PROBLEM 3.101

KNOWN: Dimensions of a plate insulated on its bottom and thermally joined to heat sinks at its ends. Net heat flux at top surface.

FIND: (a) Differential equation which determines temperature distribution in plate, (b) Temperature distribution and heat loss to heat sinks.

SCHEMATIC:

ASSUMPTIONS: (1) Steady-state, (2) One-dimensional conduction in x \((W,L>>t)\), (3) Constant properties, (4) Uniform surface heat flux, (5) Adiabatic bottom, (6) Negligible contact resistance.

ANALYSIS: (a) Applying conservation of energy to the differential control volume, \(q_x + dq_x = q_x + dx\), where \(q_x+dx = q_x + (dq_x/dx) dx\) and \(dq = q_0 (W \cdot dx)\). Hence, \((dq_x/dx) - q_0 W = 0\).

From Fourier’s law, \(q_x = -k (t \cdot W) dT/dx\). Hence, the differential equation for the temperature distribution is

\[
-\frac{d}{dx} \left[ kT \frac{dT}{dx} \right] - q_0 W = 0
\]

\[
\frac{d^2 T}{dx^2} + \frac{q_0}{kt} = 0.
\]

(b) Integrating twice, the general solution is,

\[
T(x) = -\frac{q_0}{2kt} x^2 + C_1 x + C_2
\]

and appropriate boundary conditions are \(T(0) = T_o\), and \(T(L) = T_o\). Hence, \(T_o = C_2\), and

\[
T_o = -\frac{q_0}{2kt} L^2 + C_1 L + C_2
\]

\[
C_1 = \frac{q_0 L}{2kt}.
\]

Hence, the temperature distribution is

\[
T(x) = -\frac{q_0 L}{2kt} (x^2 - Lx) + T_o.
\]

Applying Fourier’s law at \(x = 0\), and at \(x = L\),

\[
q(0) = -k \left( Wt \cdot \frac{dT}{dx} \right)_{x=0} = -kWt \left[ -\frac{q_0}{kt} \right] \left[ x - \frac{L}{2} \right] x=0 = -\frac{q_0 WL}{2}
\]

\[
q(L) = -k \left( Wt \cdot \frac{dT}{dx} \right)_{x=L} = -kWt \left[ -\frac{q_0}{kt} \right] \left[ x - \frac{L}{2} \right] x=L = +\frac{q_0 WL}{2}
\]

Hence the heat loss from the plates is \(q = 2(q_0 WL/2) = q_0 WL\).

COMMENTS: (1) Note signs associated with \(q(0)\) and \(q(L)\). (2) Note symmetry about \(x = L/2\). Alternative boundary conditions are \(T(0) = T_o\) and \(dT/dx)_{x=L/2} = 0\).