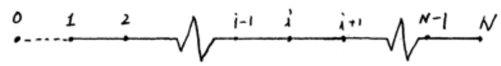
Governing eqn. : $$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad (1)$$

Initial condition : $$T(x, 0) = 100\,^\circ C \qquad (2)$$

Boundary Conditions : $$T(L, t) = 0\,^\circ C \qquad (3)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad (4)$$

A. Discretization



$$\left.\frac{\partial T}{\partial t}\right|_i \approx \frac{T_i^{p+1} - T_i^p}{\Delta t} \qquad (5)$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_i \approx \frac{T_{i+1}^p - 2T_i^p + T_{i-1}^p}{\Delta x^2} \qquad (6)$$

Substituting (5) & (6) into (1) gives

$$\frac{T_i^{p+1} - T_i^p}{\Delta t} = \alpha \frac{T_{i+1}^p - 2T_i^p + T_{i-1}^p}{\Delta x^2}$$

$$\Rightarrow \quad T_i^{p+1} = T_i^p + \frac{\alpha \Delta t}{\Delta x^2}\left(T_{i+1}^p - 2T_i^p + T_{i-1}^p\right) \qquad (7)$$

$$= \alpha \frac{\Delta t}{\Delta x^2}\left(T_{i+1}^p + T_{i-1}^p\right) + \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right) T_i^p$$

Initial Condition : $T_i^0 = 100\,^\circ C \qquad i = 1, 2, 3 \ldots N-1.$

$T_N^0 = 20\,^\circ C$

Boundary Conditions : $T_N = 0$

T_1 : $$\frac{T_1^{p+1} - T_1^p}{\Delta t} = \alpha \frac{T_2^p - 2T_1^p + T_0^p}{\Delta x^2}$$

$$\Rightarrow \quad T_1^{p+1} = T_1^p + \alpha \frac{\Delta t}{\Delta x^2}\left(T_2^p - 2T_1^p + T_0^p\right)$$

$$= 2\alpha \frac{\Delta t}{\Delta x^2} T_2^p + \left(1 - 2\alpha \frac{\Delta t}{\Delta x^2}\right) T_1^p$$

* This B.C. is obtained using a ghost node 0 with the fact that $T_2 = T_0$.

material properties

Copper : $\rho = 8933$ kg/m^3

$C_p = 385$ $J/kg\cdot K$

$k = 401$ $W/m\cdot K$

glass : $\rho = 2500$ kg/m^3

$C_p = 750$ $J/kg\cdot K$

$k = 1.4$ $W/m\cdot K$

We know that $F_o = \alpha \frac{\Delta t}{\Delta x^2}$. For explicit method, stability criterion requires

$$1 - 2F_o \le 0$$

$$\Rightarrow F_o \le \frac{1}{2}.$$

Because of different boundary condition, the analytic solution in the textbook (§5.5.1) can not be used. The analytic solution, however, can be obtained by using Fourier's method, which gives

$$T = \sum_{n=1}^{\infty} \frac{400}{(2n-1)\pi} \exp\left[-\frac{(2n-1)^2\pi^2}{4L^2}\alpha t\right] \cos\frac{(2n-1)\pi}{2L}x.$$

For midplane,

$$T_0 = \sum_{n=1}^{M} \frac{400}{(2n-1)\pi} \exp\left[-\frac{(2n-1)^2\pi^2}{4L^2}\alpha t\right]$$

The analytic solution for midplane with copper is shown in figure 2a. The inset is the blow-up view of the results within the square. Note that when M >= 10 the agreement between numerical & analytic solutions is very good!

Discussion

Based on the stability criterion for explicit method, it can be shown that Fo≤0.5 is required.

Figure 1 shows the temperature distribution in half of the slab. There are 11 nodes. The Fourier number is 0.25 for both copper and glass. The plot shows the data for every 50 time steps as the increase of the time is in the direction of the arrow. Note that the temperature distribution is the same for both materials only in terms of number of time steps. The real time difference between every two data lines is actually different. For the same Fo=0.25, the time step is 0.0134 second for copper and 2.0926 second for glass. Obviously it takes much longer time for glass to reach the same temperature as compared to copper. This is evident if one looks at the time histories of the temperature variation in the midplane of the slab which are shown in figure 2. It is found that while it takes 15.5 seconds for copper to reach 0.1 ^{o}C in the midplane, it takes 2419 seconds for glass.

The difference of the behavior is caused by the value of diffusivity α of two material. It is 1.17E-4 for copper while 7.47E-7 for glass.

To increase the time step so that Fo=0.51 that is greater than the critical value, one can see the oscillation of the numerical solution which obviously is not correct. The results are shown in figure 3

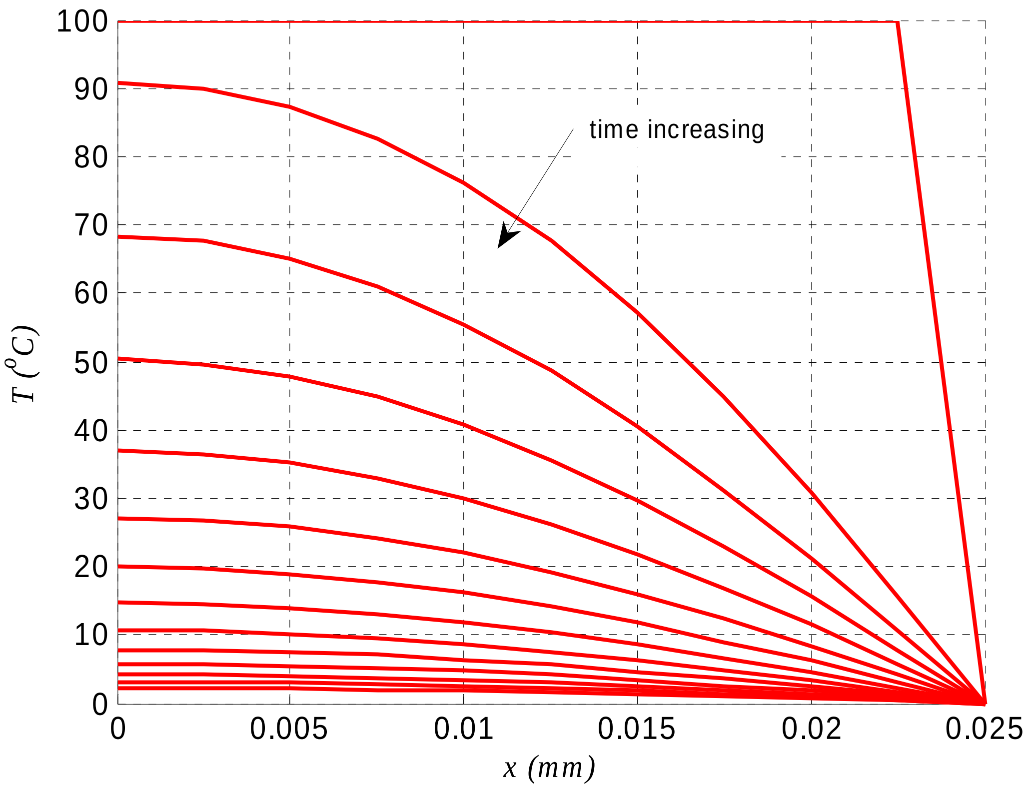
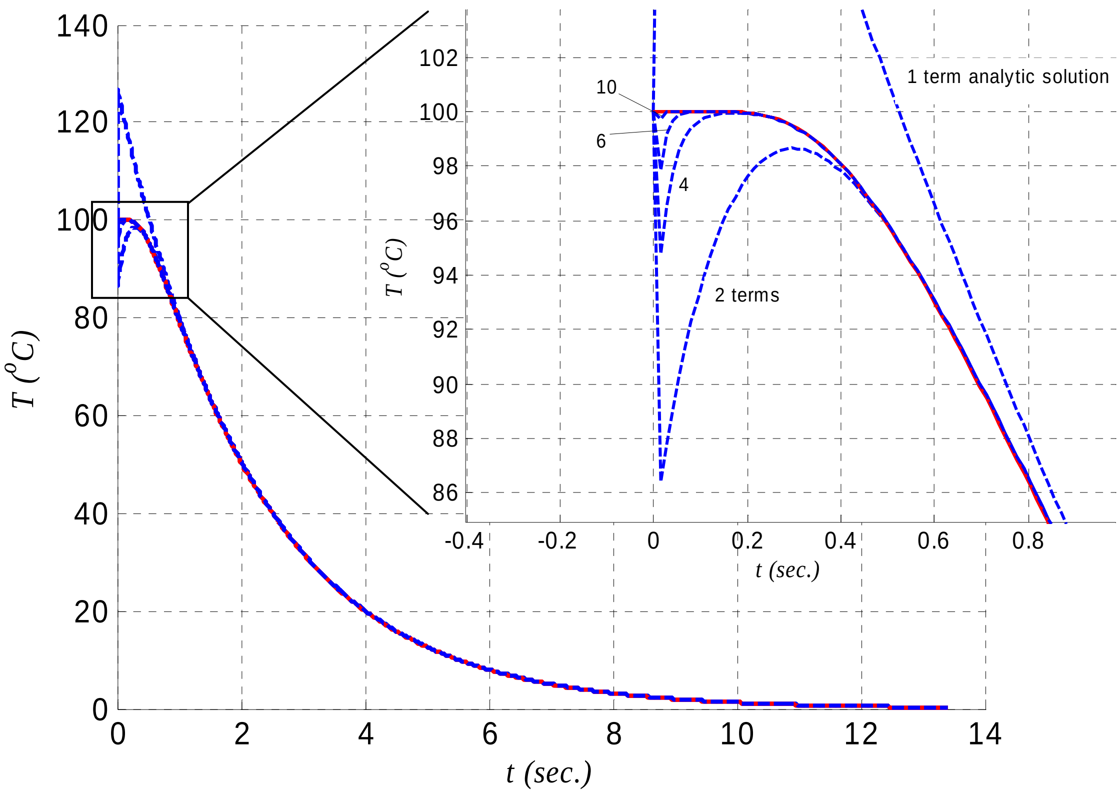
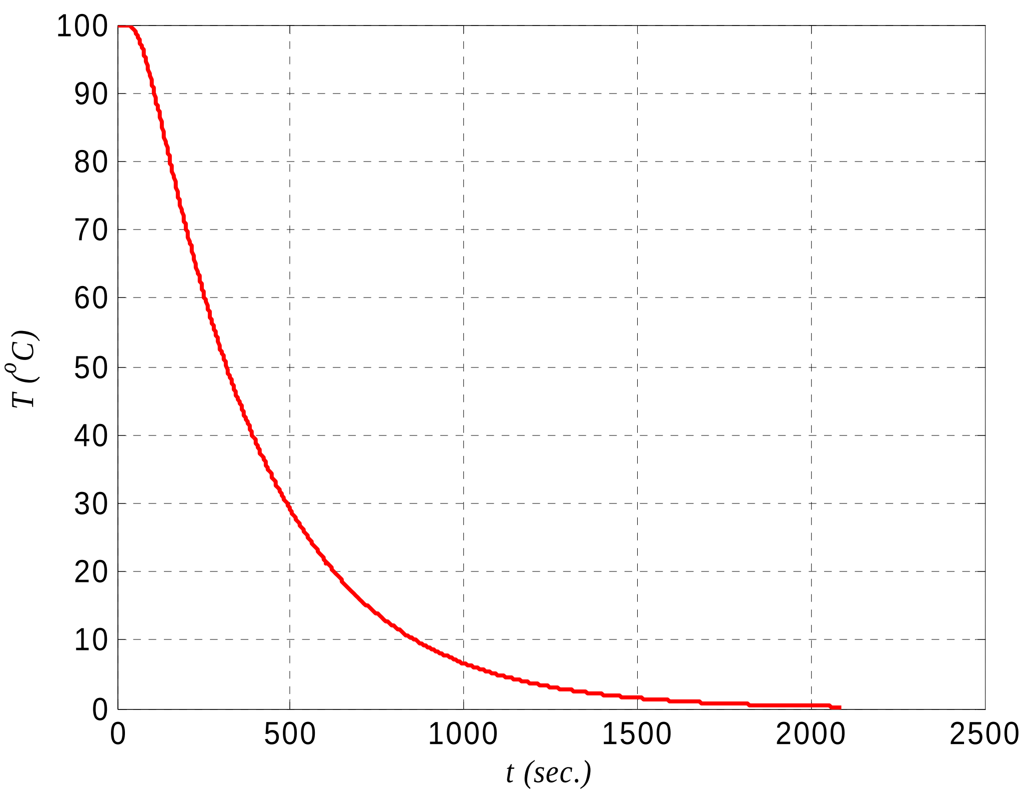


Figure 1. Temperature distribution in the slab. Fo=0.25



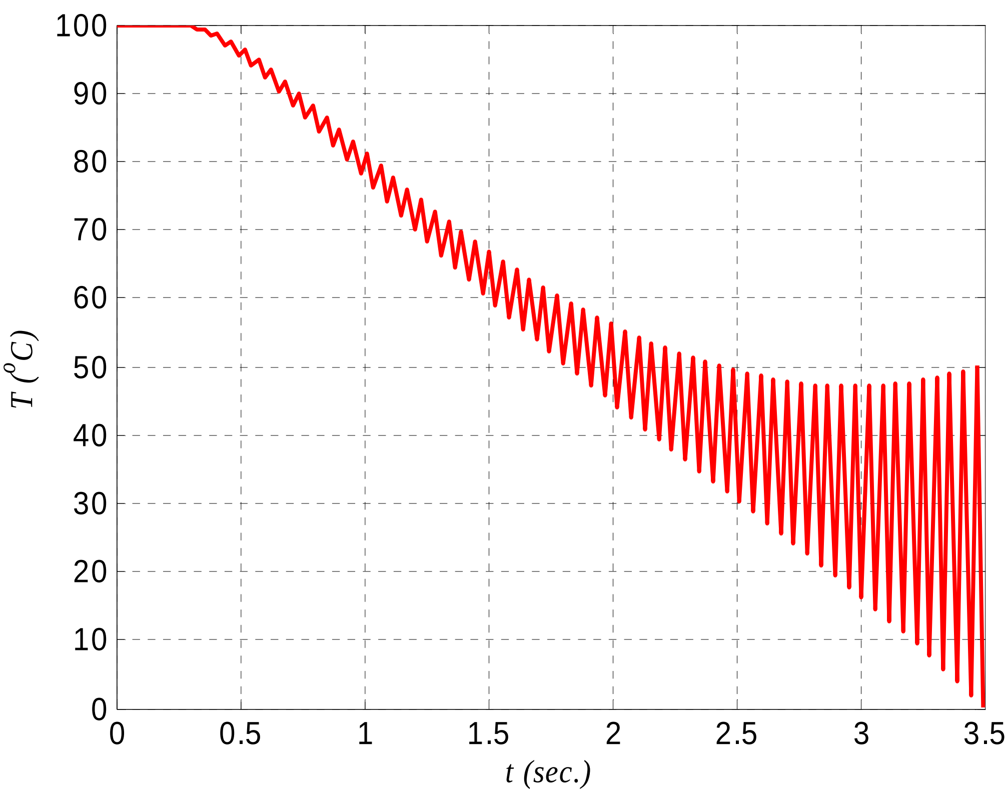
a. copper



b. glass

Figure 2. Time history of temperature variation in the midplane.

a. Temperature distribution



b. Time history of midplane temperature.

Figure 3. Results for Fo=0.51