Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

THE CHARACTERISTIC EQUATION

Corresponding to the differential equation

$$y'' + a_1 y' + a_0 y = 0 \tag{8.1}$$

in which a_1 and a_0 are constants, is the algebraic equation

$$\lambda^2 + a_1\lambda + a_0 = 0 \tag{8.2}$$

which is obtained from Eq. (*8.1*) by replacing y'', y', and y by λ^2, λ^1, and $\lambda^0 = 1$, respectively. Equation (*8.2*) is called the *characteristic equation* of (*8.1*).

Example 8.1. The characteristic equation of $y'' + 3y' - 4y = 0$ is $\lambda^2 + 3\lambda - 4 = 0$; the characteristic equation of $y'' - 2y' + y = 0$ is $\lambda^2 - 2\lambda + 1 = 0$.

Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j ($j = 0, 1, 2$).

The characteristic equation can be factored into

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \tag{8.3}$$

THE GENERAL SOLUTION

The general solution of (*8.1*) is obtained directly from the roots of (*8.3*). There are three cases to consider.

Case 1. λ_1 and λ_2 both real and distinct. Two linearly independent solutions are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$, and the general solution is (Theorem 7.2)

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{8.4}$$

In the special case $\lambda_2 = -\lambda_1$, the solution (*8.4*) can be rewritten as $y = k_1 \cosh \lambda_1 x + k_2 \sinh \lambda_1 x$.

Case 2. $\lambda_1 = a + ib$, a complex number. Since a_1 and a_0 in (*8.1*) and (*8.2*) are assumed real, the roots of (*8.2*) must appear in conjugate pairs; thus, the other root is $\lambda_2 = a - ib$. Two linearly independent solutions are $e^{(a+ib)x}$ and $e^{(a-ib)x}$, and the general complex solution is

$$y = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x} \tag{8.5}$$

which is algebraically equivalent to (see Problem 8.16)

$$y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \tag{8.6}$$

Case 3. $\lambda_1 = \lambda_2$. Two linearly independent solutions are $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$, and the general solution is

$$y = c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x} \tag{8.7}$$

Warning: The above solutions *are not valid* if the differential equation is not linear or does not have constant coefficients. Consider, for example, the equation $y'' - x^2y = 0$. The roots of the characteristic equation are $\lambda_1 = x$ and $\lambda_2 = -x$, but the solution is *not*

$$y = c_1e^{(x)x} + c_2e^{(-x)x} = c_1e^{x^2} + c_2e^{-x^2}$$

Linear equations with variable coefficients are considered in Chapters 23 and 24.

Solved Problems

8.1. Solve $y'' - y' - 2y = 0$.

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, which can be factored into $(\lambda + 1)(\lambda - 2) = 0$. Since the roots $\lambda_1 = -1$ and $\lambda_2 = 2$ are real and distinct, the solution is given by (*8.4*) as

$$y = c_1e^{-x} + c_2e^{2x}$$

8.2. Solve $y'' - 7y' = 0$.

The characteristic equation is $\lambda^2 - 7\lambda = 0$, which can be factored into $(\lambda - 0)(\lambda - 7) = 0$. Since the roots $\lambda_1 = 0$ and $\lambda_2 = 7$ are real and distinct, the solution is given by (*8.4*) as

$$y = c_1e^{0x} + c_2e^{7x} = c_1 + c_2e^{7x}$$

8.3. Solve $y'' - 5y = 0$.

The characteristic equation is $\lambda^2 - 5 = 0$, which can be factored into $(\lambda - \sqrt{5})(\lambda + \sqrt{5}) = 0$. Since the roots $\lambda_1 = \sqrt{5}$ and $\lambda_2 = -\sqrt{5}$ are real and distinct, the solution is given by (*8.4*) as

$$y = c_1e^{\sqrt{5}x} + c_2e^{-\sqrt{5}x}$$

8.4. Rewrite the solution of Problem 8.3 in terms of hyperbolic functions.

Using the results of Problem 8.3 with the identities

$$e^{\lambda x} = \cosh \lambda x + \sinh \lambda x \qquad \text{and} \qquad e^{-\lambda x} = \cosh \lambda x - \sinh \lambda x$$

we obtain,

$$\begin{aligned} y &= c_1e^{\sqrt{5}x} + c_2e^{-\sqrt{5}x} \\ &= c_1(\cosh \sqrt{5}x + \sinh \sqrt{5}x) + c_2(\cosh \sqrt{5}x - \sinh \sqrt{5}x) \\ &= (c_1 + c_2)\cosh \sqrt{5}x + (c_1 - c_2)\sinh \sqrt{5}x \\ &= k_1 \cosh \sqrt{5}x + k_2 \sinh \sqrt{5}x \end{aligned}$$

where $k_1 = c_1 + c_2$ and $k_2 = c_1 - c_2$.

8.5. Solve $\ddot{y} + 10\dot{y} + 21y = 0$.

Here the independent variable is t. The characteristic equation is

$$\lambda^2 + 10\lambda + 21 = 0$$

which can be factored into

$$(\lambda + 3)(\lambda + 7) = 0$$