convex set since \( [tv, (1-t)v_2] = u_0 \) \( (t, v_2) \in K \)

Variational formulation:

Find \( u \in K : \int_\Omega \nabla u \cdot \nabla v + \int_\Omega g(u)v = \int_\Omega f, \forall v \in K \)

Minimization problem:

\[
\min_{u \in K} J(u), \quad J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \int_\Omega g(u) - \int_\Omega f u
\]

where \( g(z) = \int_0^z g(t) \, dt \)

\( \rightarrow \) assume that \( J \) is well-defined and \( J \in C^1(K) : J'(u)v = \int_\partial \nabla u \cdot \nabla v - \int_\Omega g(u)v \)
Sufficient conditions for the existence of the minimizer

\[ J \text{ is coercive (} \lim_{\|u\| \to \infty} J(u) = +\infty \) \]

and one of the following:

(a) \( G: \mathbb{R} \to \mathbb{R} \) is convex

(b) \( J: \mathbb{R} \to \mathbb{R} \) is bounded

Sufficient conditions for the uniqueness of the minimizer. Each one of the following:

(c) \( G: \mathbb{R} \to \mathbb{R} \) is convex.

(d) \( J: \mathbb{R} \to \mathbb{R} \) is increasing (this is equivalent to (c)). More general, if \( g \) is increasing in the interval \((c_1, c_2)\) then there is at most one solution \( u(x) \) such that \( c_1 \leq u(x) \leq c_2, \ x \in \Omega \).

(E) \( g: \mathbb{R} \to \mathbb{R} \) is such that \( g'(t) > -\lambda \), where \( \lambda \) is the principal eigenvalue of \[
\begin{cases}
-\Delta e_i = \lambda e_i, & \text{in } \Omega \\
e_i |_{\partial \Omega} = 0
\end{cases}
\]
More general, if \( g'(t) > -2 \), for \( c < t < c_2 \) then there is at most one solution such that \( c_1 < u(x) < c_2 \), \( x \in \Omega \).

**Proof of (E)**

Let \( u_1, u_2 \) solutions, \( u_1 \neq u_2 \)

\[
-\Delta u_1 + g(u_1) = f \\
-\Delta u_2 + g(u_2) = f
\]

Then

\[
\begin{align*}
\int_{\Omega} \left[ g(u_1) - g(u_2) \right] (u_1 - u_2) &= 0 \\
\int_{\Omega} g'(u) (u_1 - u_2)^2 &= 0 \\
\int_{\Omega} g'(u_2) (u_1 - u_2)^2 &= 0
\end{align*}
\]

\[
\int_{\Omega} g'(u) (u_1 - u_2)^2 < 0
\]

Contradiction with \( \lambda_1 = \min_{u \in H_0^1} \frac{\int_{\Omega} \Delta u^2}{\int_{\Omega} u^2} \).
Example \[ \begin{aligned} -\Delta u + |u|^{p-1} u &= f & \text{in } \Omega, \quad p > 1 \\ u |_{\partial \Omega} &= u_0 
 \end{aligned} \] 

\((V)\) \[ \begin{aligned} &\int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} |u|^{p-1} u v - \int_{\Omega} fu = 0, \quad \forall v \in H_0^1(\Omega) \n \end{aligned} \]

\((M)\) \[ \begin{aligned} &\min \left\{ u \in H_0^1(\Omega) \right\} J(u), \quad J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{p+1} \int_{\Omega} |u|^{p+1} - \int_{\Omega} fu 
 \end{aligned} \]

Here \[ g(u) = |u|^{p-1} u, \quad G(u) = \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \]

Remark: If \( n \leq 2 \) then \((V)\) and \((M)\) are well-defined for any \( p > 1 \).

If \( n > 3 \) we require \( 1 < p \leq \frac{n+2}{n-2} \)

Since \[ H_0^1(\Omega) \xrightarrow{\text{compact}} L^2 \], \( 1 \leq q \leq 2^* = \frac{2n}{n-2} \)

\[ H_0^1(\Omega) \xrightarrow{\text{continuous}} L^{2^*} \]

Remark: \( G(z) = \frac{1}{p+1} |z|^{p+1} \) is convex for \( p > 1 \)

\( G'(z) = |z|^{p-1} z \), \( G''(z) = p|z|^{p-1} > 0 \).

Thus there is a unique minimizer. Why is \( J \) coercive on \( K \)? (Poincare with B.C.)
Lagrange multipliers theory

References: Evans §4, McOwen 7.2, 13.3

Background: Implicit Function Theorem

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be continuously differentiable, \( f \in C^1(\mathbb{R}^2, \mathbb{R}) \) and \((\bar{x}, \bar{y})\) such that \( f(\bar{x}, \bar{y}) = 0 \). Assume that \( f_y(\bar{x}, \bar{y}) \neq 0 \).

Then there is a vicinity \( B_\varepsilon(\bar{x}, \bar{y}) \) of \((\bar{x}, \bar{y})\), a vicinity \( B_\varepsilon(\bar{x}) \) of \(\bar{x}\), a function \( \varphi : B_\varepsilon(\bar{x}) \to \mathbb{R} \) such that

\[
\begin{align*}
\varphi(\bar{x}) & = \bar{y} \\
f(x, \varphi(x)) & = 0, \quad \forall x \in B_\varepsilon(\bar{x})
\end{align*}
\]

in addition, \( \varphi \) is unique \( \{ (x, y) \in B_\varepsilon(\bar{x}, \bar{y}) \mid f(x, y) = 0, \ y = \varphi(x) \} \) and

\[
\varphi'(x) = -\frac{f_y(x, \varphi(x))}{f_x(x, \varphi(x))} f_y(x, \varphi(x))^{-1}, \quad \forall x \in B_\varepsilon(\bar{x}).
\]
Lagrange multipliers Theorem

Let $X$ be a Banach space, $F: X \to \mathbb{R}$, $G: X \to \mathbb{R}$ be continuously differentiable functions, $F, G \in \mathcal{C}^1(X, \mathbb{R})$. Consider the problem of minimizing $F(x)$ subject to the constraint $G(x) = 0$.

$$\min_{x \in X} F(x) \quad \text{subject to} \quad G(x) = 0$$

Theorem 1: If $x^*$ is a local minimum point to $F$ subject to the constraint $G(x) = 0$, then either

i) $G'(x^*) = 0$ (that is, $G'(x^*) y = 0$, $\forall y \in X$)

or

ii) there is a constant $\lambda \in \mathbb{R}$ such that

$$F'(x^*) = \lambda G'(x^*) \quad \text{(that is, } F'(x^*) y = \lambda G'(x^*) y, \quad \forall y \in X)$$

$\lambda$ is called the Lagrange multiplier.
Proof. Let \( x^* \) local min point and assume that \( G'(x^*) \neq 0 \). Then there is \( y \in X : G'(x^*)y \neq 0 \).

We show that in this case \( y \) is such that \( F'(x^*) w = G'(x^*) w, \forall w \in X \).

Thus (i) holds.

Let \( w \in X \) arbitrary fixed.

Define \( g : \mathbb{R}^2 \to \mathbb{R}, \ g(t, s) = G(x^* + tw + sy) \).

Then \( g(0, 0) = G(x^*) = 0 \)

\( g_t (t, s) = G'(x^* + tw + sy) w \)

\( g_s (t, s) = G'(x^* + tw + sy) y \)

such that \( g_t (0, 0) = G'(x^*) w \)

\( g_s (0, 0) = G'(x^*) y \neq 0 \).

Implicit Function Theorem: There is \( \varepsilon > 0 \) and \( \phi : (-\varepsilon, \varepsilon) \to \mathbb{R} \) such that

\( \phi(0) = 0 \)

\( g(t, \phi(t)) = 0, \quad \forall t \in (-\varepsilon, \varepsilon) \)

\( \phi'(t) = -\frac{g_t (t, \phi(t))}{g_s (t, \phi(t))}, \quad \forall t \in (-\varepsilon, \varepsilon) \).
Then \( \varphi'(0) = - \frac{g_t(0, \varphi(0))}{g_s(0, \varphi(0))} = - \frac{G'(x^*)w}{G'(x^*)y} \) \hspace{1cm} (**)

Remark \( g_t(t, \varphi(t)) = 0, \quad \forall \; t \in (-\varepsilon, \varepsilon) \) implies

\( G(x + tw + \varphi(t)y) = 0, \quad \forall \; t \in (-\varepsilon, \varepsilon) \)

and therefore

\( x + tw + \varphi(t)y \) is admissible (feasible) point for \( \forall \; t \in (-\varepsilon, \varepsilon) \).

Define \( \psi : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}, \quad \psi(t) = F(x^* + tw + \varphi(t)y) \)

Then \( \psi(t) = F(x^*) \) so \( t = 0 \) is local min point to \( \psi \), thus \( \psi'(0) = 0 \).

Notice \( \psi'(t) = F'(x^* + tw + \varphi(t)y)(w + \varphi'(t)y) \).

Thus \( \psi'(0) = F'(x^*)(w + \varphi'(0)y) = 0 \quad \Rightarrow \quad \psi'(0) = F'(x^*)(w + \varphi'(0)y) = 0 \quad \Rightarrow \quad F'(x^*)w = -\varphi'(0)F'(x^*)y \) \hspace{1cm} (***)

\( \Rightarrow \quad F'(x^*)w = \frac{F'(x^*)y}{G'(x^*)} G'(x^*)w \)

Define \( \gamma = \frac{F'(x^*)y}{G'(x^*)} \) \hspace{1cm} Then \( F'(x^*)w = \gamma G'(x^*)w \quad \forall \; w \in \mathbf{X} \)
Applications

Existence of the eigenvalues

Consider \( \min_{u \in H^1(\Omega)} \left( \int \nabla u \cdot \nabla (u^2) \right) \) with \( \|u\|_0^2 = 1 \).

Here the constraint function is \( G : H^1 \to \mathbb{R}, \ G(u) = \|u\|_0^2 - 1, \ G'(u)v = 2(u, v) \).

and the function to minimize is \( J : H^1 \to \mathbb{R} \)

\( J(u) = \int \nabla u \cdot \nabla (u^2), \ J'(u)v = 2 \int \nabla u \cdot \nabla v \)

if \( u^* \) is solution to (W2) then
\( G'(u^*) v = 2(u^*, v), \ G(u^*) = 0 \)

Thus \( \|u^*\|_0 = 1 \). Therefore \( u^* \neq 0 \) thus \( G'(u^*) \neq 0 \). There is \( \lambda \in \mathbb{R} \) such that
\( J'(u^*) v = 2 G'(u^*) v, \forall v \in H^1 \)

\( \Rightarrow \int \nabla u^* \cdot \nabla v = 2 \int uv, \forall v \in H^1 \)

Remark: The existence of the minimizer \( u^* \) follows from \( J \) coercive and
\( J(\varepsilon v) : \|v\|_0^2 - 1 = 0 \) is weakly closed.
Consider \(-
abla u - |u|^{p-1} u = 0, u \geq 0, u \partial_2 = 0\)

where \(1 < p < \frac{n+2}{n-2}\) if \(n > 3\).

**Theorem** There is a nontrivial weak solution \(u \in H_0^1(\Omega)\):

\[\int_\Omega \nabla u \nabla v - \int_\Omega |u|^{p-1} u v = 0, \forall v \in H_0^1(\Omega)\]

**Proof** Let

\[F(u) = \frac{1}{2} \int_\Omega |u|^2, \quad G(u) = \frac{1}{p+1} \int_\Omega |u|^{p+1} - 1\]

Consider

\[\min_{u \in H_0^1} \begin{cases} F(u) \\ G(u) = 0 \end{cases} \quad (\star)\]

**Existence of the minimizer**

**Remark:** \(V = \{ v \in H_0^1 : G(v) = 0 \} \) is weakly closed in \(H_0^1\) since \(H_0^1 \xrightarrow{\text{compact}} L^{p+1}\), thus

\[v_k \to v \Rightarrow G(v_k) \to G(v)\]

and \(F\) is coercive and convex, thus w.l.o.g.
Let \( u \in V \) denote the minimizer, solution to (4). Notice \( G'(u) v = \int u |u|^{p-1} v \).

Thus \( G'(u) u = \int u |u|^{p+1} = p+1 \neq 0 \), therefore \( G'(u) \neq 0 \). Then there is a Lagrange multiplier \( \lambda \) such that

\[
F'(u) v = G'(u) v, \forall v \in H^1_0.
\]

\[
\Rightarrow \int u v \lambda = \int |u|^{p-1} u v, \forall v \in H^1_0.
\]

Look for the solution \( u^* \) as \( u^* = \lambda u, \lambda > 0 \).

\[
\int u = \frac{1}{\lambda} \int u^*
\]

\[
|u|^{p-1} u = \frac{1}{\lambda} |u^*|^{p-1} u^*
\]

\[
\Rightarrow \frac{1}{\lambda} \int |u^*| \, \lambda v = \frac{2}{\lambda} \int |u^*|^{p-1} u^* v
\]

Let \( 2^{p-1} = 2 \), or \( \lambda = 2^{\frac{1}{p-1}} \).

Remark: \( \lambda \neq 0 \) otherwise at \( u = v \) we have \( \int |u|^2 u^2 = 0 \) thus \( u \equiv 0 \) in contradiction with \( \|u\|_{L^{p+1}} = p+1 \).
Nonlinear eigenvalue problem.

Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) smooth function. Consider the problem
\[
\begin{align*}
-\Delta u &= g(u) \quad \text{in } \mathbb{R}, \\
(\frac{\partial u}{\partial r})_r &= 0
\end{align*}
\]

A pair \((u, \lambda)\) that solves (*), and such that \(\lambda \neq 0\) is an eigenpair.

Variational formulation: Find \( u \in H^1(\mathbb{R}) \):
\[
\int_{\mathbb{R}} \frac{1}{2} |\nabla u|^2 = \int_{\mathbb{R}} g(u)u, \quad \forall u \in H^1(\mathbb{R}).
\]

Assume \( |g(t)| \leq C(1 + |t|^p + 1) \) with \( 1 \leq p < \frac{n+2}{n-2} \) for \( n \geq 3 \).

Then \( G(t) = \int_0^t g(t) dt \) satisfies
\[
|G(t)| \leq \int_0^{|t|} |g(t)| dt \leq C(|t|^{p+1} + |t|)
\]

Thus, \( J : H^1(\mathbb{R}) \rightarrow \mathbb{R}, J(u) = \int_0^{|u|} G(t) dt \) is well-defined, continuously differentiable.

Let \( \lambda \) arbitrary, fixed. \( J'(u) \) is weakly closed since \( v_k \rightarrow u \) in \( L^p(\mathbb{R}) \) implies \( J'(v_k) \rightarrow J'(v) \).