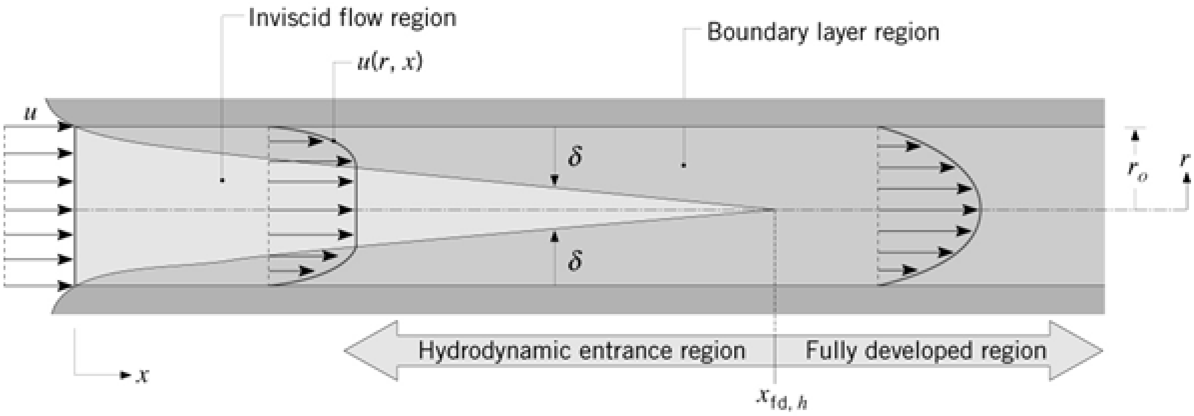
Hydrodynamic boundary layer

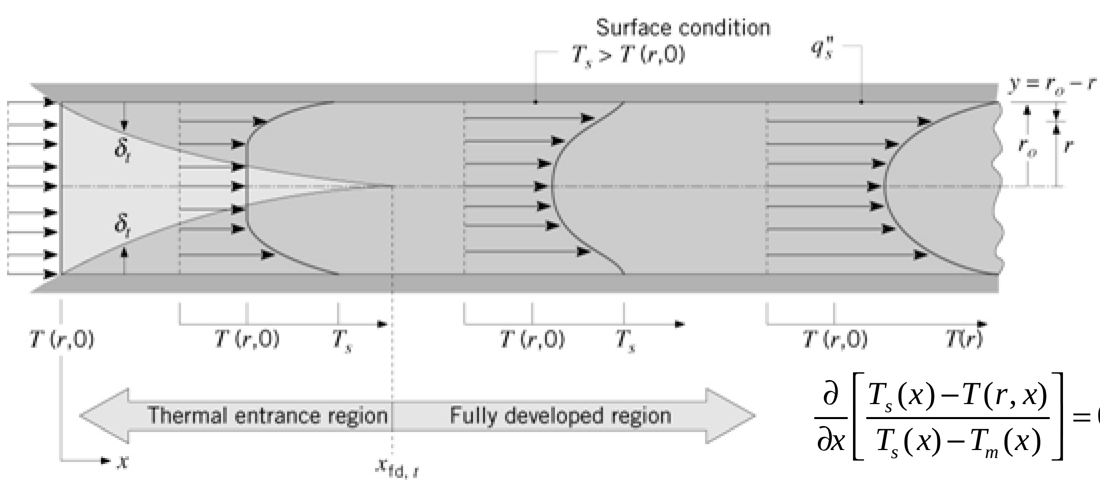


Laminar flow: $\left(\frac{x_{fd,h}}{D}\right)_{lam} \approx 0.05 Re_D$

Turbulent flow: $10 \le \left(\frac{x_{fd,h}}{D}\right)_{turb} \le 60$

Laminar flow of an incompress ible, constant property fluid $\frac{\partial u}{\partial x} = 0$

Thermal boundary layer



$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0$$

Laminar flow: $\left(\frac{x_{fd,t}}{D}\right)_{lam} \approx 0.05 Re_D Pr$

Turbulent flow: $\left(\frac{x_{fd,t}}{D}\right)_{turb} = 10$

* Internal flow

mean velocity U_m is defined such that

$$\dot{m} = \rho U_m A_c$$

$$Re_D = \frac{\rho U_m D}{\mu} = \frac{4 \rho U_m \frac{1}{4} \pi D^2}{\pi D \mu}$$

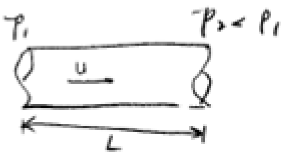
$$= \frac{4 \dot{m}}{\pi D \mu}$$

* Fully developed region

$$v = 0, \quad \frac{\partial u}{\partial x} = 0.$$

$$\Rightarrow u(x, r) = u(r)$$

* Pressure drop



$$f \equiv \frac{-\frac{dp}{dx} D}{\frac{1}{2} \rho u_m^2}$$

$$\Rightarrow p_1 - p_2 = f \cdot \frac{L}{D} \frac{1}{2} \rho u_m^2$$

For fully developed laminar flow in circular tube.

$$f = \frac{64}{Re_D}$$

For fully developed turbulent flow.

$$f = 0.316 \, Re_D^{-1/4} \qquad Re_D \lesssim 2 \times 10^4$$

$$f = 0.184 \, Re_D^{-1/5} \qquad Re_D \gtrsim 2 \times 10^4$$

* Thermal entry length

$$\left(\frac{x_{fd,t}}{D}\right)_{lam} \approx 0.05 \, Re_D \, Pr, \quad \left(\frac{x_{fd,h}}{D}\right)_{lam} \approx 0.05 \, Re_D$$

$$Pr = \frac{\nu}{\alpha}$$

* Mean temp.

$$q = \dot{m} c_p (T_{out} - T_{in})$$

$$\dot{m} c_p T_m = \int_{A_c} \rho u c_p T \, dA_c$$

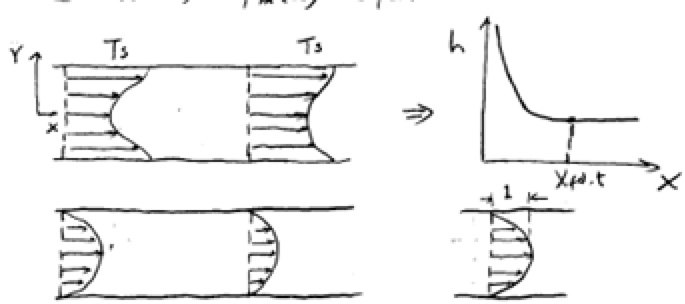
* Newton's cooling law

$$q_s'' = h (T_s - T_m)$$ ← const q'' and const T_s can not be imposed simultaneously.

* $\frac{dT_m}{dx} \neq 0$.

* Thermally fully developed

$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)} \right]_{fd,t} = 0$$



$T_s(x) - T(r,x)$ $\qquad$ $\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}$

relative shape of profile no longer changes.

* in essence, the location of $T_m(x)$ is always at the same r value.

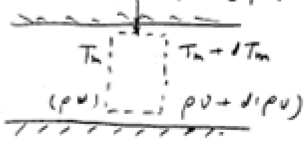
surface q'' const. $\quad \left.\frac{dT_s}{dx}\right|_{fd,t} = \left.\frac{dT_m}{dx}\right|_{fd,t}$

T_s const. $\quad \left.\frac{\partial T}{\partial x}\right|_{fd,t} = \frac{T_s - T}{T_s - T_m} \left.\frac{dT_m}{dx}\right|_{fd,t}$

EXAMPAD

III. Energy balance — to determine $T_m(x)$

$dq_{conv} = q_s'' P dx$



$$dq_{conv} + \dot{m}(c_v T_m + pv) - \left[\dot{m}(c_v T_m + pv) + \dot{m}\frac{d(c_v T_m + pv)}{dx}dx\right] = 0$$

(dq_{conv}: energy input; $c_v T_m$: internal energy; pv: flow work)

$\Rightarrow \quad \dot{m}\, d(c_v T_m + pv) = dq_{conv} = \dot{m} c_p dT_m$ (incompressible fluid)

pressure drop ignored $\Rightarrow \quad q_{conv} = \dot{m} c_p (T_{m,o} - T_{m,i})$

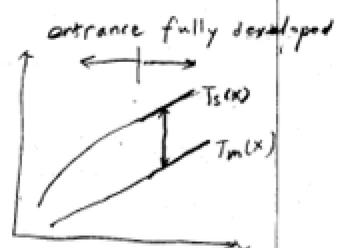
important $\rightarrow$ $\Rightarrow \quad \dfrac{dT_m}{dx} = \dfrac{q_s'' P}{\dot{m} c_p} = \dfrac{P}{\dot{m} c_p} h (T_s - T_m)$

— $T_s > T_m$, $\dfrac{dT_m}{dx} > 0$. heat is added to fluid.

— constant surface heat flux q_s''

$$\left(\frac{dT_s}{dx} = \right)\frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p} \neq f(x)$$

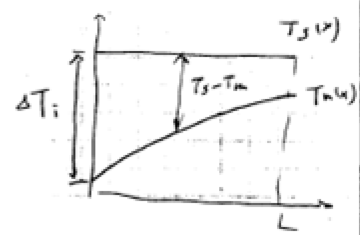
$$\Rightarrow \quad T_m(x) = T_{m,i} + \frac{q_s'' P}{\dot{m} c_p} x$$



— constant surface temperature

$$\frac{T_s - T_m(x)}{T_s - T_{m,i}} = \exp\left(-\frac{Px}{\dot{m} c_p}\bar{h}\right) \qquad \bar{h} : \text{average.}$$

$$q_{conv} = \bar{h} A_s \Delta T_{lm}$$



log mean temp. $\Delta T_{lm} \equiv \dfrac{\Delta T_o - \Delta T_i}{\ln(\Delta T_o / \Delta T_i)}$

$\Delta T_i = T_s - T_{m,i}$

$\Delta T_o = T_s - T_{m,o}$

IV. Convection correlation — laminar flow

energy eqn. $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \alpha\frac{\partial^2 T}{\partial x^2}$

$$\frac{u(r)}{u_m} = 2\left[1-\left(\frac{r}{r_o}\right)^2\right]$$

in fully developed region:

$$v = 0, \quad \frac{\partial v}{\partial x} = 0.$$

— constant surface heat flux

$$\frac{\partial^2 T}{\partial x^2} = 0. \qquad \left(\frac{\partial T}{\partial x} = \frac{dT_m}{dx} + f(x)\right)$$

B.C. T finite at $r=0$

$T = T_s$ at $r = r_o$

$$\Rightarrow T = T_s - \frac{2u_m r_o^2}{\alpha}\left(\frac{dT_m}{dx}\right)\left[\frac{3}{16} + \frac{1}{16}\left(\frac{r}{r_o}\right)^4 - \frac{1}{4}\left(\frac{r}{r_o}\right)^2\right]$$

$$\Rightarrow T_m = T_s(x) - \frac{11}{48}\left(\frac{u_m r_o^2}{\alpha}\right)\left(\frac{dT_m}{dx}\right)$$

$$\frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p}, \quad \dot{m} = \rho u_m (\pi D^2/4)$$

$$\Rightarrow T_m(x) - T_s(x) = -\frac{11}{48}\frac{q_s'' D}{k}$$

$$q_s'' = h(T_m - T_s)$$

Important!
in a circular tube characterized by uniform surface heat flux and laminar fully developed conditions, Nu is a constant.

$$\Rightarrow h = \frac{48}{11}\left(\frac{k}{D}\right)$$

$$\Rightarrow \underline{Nu_D \equiv \frac{hD}{k} = 4.36}$$

— constant surface temp. (little more complex).

$$\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial r^2}, \quad \text{no closed form sol.}$$

$$\Rightarrow \underline{Nu_D = 3.66}$$

to determine h, k should be evaluated at T_m

8.4.2 The Entry Region

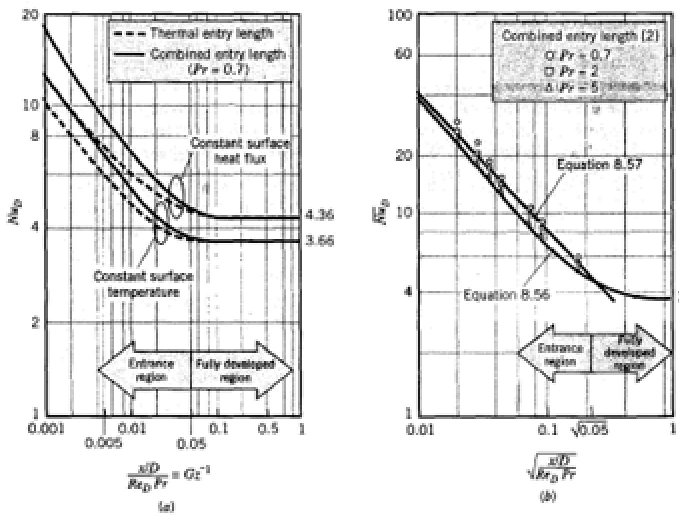


FIGURE 8.10 Results obtained from entry length solutions for laminar flow in a circular tube: (*a*) local Nusselt numbers and (*b*) average Nusselt numbers [2].

$$\overline{Nu}_D = 3.66 + \frac{0.0668(D/L)Re_D Pr}{1 + 0.04[(D/L)Re_D Pr]^{2/3}} \tag{8.56}$$

$$\begin{bmatrix} \text{thermal entrance length} \\ \textit{or} \\ \text{combined entrance length with } Pr \gtrsim 5 \end{bmatrix}$$

$$\overline{Nu}_D = 1.86\left(\frac{Re_D Pr}{L/D}\right)^{1/3}\left(\frac{\mu}{\mu_s}\right)^{0.14} \tag{8.57}$$

$$0.60 \lesssim Pr \lesssim 5$$

$$0.0044 \lesssim \left(\frac{\mu}{\mu_s}\right) \lesssim 9.75$$

— Noncircular tubes.

a) hydraulic diameter

$$D_h = \frac{4A_c}{P}$$

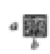
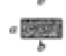
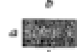
A_c : cross-sectional area.

P : wetted perimeter.

$$Nu_{D_h} \equiv \frac{h D_h}{k} \quad , \quad Re_{D_h} = \frac{u_m D_h}{\nu}$$

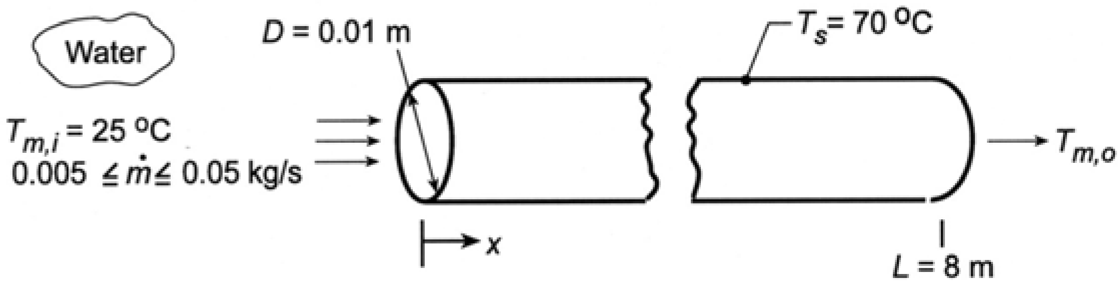
b) Nu_D see Table 8.1

TABLE 8.1 Nusselt numbers and friction factors for fully developed laminar flow in tubes of differing cross section

Cross Section	$\frac{b}{a}$	$Nu_D = \frac{hD_h}{k}$ (Uniform q_s'')	(Uniform T_s)	$f\,Re_{D_h}$
	—	4.36	3.66	64
	1.0	3.61	2.98	57
	1.43	3.73	3.08	59
	2.0	4.12	3.39	62
	3.0	4.79	3.96	69
	4.0	5.33	4.44	73
	8.0	6.49	5.60	82
	∞	8.23	7.54	96
 Heated / Insulated	∞	5.39	4.86	96
	—	3.11	2.49	53

Used with permission from W. M. Kays and M. E. Crawford, *Convection Heat and Mass Transfer*, 3rd ed. McGraw-Hill, New York, 1993.

Determine the effect of flow rate on outlet temperature and heat rate for water flow through the tube of a flat-plate solar collector.



KNOWN: Diameter and length of copper tubing. Temperature of Collector plate to which tubing is soldered. Water inlet temperature And flow rate.

FIND: Water outlet temperature and heat rate for $\dot{m} = 0.01\text{kg/s}$

ASSUMPTIONS:
1) Straight tube with smooth surface,
2) Negligible kinetic/potential energy and flow work changes,
3) Negligible thermal resistance between plate and tube inner surface
4) $\text{Re}_{D,c} = 2300$

PROPERTIES: Table A.6, water (assume

$$\overline{T}_m = (T_{m,i} + T_s)/2 = 47.5°C = 320.5K) : r = 986\ \text{kg/m}^3,$$

$$c_p = 4180\ \text{J/kg}\cdot\text{K}, m = 577\times10^{-6}\ \text{N}\cdot\text{s/m}^2,$$

$$k = 0.640\ \text{W/m}\cdot\text{K}, \text{Pr} = 3.77$$

Table A.6 water $(T_s = 343K) : m_s = 400\times10^{-6}\ \text{N}\cdot\text{s/m}^2$

ANALYSIS: for $\dot{m} = 0.01\text{kg/s}$

$$\text{Re}_D = 4\dot{m}/(pDm) = 2200,$$

in which case the flow may be assumed to be laminar.

With $x_{fd,t}/D \approx 0.5\,\text{Re}_D\,\text{Pr} = 0.05 \times 2200 \times 3.77 = 415$ and $L/D = 800$, the flow is fully developed over approximately 50% of the tube length.

With $(m/m_s)^{0.14} = 1.05$, therefore Eq. 8.57 may be used to compute the average coefficient

$$\overline{\text{Nu}}_\text{D} = 1.86\left(\frac{\text{Re}_D\,\text{Pr}}{L/D}\right)^{1/3}\left(\frac{m}{m_s}\right)^{0.14} = 4.27$$

$$\overline{\text{h}} = (\text{k/D})\,\overline{\text{Nu}}_\text{D} = 4.27 \times 0.64/0.01 = 273\ \text{W/m}^2 \cdot \text{K}$$

From Eq. 8.41b,

$$\frac{T_\text{s} - T_\text{m,o}}{T_\text{s} - T_\text{m,i}} = \exp\left(-\frac{pDL}{\dot{m}c_p}\bar{h}\right) = \exp\left(-\frac{p \times 0.01 \times 8 \times 273}{0.01 \times 4186}\right)$$

$$\text{T}_\text{m,o} = \text{T}_\text{s} - 0.194(T_\text{s} - T_\text{m,i}) = 61.3°\text{C}$$

$$q = \dot{m}c_p\left(T_\text{m,o} - T_\text{m,i}\right) = 0.01 \times 4186 \times (61.3 - 25) = 1519\text{W}$$